Labellings of Graphs and Incidence Algebras

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Abstract
Ancykutty Joseph, On Incidence Algebras and Directed Graphs, IJMMS, 31:5(2002), 301-305, studied the incidence algebras of directed graphs. We have extended it to undirected graphs also. In this paper, we have also established a relation between incidence algebras and the labelings and index vectors introduced by R.H. Jeurissen in Incidence Matrix and Labelings of a Graph, Journal of Combinatorial Theory, Series B, Vol 30, Issue 3, June 1981, 290-301.

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1 Introduction

Ancykutty Joseph introduced the concept of incidence algebras of directed graphs in [1]. She used the number of directed paths from one vertex to another for introducing the incidence algebras of directed graphs. Foldes and Meletiou [4] has discussed the incidence algebras of pre-orders also. This is the motivation for our study on the incidence algebras of undirected graphs. In this paper we use the number of paths for introducing the concept of incidence algebras of undirected graphs. We also establish a relation between the labelings and index vectors of a graph and incidence algebras. For this we use the concepts given by Jeurissen in [6] based on the works of Brouwer[2],
Doob[3] and Stewart[8]. We prove that the collection of all labelings for the collection of all admissible (as defined by Jeurissen) index vectors, the collection of all labelings for the index vector 0 and the collection of all labelings for the index vector $\lambda_j$, ($\lambda \in F, F$, a commutative ring $j$ an all 1-vector) of a graph $G = (V, E)$ are subalgebras of the incidence algebra $I(V, F)$.

2 Preliminary Notes

We first go through the definitions of commutative ring, partially ordered set, pre-ordered set etc.
Throughout this paper, by a ring we mean an associative ring with identity.

The following definitions are adapted from [9].

Definition 2.1 A (left) $A$-module is an additive abelian group $M$ with the operation of (left) multiplication by elements of the ring $A$ that satisfies the following properties.

i. $a(x + y) = ax + ay$ for any $a \in A, x, y \in M$.

ii. $(a + b)x = ax + bx$ for any $a, b \in A, x \in M$.

iii. $(ab)x = a(bx)$ for any $a, b \in A, x \in M$.

iv. $1x = x$ for an $x \in M$.

By an $A$-module, we mean a left $A$-module.

Definition 2.2 A set $\{x_1, x_2, ..., x_n\}$ of elements of $M$ is a basis for $M$ if

i. $a_1x_1 + a_2x_2 + ... + a_nx_n = 0$ for $a_i \in A$ only if $a_1 = a_2 = ... = a_n = 0$.

and ii. $M$ is generated by $\{x_1, x_2, ..., x_n\}$, ie., $M$ is the collection of all linear combinations of $\{x_1, x_2, ..., x_n\}$ with scalars from $A$.

Definition 2.3 A finitely generated module that has a basis is called free.

Definition 2.4 An algebra $A$ is a set over a field $K$ with operations of addition, multiplication and multiplication by elements of $K$ that have the following properties.

i. $A$ is a vector space with respect to addition and multiplication by elements of the field.

ii. $A$ is a ring with respect to addition and multiplication.

iii. $(\lambda a)b = a(\lambda b) = \lambda(ab)$ for any $\lambda \in K, a, b \in A$.

Definition 2.5 A subset $S$ of an algebra $A$ is called a subalgebra if it is simultaneously a subring and a subspace of $A$.

Definition 2.6 [7] A set $X$ with a binary relation $\leq$ is a pre-ordered set if $\leq$ is reflexive and transitive. If $\leq$ is reflexive, transitive and antisymmetric, then $X$ is a partially ordered set (poset).

**Definition 2.7** [7] The incidence algebra $I(X, R)$ of the locally finite partially ordered set $X$ over the commutative ring $R$ with identity is $I(X, R) = \{f : X \times X \to R | f(x, y) = 0 \text{ if } x \text{ is not less than or equal to } y\}$ with operations given by

\[
(f + g)(x, y) = f(x, y) + g(x, y)
\]

\[
(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)
\]

\[
(rf)(x, y) = rf(x, y)
\]

for $f, g \in I(X, R)$ with $r \in R$ and $x, y, z \in X$.

### 3 An incidence algebra of graphs

Ancykutty Joseph [1] has established the relation between incidence algebras and directed graphs. The incidence algebra $I(G, Z)$ for digraph without cycles and multiple edges $(G, \leq)$ representing the finite poset $(V, \leq)$ is defined in [1] as follows.

**Definition 3.1** [1] For $u, v \in V$, let $p_k(u, v)$ denote the number of directed paths of length $k$ from $u$ to $v$ and $p_k(v, u) = -p_k(u, v)$. For $i = 0, 1, ..., n - 1$, define $f_i, f_i^* : V \times V \to Z$ by $f_i(u, v) = p_i(u, v), f_i^*(u, v) = -p_i(u, v)$.

The incidence algebra $I(G, Z)$ of $(G, \leq)$ over the commutative ring $Z$ with identity is defined by $I(G, Z) = \{f_i, f_i^* : V \times V \to Z, i = 0, 1, ..., n - 1\}$ with operations defined as

i. for $f \neq g$, $(f + g)(u, v) = f(u, v) + g(u, v)$.

ii. $(fg)(u, v) = \sum w f(u, w)g(w, v)$.

iii. $(zf)(u, v) = z f(u, v) \forall z \in Z; f, g \in I(G, Z)$.

In [4], Stefan Foldes and Gerasimos Meletiou says about incidence algebra of pre-order as follows.

**Definition 3.2** [4] Given a field $F$, the incidence algebra $A(\rho)$, of a pre-ordered set $(S, \rho), S = \{1, 2, ..., n\}$ over $F$ is the set of maps $\alpha : S^2 \to F$ such that $\alpha(x, y) = 0$ unless $x \rho y$. The addition and multiplication in $A(\rho)$ are defined as matrix sum and product.

Replacing field $F$ by a commutative ring $R$ with identity and following the definition of Foldes and Meletiou[4], we can find in graphs an analogue of the incidence algebra of directed graph given by Ancykutty Joseph[1].
**Theorem 3.3** Let $G = (V, E)$ be a graph without cycles and multiple edges with $V$ and $E$ finite. For $u, v \in V$, let $f_i(u, v)$ be the number of paths of length $i$ between $u$ and $v$. Let $f_i^*(u, v) = -f_i(u, v)$. Then \( \{f_i, f_i^* : V \to \mathbb{Z}, i = 0, 1, 2, ..., n-1\} \) is an incidence algebra of \((G, \rho)\) denoted by $I(G, \mathbb{Z})$ over the commutative ring $\mathbb{Z}$ with identity.

**Proof**

Let $f_i$ and $f_j$ be elements of $I(G, \mathbb{Z}) = \{f_i(u, v) : f_i$ is the number of paths of length $i$ from $u$ to $v\}$. Then for $f_i \neq f_j$,

\[
\begin{align*}
(f_i + f_j)(u, v) &= \text{number of paths of length either } i \text{ or } j = f_i(u, v) + f_j(u, v). \\
(f_i, f_j)(u, v) &= \text{number of paths of length } i + j = \sum_{u \rho w \rho v} f_i(u, w)f_j(w, v). \\
(zf_i)(u, v) &= z.f_i(u, v) \forall z \in \mathbb{Z}; f_i, f_j \in I(G, \mathbb{Z}).
\end{align*}
\]

Hence $I(G, \mathbb{Z})$ is an incidence algebra of \((G, \rho)\).

4 Labelings and incidence algebras

We can have other incidence algebras which are subalgebras of $I(V, R)$ using the concept of labeling and index vectors given in [6] of R.H. Jeurissen.

**Definition 4.1** [6] Let $G = (P, L)$ be a finite connected graph without loops and multiple edges with $P = \{x_0, x_1, ..., x_{p-1}\}$ as vertex set and $L = \{m_1, m_2, ..., m_q\}$ as edge-set. Let $F$ be an abelian group or a ring. A function $r : P \to F$ is called an index vector with components $r(x_i)$. Let $G = (P, L)$ be a finite connected graph without loops and multiple edges with $P = \{x_0, x_1, ..., x_{p-1}\}$ as vertex set and $L = \{m_1, m_2, ..., m_q\}$ as edge-set. Let $F$ be an abelian group or a ring. A function $x : L \to F$ is called a labeling with components $x(m_j), j = 1, 2, ..., q$.

A labeling $x : L \to F$ is called a labeling for the index vector $r$ if

\[
\sum_{m \in E_i} x(m) = r(x_i), i = 0, 1, 2, ..., p-1,
\]

where $E_i$ is the set of edges incident with $x_i$. The index vectors for which a labeling exists are called admissible index vectors.

Now we go through some of the results proved by Jeurissen [6] using column operations on incidence matrix of a graph. Addition and scalar multiplication are defined as follows.

\[
\begin{align*}
(r_1 + r_2)(v_i) &= r_1(v_i) + r_2(v_i) \\
(x_1 + x_2)(e_j) &= x_1(e_j) + x_2(e_j) \\
(fr)(v_i) &= f.r(v_i) \\
(fx)(e_j) &= f.x(e_j)
\end{align*}
\]
Now we define multiplication of index vectors and labelings as follows.

**Definition 4.3** Let \( x_1 \) and \( x_2 \) be two labelings and \( r_1 \) and \( r_2 \) be two index vectors of \( G \). Then \((r_1, r_2)(v_i) = \sum_{s: (v_i, v_s) \in E} r_1(v_i)r_2(v_s)\)

\((x_1 \cdot x_2)(v_i, v_j) = 2 \sum_{s: (v_i, v_s) \in E, (v_s, v_j) \in E} x_1(v_i, v_s) x_2(v_s, v_j)\).

**Remark:** Multiplication by 2 in the above definition is for adjusting the symmetric property of the edges of the graph.

Now we prove some results on labelings and incidence algebras.

**Theorem 4.4** The set of labelings for all admissible index vectors of a graph \( G = (V, E) \) is a subalgebra of the incidence algebra \( I(V, F) \).

**Proof**
Let \( A \) be the set of all admissible index vectors and \( I_{L(A)}(V, F) \) the set of all labelings for elements of \( A \).

Let \( x_1, x_2 \in I_{L(A)}(V, F) \). Then there exist \( r_1, r_2 \in A \) such that

\[
r_1(v_i) = \sum_{j: (v_i, v_j) \in E} x_1(v_i, v_j) \quad \text{and} \quad r_2(v_i) = \sum_{j: (v_i, v_j) \in E} x_2(v_i, v_j).
\]

\[
(r_1 + r_2)(v_i) = \sum_{j: (v_i, v_j) \in E} x_1(v_i, v_j) + \sum_{j: (v_i, v_j) \in E} x_2(v_i, v_j) = \sum_{j: (v_i, v_j) \in E} (x_1 + x_2)(v_i, v_j).
\]

Therefore \( x_1 + x_2 \) is a labeling for \( r_1 + r_2 \), i.e., \( x_1 + x_2 \in I_{L(A)}(V, F) \).

\[
(r_1 \cdot r_2)(v_i) = \sum_{s: (v_i, v_s) \in E} r_1(v_i)r_2(v_s)
\]

\[
= \sum_{s: (v_i, v_s) \in E} \left[ \sum_{j: (v_i, v_j) \in E} x_1(v_i, v_j) \sum_{k: (v_s, v_k) \in E} x_2(v_s, v_k) \right]
\]

\[
= 2 \sum_{s: (v_i, v_s) \in E} \sum_{k: (v_s, v_k) \in E} x_1(v_i, v_s) x_2(v_s, v_k)
\]

\[
= \sum_{(v_i, v_j) \in E} (x_1 \cdot x_2)(v_i, v_j)
\]

i.e., \( x_1 \cdot x_2 \) is a labeling for \( r_1 \cdot r_2 \). Hence \( x_1 \cdot x_2 \in I_{L(A)}(V, F) \).

\[
(f r_1)(v_i) = f \cdot r_1(v_i)
\]
\[ f = \sum_{j:(v_i,v_j)\in E} x_1(v_i,v_j) \]
\[ = \sum_{j:(v_i,v_j)\in E} f.x_1(v_i,v_j) \]
\[ = \sum_{j:(v_i,v_j)\in E} (f x_1)(v_i,v_j) \]

Therefore \( f x_1 \) is a labeling for \( f r_1 \). Hence \( f x_1 \in I_{L(A)}(V,F) \).
So the collection \( I_{L(A)}(V,F) \) is a subalgebra of \( I(V,F) \).

**Theorem 4.5** The set of labelings for \( \lambda_j, \lambda \in F, j-an all 1 \) vector, of a graph \( G = (V,E) \) is a subalgebra of the incidence algebra \( I(V,F) \).

Let \( I_{L(\lambda)} \) be the collection of labelings for \( \lambda_j, \lambda \in F, j-an all 1 \) vector. By theorem 4.1, \( I_{L(\lambda)}(V,F) \) is a submodule of \( F^q \). So it is enough if we prove that \( x_1 x_2 \in I_{L(\lambda)}(V,F) \forall x_1, x_2 \in I_{L(\lambda)}(V,F) \). Let \( x_1, x_2 \in I_{L(\lambda)}(V,F) \). Then there exist \( \lambda_1, \lambda_2 \in F \) such that
\[ \lambda_1(v_i) = \sum_{j:(v_i,v_j)\in E} x_1(v_i,v_j) \text{ and } \lambda_2(v_i) = \sum_{j:(v_i,v_j)\in E} x_2(v_i,v_j). \]

\[ (\lambda_1, \lambda_2)(v_i) = \sum_{s:(v_i,v_s)\in E} \lambda_1(v_i) \lambda_2(v_s) \]
\[ = \sum_{s:(v_i,v_s)\in E} \left[ \sum_{j:(v_i,v_j)\in E} x_1(v_i,v_j) \sum_{k:(v_s,v_k)\in E} x_2(v_s,v_k) \right] \]
\[ = 2 \sum_{s:(v_i,v_s)\in E} \sum_{s:(v_s,v_k)\in E} x_1(v_i,v_s) x_2(v_s,v_k) \]
\[ = \sum_{(v_i,v_j)\in E} (x_1 x_2)(v_i,v_j) \]

ie., \( x_1 x_2 \) is a labeling for \( \lambda_1, \lambda_2 \). Hence \( x_1 x_2 \in I_{L(\lambda)}(V,F) \). So the collection \( I_{L(\lambda)}(V,F) \) is a subalgebra of \( I(V,F) \).

**Theorem 4.6** The set of labelings of 0 of a graph \( G = (V,E) \) is a subalgebra of the incidence algebra \( I(V,F) \).

Let \( I_{L(0)}(V,F) \) be the collection of all labelings for 0.
By theorem 4.1, \( I_{L(0)}(V,F) \) is a submodule of \( F^q \). So it is enough if we prove that \( x_1 x_2 \in I_{L(0)}(V,F) \forall x_1, x_2 \in I_{L(0)}(V,F) \).
Let \( x_1, x_2 \in I_{L(0)}(V,F) \). Then \( \sum_{j:(v_i,v_j)\in E} x_1(v_i,v_j) = 0(v_i) \)
and \( \sum_{j:(v_i,v_j)\in E} x_2(v_i,v_j) = 0(v_i) \).

\[ \sum_{j:(v_i,v_j)\in E} x_1 x_2(v_i,v_j) = 2 \sum_{j:(v_i,v_j)\in E} \sum_{s:(v_i,v_s)\in E} \sum_{(v_s,v_j)\in E} [x_1(v_i,v_s) x_2(v_s,v_j)] \]
\[
\begin{align*}
&= 2 \sum_{s(v_i, v_s) \in E} \sum_{j(v_s, v_j) \in E} x_1(v_i, v_s) x_2(v_s, v_j) \\
&= \sum_{s(v_i, v_s) \in E} x_1(v_i, v_s) \sum_{j(v_s, v_j) \in E} x_2(v_s, v_j) \\
&= \sum_{s(v_i, v_s) \in E} x_1(v_i, v_s) 0(v_s) \\
&= 0(v_i) 0(v_s) \\
&= 0(v_i)
\end{align*}
\]

Therefore \( I_{L(0)}(V, F) \) is a subalgebra of the incidence algebra \( I(V, F) \).

**Remark**

For a directed graph, the labeling of directed edges is similar except that they are taken negative at the initial vertex. ie., \( x(v, u) = -x(u, v) \). So we take \( f(u, v) = x(u, v) \) and \( f^*(u, v) = -x(u, v) \) where \( x \) is a labeling. This is similar to the definition of \( f^* \) in [1]. Then \( \{ f, f^*: E \to \mathbb{Z} \} \) is a subalgebra of \( I(V, F) \).

**References**


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