Insensitive Arc in Domination of Fuzzy Graph

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Abstract

A connected fuzzy graph is arc insensitive if the domination number is unchanged when any single arc is removed. The minimum number of arcs required by such arc insensitive fuzzy graph is determined. Similarly we determine the minimum number of arcs required for the fuzzy graph to remain connected after the removal of any arc, when the dominating set remain fixed.

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1. INTRODUCTION


Among the various applications to the theory of domination in fuzzy graph, here we consider the fault tolerant property in communication network. With smallest number of stations we will communicate with other stations. Our problem is to build a fault tolerant communication network, that is, even if any communication link to a station is failed, still it can communicate the message to that station. In section 3 the fault tolerant is represented in arc insensitive fuzzy graph. In the section 4 we determine the number of arcs required by the fuzzy graph to have this property.

In section 5 we consider the fuzzy graph to hold only the connectedness property and the minimum number of arcs for such fuzzy graph.

2. PRELIMINARIES

A fuzzy subset of a nonempty set V is a mapping \( \sigma : V \rightarrow [0,1] \). A fuzzy relation on V is a fuzzy subset of V \( \times \) V. A fuzzy graph \( G = (\sigma, \mu) \) is a pair of function \( \sigma : V \rightarrow [0,1] \) and \( \mu : V \times V \rightarrow [0,1] \), where \( \mu(u, v) \leq \sigma(u) \wedge \sigma(v) \) for all \( u, v \in V \). The underlying crisp graph of \( G = (\sigma, \mu) \) is denoted by \( G^* = (V,E) \), where \( V = \{ u \in V : \sigma(u) > 0 \} \) and \( E = \{ (u, v) \in V \times V : \mu(u, v) > 0 \} \). The order p and size q of fuzzy graph \( G = (\sigma, \mu) \) are defined by \( p = \sum_{v \in V} \sigma(v) \) and \( q = \sum_{(u,v) \in E} \mu(u, v) \). The graph \( G = (\sigma, \mu) \) is denoted by \( G \), if unless otherwise mentioned.

The strength of connectedness between two nodes \( u, v \) in a fuzzy graph G is \( \mu^\infty(u, v) = \sup \{ \mu^k(u, v) : k=1,2,3,... \} \) where \( \{ \mu^k(u, v) = \sup \{ \mu(u, u_1) \wedge \mu(u_1, u_2) \wedge ... \wedge \mu(u_{k-1}, v) \} \}. An arc \( (u, v) \) is said to be a strong arc if \( \mu(u, v) \geq \mu^\infty(u, v) \) and the node \( v \) is said to be a strong neighbor of \( u \). If \( \mu(u, v) = 0 \) for every \( v \in V \), then \( u \) is called isolated node.

Let \( u \) be a node in fuzzy graph G then \( N(u) = \{ v : (u, v) \text{ is a strong arc} \} \) is called neighborhood of \( u \) and \( N[u] = N(u) \cup \{ u \} \) is called closed neighborhood of \( u \). Neighborhood degree of the node is defined by the sum of the weights of the strong neighbor node of \( u \) and is denoted by \( d_N(u) = \sum_{v \in N(u)} \sigma(v) \). Minimum neighborhood degree of a fuzzy graph G is defined by \( \delta_N(G) = \min \{ d_N(u) : u \in V(G) \} \) and maximum neighborhood degree of G is by \( \Delta_N(G) = \max \{ d_N(u) : u \in V(G) \} \).

3. FUZZY DOMINATING SET

**Definition 3.1:** Let G be a fuzzy graph and u be a node in G then there exists a node v such that \((u,v)\) is a strong arc then u dominates v.

**Definition 3.2:** Let G be a fuzzy graph. A subset D of V is said to be a fuzzy
dominating set if for every \( v \in V - D \), there exists \( u \in D \) such that \( u \) dominates \( v \).

The domination number of \( G \) is the minimum cardinality taken over all dominating sets in \( G \) and is denoted by \( \gamma(G) \), where \( \gamma(G) = \sum_{v \in D} \sigma(v) \). A dominating set with cardinality \( \gamma(G) \) is called \( \gamma \)-set of \( G \).

**Definition 3.3:** Let \( G \) be a fuzzy graph. Let \( S \) be a set of vertices in \( G \). Let \( u \in S \) then the private neighborhood of \( u \) is \(pn[u,S] = \{v : N(u) \cap S = \{u\}\} \).

### 4. FUZZY INSENSITIVE ARC

**Definition 4.1:** The fuzzy graph \( H = (\tau, \rho) \) is called a fuzzy subgraph of \( G \) if \( \tau(x) \leq \sigma(x) \) for all \( x \in V \) and \( \rho(x, y) \leq \mu(x, y) \) for all \( x, y \in V \).

**Definition 4.2:** A fuzzy subgraph \( H = (\tau, \rho) \) is said to be a spanning fuzzy subgraph of \( G = (\sigma, \mu) \) if \( \tau(x) = \sigma(x) \) for all \( x \).

**Definition 4.3:** A fuzzy graph \( G = (\sigma, \mu) \) is said to be connected if there exists a strongest path between any two nodes of \( G \).

**Definition 4.4:** A fuzzy graph \( G = (\sigma, \mu) \) is fuzzy bipartite if it has a spanning fuzzy subgraph \( H = (\tau, \pi) \) which is bipartite where for all edges \((u,v)\) not in \( H = (\tau, \pi) \) weight of \((u,v)\) in \( G \) is strictly less than the strength of pair \((u,v)\) in \( H \).

**Example 4.5:**

![Graph](https://via.placeholder.com/150)

Fig. 4.1

From Fig. 4.1., the fuzzy graph \( G \) is fuzzy bipartition with \( V_1 = \{x,z\} \) and \( V_2 = \{y,w\} \) of the node set \( P \).

**Definition 4.6:** The fuzzy graph \( G \) is said to be arc insensitive if \( \gamma(G) = \gamma(G-e) \) for any arc \( e \) of \( G \). We say as \( \gamma \)-insensitive when the domination number is \( \gamma \).

Throughout this paper we consider the fuzzy graph \( G \) to be connected. Let \( D_1 \)
be the minimum dominating set of $G$ with $k$ nodes where $k<n$ and domination number $\gamma(G)$. Let $D_2$ be the complement of $D_1$ (i.e.) $D_2 = V - D_1$.

Example 4.7:

From Fig. 4.2., $D_1 = \{a,f,e\}, D_2 = \{b,c,d\}$. $\gamma = 0.6$ and this fuzzy graph is arc insensitive. i.e., $\gamma(G-e) = \gamma(G)$.

**Theorem 4.8:** A fuzzy graph $G$ is arc insensitive if and only if for each $e = (u, v)$ of $G$ and the minimum dominating set $D_1$ with domination number $\gamma$ then one of the following conditions is satisfied :

(i) $u, v \in D_1$
(ii) $u, v \in D_2$
(iii) $u \in D_1, v \in D_2$ and $N(v) \cap D_1 = \{u\}$.

**Proof:**
Consider the fuzzy graph $G$ is arc insensitive it means that the removal of any arc $e = (u, v)$ will not affect the dominating set $D_1$. If the arc $e$ is adjacent to any two nodes $u$ and $v$ of $G$, then if $u, v$ satisfies (i) and (ii), then the removal will not affect $\gamma$ and it is obvious. If $u \in D_1, v \in D_2$ and $N(v) \cap D_1 = \{u\}$, then the removal of this arc $e$ will affect $\gamma$ which contradicts to our assumption. Hence $N(v)$ contains at least two nodes in $D_1$. Converse of the theorem is trivial.

**Theorem 4.9:** A fuzzy graph $G$ is arc insensitive then it is bipartite with bipartitions $D_1$ and $D_2$.

**Proof:**
Let the fuzzy graph $G$ be arc insensitive. Since $D_1$ is the fuzzy minimum dominating set, dominance is unaffected by the removal of any arc between any two nodes of $D_1$ or any two nodes of $D_2$ and these arcs are unnecessary. Hence $G$ is bipartite with bipartitions $D_1$ and $D_2$.

5. **MINIMUM ARCS IN INSENSITIVE FUZZY GRAPH**

In this section we find the minimum number of arcs required to satisfy the arc
Insensitive arc in domination of fuzzy graph

Theorem 5.1: Arc insensitive fuzzy graph exist for $k \geq 2$ and $p \geq 2\gamma$.

Proof:
Let $G$ be a fuzzy graph with $n$ nodes and size $p$. Let $D_1$ be a minimum dominating set with $k$ nodes and domination number $\gamma$. Assume $G$ as arc insensitive (i.e) $\gamma(G-e) = \gamma(G)$. If suppose $k = 1$, then it implies a single node dominating all other nodes of $D_2$, which contradicts our assumption. If $p < 2\gamma$ then dominance property fails. Hence $k \geq 2$ and $p \geq 2\gamma$.

Theorem 5.2: Any arc insensitive fuzzy graph $G$ must have $A_i = 2n - 2k$ for $k \geq 2$ and $p \geq 2\gamma$.

Proof:
Let $G$ be an arc insensitive fuzzy graph. Let $D_1$ be a dominating set with $k$ nodes and $D_2$ be the complement of $D_1$ with $n-k$ nodes and cardinality $p-\gamma$. Since by theorem 4.8 every node in $D_2$ has at least two neighbors in $D_1$ and hence the minimum number of arcs required for arc insensitive is twice $(n-k)$. That is $A_i = 2(n-k)$.

In the Example 4.7 we see $n = 6$, $k = 3$, $\gamma = 0.6$, $p = 3.3 > 2\gamma$ and $A_i = 6$.

6. MINIMUM ARCS IN CONNECTED FUZZY GRAPH

In this section we see when a fuzzy graph will remain connected even after the removal of any arc and the minimum number of arcs required for such graph where the domination number remains fixed. The minimum number of arcs is denoted by $A_c$.

Theorem 6.1: Let $G$ be a connected fuzzy graph with $n > 3$ and let $D_1$ be a dominating set with a single node then $G$ remains connected after the removal of any arc if no node in $D_2$ is pendent node, where the domination remains fixed.

Proof:
Let $G$ be a connected fuzzy graph. $D_1$ is a dominating set with $k = 1$ and domination number $\gamma$. Since $k = 1$, this node dominates all other $n-k$ nodes. We know every node in $D_2$ is adjacent with $D_1$ and if $D_2$ has a pendent node then the removal of the arc adjacent to the pendent node will increase the domination number $\gamma$. Hence $D_2$ will not have a pendent node.

Theorem 6.2: Let $G$ be a connected fuzzy graph with $n > 3$ and let $D_1$ be a dominating set with $k = 1$. The minimum number of arcs $A_c$ required for the fuzzy graph to remain connected after the removal of any arc is $A_c = n-1 + \lceil n-1/2 \rceil$.

Proof:
Let $G$ is a connected fuzzy graph with $n > 3$. Since $k = 1$, $D_1$ has a dominating node
say \( \{v\} \); where \( 0 \leq \sigma(v) \leq 1 \). Then \( \{v\} \) dominates remaining \((n-1)\) nodes of \( D_2 \). By theorem 6.1., \( D_2 \) will not have a pendent node; hence every node in \( D_2 \) is adjacent to at least one node in \( D_2 \) itself. Since \( D_2 \) has \( n-1 \) nodes its adjacency is \([n-1/2]\). Therefore the minimum number of arcs required to have this property is the sum of dominating arcs and arcs between \( D_2 \). Hence \( A_c = (n-1) + [n-1/2] \).

Example 6.3:
(i) 

From Fig. 6.1., this fuzzy graph remains connected after the removal of any arc. Here \( n = 5 \); \( D_1 = \{t\} \); \( \gamma = 0.3 \); \( p = 2.2 \); Since \( n \) is odd \( A_c = 6 \).

(ii) 

From Fig. 6.2., this graph does not hold the connectedness property since it has a pendent node.


References


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