Coefficient Estimates for Concave Partial Sums of Univalent Functions

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Abstract

In this paper, we consider the class $(\Sigma_\alpha)$, $\alpha \geq 0$, of analytic functions $f : U \to \overline{C}$, where $U$ is an open unit disk, satisfying the standard normalization $f(0) = f'(0) - 1 = 0$ take the form

$$f(z) = z + \sum_{n=2}^{\infty} A_n(f)z^{n+\alpha}, \alpha \geq 0.$$ 

And assume that its partial sums $\sigma(f)$ satisfies the following conditions:

(i) $\sigma(f) \in \Sigma_\alpha$ is meromorphic in $U$ and has a simple pole at the point $p$.
(ii) $\sigma(f)(0) = \sigma(f)'(0) - 1 = 0$.
(iii) $\sigma(f)$ maps $U$ conformally onto a set whose complement with respect to $\overline{C}$ is convex.

We call such functions concave univalent functions. We prove some coefficient estimates for the partial sums $\sigma(f)$ such that

$$\sigma(f)(z) = \sum_{n=0}^{\infty} a_n(z-p)^{n+\alpha}, \quad |z-p| < 1-p, \quad p \in (0,1), \quad \alpha \geq 0.$$ 

Moreover the pre-Schwarzian derivative of $f$ are determined.

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1 Introduction and preliminaries.

In the theory of univalent functions the most interesting problem is to find the coefficient estimates for functions

\[ f(z) = z + \sum_{n=2}^{\infty} a_n(f)z^n \] (1.1)

that are analytic and univalent in the unit disk \( U := \{ z : |z| < 1 \} \). Let Co\((p)\) be the family of functions \( f : U \to \mathbb{C} \) where \( p \in (0, 1) \) that satisfy the following assumption

**Assumption (A):**

(i) \( f \) is meromorphic in \( U \) and has a simple pole at the point \( p \).
(ii) \( f(0) = f'(0) - 1 = 0 \).
(iii) \( f \) maps \( U \) conformally onto a set whose complement with respect to \( \mathbb{C} \) is convex.

The family Co\((p)\) has been investigated recently in [1-4,18]. In [14], Livingston introduced a necessary and sufficient condition for a function \( f \) to be in Co\((p)\)

\[ \Re\{-1 + p^2 \} + 2pz - \frac{(z-p)(1-pz)f''(z)}{f'(z)} > 0, \quad \forall z \in U. \]

In [18], Avkhadiev and Wirths proved that for each \( f \in Co(p) \) with the expansion in (1.1), the inequality

\[ |a_n(f) - \frac{1 - p^{2n+2}}{p^{n-1}(1-p^4)}| \leq \frac{p^2(1-p^{2n-2})}{p^{n-1}(1-p^4)} \]

is valid. Equality is attained if and only if

\[ f(z) = \frac{z - \frac{p}{1+p^2}(1+e^{i\theta})z^2}{(1 - \frac{z}{p})(1-zp)}. \] (1.2)

Finally, Bhowmik and Pommerenke [6] obtained certain coefficient estimates for functions have the Laurent expansion

\[ f(z) = \sum_{n=-1}^{\infty} A_n(f)(z-p)^n, \quad z \in \triangle \]

where \( \triangle := \{ z \in C : |z - p| < 1 - p \} \) and \( p \in (0, 1) \),

\[ |A_{n-2} - \frac{(1-p^2A_{n-1})}{p}| \leq \frac{p}{(1-p^4)(1-p)^{n-1}}|1 - \frac{1-p^4}{p^4}|A_{n-1} + \frac{p^2}{1-p^4}, \quad n \geq 3 \]
and of the form (1.2).

In this work, we consider the class \((\Sigma_{\alpha}), \alpha > 0\), of analytic functions \(f : U \to \mathbb{C}\), where \(U\) is an open unit disk, satisfying the standard normalization \(f(0) = f'(0) - 1 = 0\) take the form

\[
f(z) = z + \sum_{n=2}^{\infty} A_n(f) z^{n+\alpha}, \quad \alpha \geq 0. \tag{1.3}
\]

Recently, in [7] the authors studied the coefficient bounds for concave functions in the class \((\Sigma_{\alpha})\). Assume that \(f \in \Sigma_{\alpha}\) satisfies the assumption (A).

Now for the functions \(f \in \Sigma_{\alpha}\) we define the following partial sums

\[
s_k(z) = \sum_{n=2}^{k} a_n z^{n+\alpha}, \quad z \in U,
\]

such that \(s_1(z) = z\), we construct the Cesáro means \(\sigma_k f(z)\) of \(f \in \Sigma_{\alpha}\) by

\[
\sigma_k f(z) = \frac{1}{k} \left[ s_1(z) + \sum_{n=2}^{k} s_n(z) \right]
\]

\[
= \frac{1}{k} \left[ s_1(z) + + s_k(z) \right]
\]

\[
= \frac{1}{k} \left[ z + (z + A_2 z^{\alpha+2}) + + (z + ... + A_k z^{\alpha+k}) \right]
\]

\[
= \frac{1}{k} \left[ k z + (k - 1) A_2 z^{\alpha+2} + + A_k z^{\alpha+k} \right] \tag{1.4}
\]

\[
= z + \sum_{n=2}^{k} \left( \frac{k - n + 1}{k} \right) A_n z^{\alpha+n}
\]

\[
= f(z) \ast \left[ z + \sum_{n=2}^{k} \left( \frac{k - n + 1}{k} \right) z^{n+\alpha} \right]
\]

\[
= f(z) \ast g_k(z)
\]

where

\[
g_k = z + \sum_{n=2}^{k} \left( \frac{k - n + 1}{k} \right) z^{n+\alpha}.
\]

Assume that \(\sigma_k f(z)\) has the expansion

\[
\sigma_k f(z) = \sum_{n=0}^{k} a_n(f) (z - p)^{n+\alpha}, \quad \alpha \geq 0, \quad z \in \Delta. \tag{1.5}
\]
and satisfies the assumption (A).

Note that the classical Cesáro means play an important role in geometric function theory (see [8, 15-17]).

We determined the coefficient bounds for the partial sums (1.4) of the functions $f \in \Sigma_\alpha$ and discuss some other properties containing (1.4) such as the per-
Schwarzian derivative.

For this purpose, we need to the following results

**Theorem 1.1.** [18] For each $f \in Co(p)$, there exists a function $\omega$ holomorphic in $U$ such that $\omega(U) \subset \overline{U}$ and

$$f(z) = \frac{z - \frac{p}{1+p^2}(1 + \omega(z))z^2}{(1 - \frac{z}{p})(1 - zp)}, \quad z \in U.$$  

(1.6)

**Lemma 1.2.** [5] Let $q \in (1, 2]$ and $f \in Co(q)$. Then there exists a function $\omega$ holomorphic in $U$ such that $\omega(U) \subset \overline{U}$ and

$$f'(z) = \frac{(1 + z\omega(z))^{q-1}}{(1 - z)^{q+1}} z \in U.$$  

2 Coefficient bounds.

In this section, we introduce some coefficient estimates for functions belong to the class $\Sigma_\alpha$ and have the expansion (1.4). Now, we state our first result

**Theorem 2.1.** Let $p \in (0, 1)$ and $\sigma_k f \in Co(p)$ has the expansion (1.5) where $f \in \Sigma_\alpha$. Then

$$|a_0| \leq \frac{1}{p(1 + p^2)}.$$  

(2.1)

The inequality is sharp.

**Proof.** Let $\sigma_k f \in Co(p)$. Then by Theorem 1.1, there exists a function $\omega(z)$ holomorphic in $U$ and $\omega(U) \subset \overline{U}$ satisfying (1.6). Assume that

$$\omega(z) = \sum_{n=0}^{\infty} c_n(z - p)^{n+\alpha}, \quad z \in \Delta.$$  

(2.2)

Using these two expansions (1.5) and (2.2), the power series formulation of (1.6) takes the form

$$\sum_{n=0}^{k} a_n(z - p)^{n+\alpha}\left[1 - 2p + zp^2\right] = 1 - \frac{p}{1 + p^2}\left[1 + \sum_{n=0}^{\infty} c_n(z - p)^{n+\alpha}\right]z.$$  

(2.3)
Comparing the coefficient of $z(z - p)^\alpha$ on both sides of (2.3) and using the inequality $|c_0| < 1$, yields the assertion (2.1).

**Theorem 2.2.** Let $p \in (0, 1)$ and $\sigma_k f \in Co(p)$ has the expansion (1.5) where $f \in \Sigma_\alpha$. Then

$$|a_n(f)| \leq \frac{1}{p(1 + p^2)(\alpha + 1)n}, \quad n \geq 2. \quad (2.4)$$

where the symbol $(\alpha + 1)_n$ is a Pochhammer symbol defined by

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \begin{cases} 1, & n = 0 \\ \alpha(\alpha + 1)...(\alpha + n - 1), & n = \{1, 2,...\}. \end{cases}$$

The inequality is sharp.

**Proof.** Let $\sigma_k f \in Co(p)$. Then by Comparing the coefficient of $z(z - p)^{n+\alpha}$ on both sides of (2.3) and using the inequality (see [8])

$$|c_n| \leq \frac{1}{(\alpha + 1)_n},$$

yields the assertion (2.4).

Consequently, the next result present sharp coefficient estimates for all $n \geq 2$ if $\sigma_k f \in Co(p)$ of the form (1.4) and has the expansion (1.5).

**Theorem 2.3.** Let $p \in (0, 1)$ and $\sigma_k f \in Co(p)$ of the form (1.4) and have the expansion (1.5). Then

$$|A_n(f)| \leq \frac{k}{p(k - n + 1)(1 + p^2)(\alpha + 1)n}, \quad n \geq 2, \quad k = 1, 2,..., \quad (2.5)$$

where $\alpha \geq 0$. The inequality is sharp.

**Proof.** Equating the right sides of (1.4) and (1.5) and applying Theorems 2.1.

**Theorem 2.4.** Let $q \in (1, 2]$ and $\sigma_k f \in Co(q)$ where $f \in \Sigma_\alpha$. Then

$$|a_n(f)| \leq \frac{1}{3(n + \alpha)(\alpha + 1)_n}, \quad n \geq 2. \quad (2.6)$$

**Proof.** Let $\sigma_k f \in Co(q)$. Then by Lemma 1.1, there exists a function $\omega(z)$ holomorphic in $U$ and $\omega(U) \subset \overline{U}$ satisfying

$$(\sigma_k f)' = \frac{(1 + z\omega(z))^{q-1}}{(1 - z)^{q+1}} \quad z \in U, \quad (2.7)$$
where
\[
\sigma_k f(z) = \sum_{n=0}^{k} a_n(f)(z - q)^{n+\alpha}, \ \alpha \geq 0, \ z \in \triangle_q,
\]
(2.8)
\[
\omega(z) = \sum_{n=0}^{\infty} d_n(z - q)^{n+\alpha-1}, \ z \in \triangle_q
\]
(2.9)
and
\[
\triangle_q := \{ z \in C : |z - q| < 2 - q, \ q \in (1, 2] \}.
\]

Now substitute (2.9) in (2.7) and using (2.8), then comparing the coefficient of \( z(z - p)^{n+\alpha-1} \) on both sides and using the inequality
\[
|d_n| \leq \frac{1}{(\alpha + 1)n},
\]
yields the assertion (2.6).

**Theorem 2.5.** Let \( q \in (1, 2] \) and \( \sigma_k f \in \text{Co}(q) \) where \( f \in \Sigma_\alpha \). Then
\[
|A_n(f)| \leq \frac{k}{3(k - n + 1)(n + \alpha)(\alpha + 1)n}, \ n \geq 2, \ k = 1, 2, ..., \quad (2.10)
\]
where \( \alpha \geq 0 \). The inequality is sharp.

**Proof.** Equating the right sides of (1.4) and (2.8) and applying Theorem 2.4, we get the required result.

### 3 Norm estimates of the pre-Schwarzian derivative

In this section, we discuss some other properties of the operator (1.3) such as the univalence of this operator by using pre-Schwarzian derivative. Let \( h \) be analytic and locally univalent in \( U \). The pre-Schwarzian derivative \( T_h \) of \( h \in \Sigma_\alpha, \ \alpha \geq 0 \) is defined by
\[
T_h(z) = \frac{h''(z)}{h'(z)}, \quad (z \in U)
\]
(3.1)
with the norm
\[
\|T_h\| = \sup_{z \in U} |T_h|(1 - |z|^{2+\alpha}).
\]
It is known that $\|T_h\| < \infty$ if and only if $h$ is uniformly locally univalent. It is also known that $\|T_h\| \leq 6$ for $h \in S$ the class of starlike functions and that $\|T_h\| \leq 4$ for $h \in K$ the class of convex functions (see [13]).

**Theorem 3.1.** Let $p \in (0, 1)$ and $\sigma_k f \in Co(p)$ of the form (1.4). Then for $z \to 0$ the pre-Schwarzian derivative of $\sigma_k f$ satisfies the inequality

$$\|T_{\sigma_k f}\| \leq \frac{(p + 2)^2 (1 + p)}{p (1 + p^2)}.$$ 

The result is sharp.

**Proof.** Let $p \in (0, 1)$ and $\sigma_k f \in Co(p)$ then in view of Theorem 1.1, $\sigma_k f$ takes the form (1.6). Differentiating both sides of (1.6) we obtain

$$\sigma_k f'(z) = \frac{H(z)(1 - W'(z)) - (z - W(z))H'(z)}{H^2(z)}$$

where $H(z) := (1 - \frac{z}{p})(1 - zp)$ and $W(z) := \frac{p}{1 + p^2}(1 + \omega(z))z^2$, or equivalent to

$$\ln \sigma_k f'(z) = \ln[H(z)(1 - W'(z)) - (z - W(z))H'(z)] - 2 \ln H(z).$$

Take the derivative for the above equality we receive

$$\frac{\sigma_k f''(z)}{\sigma_k f'(z)} = \frac{Q'(z)}{Q(z)} - \frac{2H'(z)}{H(z)}$$

where

$$Q(z) := [H(z)(1 - W'(z)) - (z - W(z))H'(z)].$$

Now for $z \to 0$ we obtain the assertion (2.10).

**Corollary 3.2.** Let $p \in (0, 1)$ and $\sigma_k f \in Co(p)$. Then $\sigma_k f$ is uniformly locally univalent when $z \to 0$.

**Proof.** By applying Theorem 3.1 we get $\|T_{\sigma_k f}\| < \infty$ hence $\sigma_k f$ is uniformly locally univalent.

**Theorem 3.3.** Let $q \in (1, 2]$ and $\sigma_k f \in Co(q)$ where $f \in \Sigma_\alpha$. Then for $z \to 0$ the pre-Schwarzian derivative of $\sigma_k f$ satisfies the inequality

$$\|T_{\sigma_k f}\| \leq 4.$$ 

The result is sharp.
**Proof.** Let \( q \in (1, 2] \) and \( \sigma_k f \in Co(q) \) where \( f \in \Sigma_{\alpha} \). Now when \( q = 2 \), (2.7) is equivalent to

\[
\ln(\sigma_k f)' = \ln(1 + z\omega(z)) - 3\ln(1 - z) \quad z \in U,
\]

take the derivative we obtain

\[
\frac{(\sigma_k f)''}{(\sigma_k f)'} = \frac{z\omega'(z) + \omega(z)}{(1 + z\omega(z))} + \frac{3}{1 - z}.
\]

Since Lemma 1.1 holds when \( \omega(z) := e^z \) (see [13]) then for \( z \to 0 \) we have

\[
\|T_{\sigma_k f}\| < 4.
\]

**Corollary 3.4.** Let the assumption of Theorem 3.3 holds. Then \( \sigma_k f \) is uniformly locally univalent when \( z \to 0 \).

**Proof.** By applying Theorem 3.3 we get \( \|T_{\sigma_k f}\| < \infty \) hence \( \sigma_k f \) is uniformly locally univalent.

**Corollary 3.5.** Let the assumption of Theorem 3.3 holds. Then \( \sigma_k f \) is convex as \( z \to 0 \).

**Proof.** By applying Theorem 3.3 we get \( \|T_{\sigma_k f}\| < 4 \) hence \( \sigma_k f \) is convex.

Note that for other work related to partial sums and also on Cesáro can be found in (see for examples [9-12]).

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**References**


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