Contact CR-Submanifolds of an Indefinite Trans-Sasakian Manifold

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Abstract

This paper is based on contact CR-submanifolds of an indefinite trans-Sasakian manifold of type \((\alpha, \beta)\). Here some properties of contact CR-submanifolds of an indefinite trans-Sasakian manifold have been studied and also the sectional curvatures of contact CR-submanifolds of an indefinite trans-Sasakian space form are discussed.

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1. Introduction

The differential geometry with contact and almost contact metric structures are very important part. Basically, contact structure has most important applications in Physics. Many authors gave their valuable and essential results on differential geometry with contact and almost contact structures on their papers. In 1973 and 1974, B.Y. Chen and K. Ogiue introduced geometry of submanifolds and totally real submanifolds in [5] and [6]. Then K. Ogiue expressed differential geometry of Kaehler submanifolds in [14]. In [3], D.E. Blair discussed contact manifolds in Riemannian geometry in 1976. A. Bejancu studied CR-submanifolds of a Kaehler manifold in [1] and [2]. In 1976 K. Yano and M. Kon introduced invariant and anti-invariant submanifolds in
Now the present paper is to study some properties of contact CR-submanifolds of an indefinite trans-Sasakian manifold. In section two and three, some properties of $D$-totally geodesic, $D^\perp$-totally geodesic, $D$-umbilic, $D^\perp$-umbilic and mixed totally geodesic contact CR-submanifolds of an indefinite trans-Sasakian manifold are discussed. Moreover some results of the sectional curvatures of contact CR-submanifolds of an indefinite trans-Sasakian space form are studied in section four.

Let $\tilde{M}$ be an $(2n + 1)$-dimensional indefinite almost contact metric manifold with indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$, where $\phi$ is a tensor field of type $(1, 1)$ of rank $2n$, $\xi$ is a vector field and $\eta$, a 1-form and $\tilde{g}$ is Riemannian metric, satisfying

\begin{align}
\phi^2 &= -I + \eta \otimes \xi, \quad \eta \circ \phi = 0, \quad \phi \circ \xi = 0, \quad \eta(\xi) = 1, \\
\tilde{g}(\phi X, \phi Y) &= \tilde{g}(X, Y) - \epsilon \eta(X)\eta(Y), \\
\tilde{g}(X, \xi) &= \epsilon \eta(X),
\end{align}

for all vector fields $X, Y$ on $\tilde{M}$ and where $\epsilon = \tilde{g}(\xi, \xi) = \pm 1$.

An indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ is called an indefinite trans-Sasakian structure if

\begin{align}
(\tilde{\nabla}_X \phi)Y &= \alpha [g(X, Y)\xi - \epsilon \eta(Y)X] + \beta [g(\phi X, Y)\xi - \epsilon \eta(Y)\phi X]
\end{align}

for functions $\alpha$ and $\beta$ on $\tilde{M}$ of type $(\alpha, \beta)$ and where $\tilde{\nabla}$ is the Levi-civita connection on $\tilde{M}$.

Also we know that,

\begin{align}
\tilde{\nabla}_X \xi &= -\alpha \epsilon \phi X + \beta [\epsilon X - \epsilon \eta(X)\xi], \text{ for any } X \in T\tilde{M}.
\end{align}
Now we define contact CR-submanifold of an indefinite trans-Sasakian manifold.

**Definition 1.1.** An $m$-dimensional Riemannian submanifold $M$ of an indefinite trans-Sasakian manifold $\tilde{M}$ is called a contact CR-submanifold if

i) $\xi$ is tangent to $M$,

ii) there exists on $M$ a differentiable distribution $D : x \mapsto D_x \subset T_x(M)$, such that $D_x$ is invariant under $\phi$; i.e., $\phi D_x \subset D_x$, for each $x \in M$ and the orthogonal complementary distribution $D^\perp : x \mapsto D_x^\perp \subset T_x(M)$ of the distribution $D$ on $M$ is totally real; i.e., $\phi D_x^\perp \subset T_x^\perp(M)$, where $T_x(M)$ and $T_x^\perp(M)$ are the tangent space and the normal space of $M$ at $x$.

We call $D$ (resp. $D^\perp$) the horizontal (resp. vertical) distribution.

Also the contact CR-submanifold of an indefinite trans-Sasakian manifold is called $\xi$-horizontal (resp. $\xi$-vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D_x^\perp$) for each $x \in M$ by [10].

The Gauss and Weingarten formulae are given by

\begin{align}
\tilde{\nabla}_XY &= \nabla_XY + h(X,Y), \\
\tilde{\nabla}_XN &= -A_NX + \nabla^\perp_XN
\end{align}

for $X, Y \in TM, N \in T^\perp M$.

where $\nabla$ is the Riemannian connection on $M$ and $\nabla^\perp$ is the connection on the normal bundle induced by $\nabla$ and $h$ (resp. $A$) is the second fundamental form (resp. tensor) of $M$ in $\tilde{M}$. Moreover $h$ satisfies the condition

\begin{align}
g(A_NX,Y) = g(h(X,Y),N).
\end{align}

The equation of Gauss is given by

\begin{align}
\tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) \\
&\quad - g(h(X,W),h(Y,Z))
\end{align}

where $\tilde{R}$ (resp. $R$) is the curvature tensor of $\tilde{M}$ (resp. $M$).

For a vector field $X$ tangent to $M$, we put

\begin{align}
X &= PX + QX
\end{align}
where $P$ and $Q$ are projectors of $D$ and $D^\perp$ respectively and i.e., $PX$ and $QX$ belong to the distributions $D$ and $D^\perp$.

Also for a vector field $N$ normal to $M$, we put

(1.11) \[ \phi N = BN + CN \]

where $BN$ (resp. CN) is vertical (resp. normal) part of $\phi N$.

Now we calculate (1.4) in two ways and comparing the horizontal, vertical and normal parts given by

(1.12) \[ P\nabla_X\phi PY - PA_{\phi QY}X = \phi P\nabla_XY + \alpha g(X,Y)P\xi \]
\[ + \beta g(\phi PX,Y)P\xi - \alpha \epsilon\eta(Y)PX \]
\[ - \beta \epsilon\eta(Y)\phi PX \]

(1.13) \[ Q\nabla_X\phi PY - QA_{\phi QY}X = Bh(X,Y) + [\alpha g(X,Y) + \beta g(\phi QX,Y)]Q\xi \]
\[ - \alpha \epsilon\eta(Y)QX \]

(1.14) \[ h(X,\phi PY) + \nabla^\perp_X\phi QY = \phi Q\nabla_XY + Ch(X,Y) - \beta \epsilon\eta(Y)\phi QX \]

Also we have from (1.5)

(1.15) \[ \nabla_X\xi = -\alpha \epsilon\phi PX + \beta [\epsilon PX - \epsilon\eta(X)\xi] \]

(1.16) \[ h(X,\xi) = -\alpha \epsilon\phi QX + \beta \epsilon QX \]

Then we obtain from (1.15) and (1.16)

(1.17) \[ \nabla_X\xi = -\beta \epsilon\eta(X)\xi, \quad \text{if } X \in D^\perp \]

(1.18) \[ h(X,\xi) = 0, \quad \text{if } X \in D \]

(1.19) \[ h(\xi,\xi) = 0, \]

(1.20) \[ A_N\xi \in D^\perp. \]

2. D-totally geodesic and $D^\perp$-totally geodesic contact CR-submanifolds of an indefinite trans-Sasakian manifold
Initially, we state the definition of $D$-totally (resp. $D^\perp$-totally) geodesic contact CR-submanifold of an indefinite trans-Sasakian manifold.

**Definition 2.1.** A contact CR-submanifold $M$ of an indefinite trans-Sasakian manifold $\tilde{M}$ is said to be $D$-totally (resp. $D^\perp$-totally) geodesic if $h(X,Y) = 0$, $\forall X, Y \in D$ (resp. $X, Y \in D^\perp$).

Then we prove

**Proposition 2.1.** Let $M$ be a contact CR-submanifold of an indefinite trans-Sasakian manifold. Then $M$ is $D$-totally geodesic if and only if $A_N X \in D^\perp$ for each $X \in D$, $N$ is normal vector field to $M$.

*Proof:* Let $M$ be $D$-totally geodesic, then by (1.8) we get

$$g(h(X,Y), N) = g(A_N X, Y) = 0.$$ 

i.e., $A_N X \in D^\perp$.

Conversely, suppose that $A_N X \in D^\perp$. Then for $X, Y \in D$ we have

$$g(A_N X, Y) = 0 = g(h(X,Y), N)$$

i.e., $h(X,Y) = 0$ $\forall$ $X, Y \in D$

which implies that $M$ is $D$-totally geodesic.

This completes the proof. $\Box$

Similarly, we can easily express the proposition 2.2:

**Proposition 2.2.** Let $M$ be a contact CR-submanifold of an indefinite trans-Sasakian manifold $\tilde{M}$. Then $M$ is $D^\perp$-totally geodesic if and only if $A_N X \in D$ for each $X \in D^\perp$, $N$ is normal vector field to $M$.

Now for the integrability conditions of $D$ and $D^\perp$, we obtain

**Theorem 2.1.** Let $M$ be a contact CR-submanifold of an indefinite trans-Sasakian manifold. If $M$ is $\xi$-horizontal then the distribution $D$ is integrable iff

$$h(X, \phi Y) = h(\phi X, Y), \quad \forall \ X, Y \in D$$

(2.1)
and if \( M \) is \( \xi \)-vertical then the distribution \( D^\perp \) is integrable iff

(2.2) \[
A_{\phi X}Y - A_{\phi Y}X = \alpha \epsilon[\eta(X)Y - \eta(Y)X], \quad \forall \ X, Y \in D^\perp.
\]

**Proof:** If \( M \) is \( \xi \)-horizontal, then from (1.14) we get

\[
h(X, \phi Y) = \phi Q\nabla_X Y + Ch(X, Y) - \beta \epsilon \eta(Y)\phi Q X, \quad \text{for} \quad X, Y \in D.
\]

Since \([X, Y] \in D\), we obtain iff \( h(X, \phi Y) = h(Y, \phi X) \).

Now for the remaining part of (1.14), we write

(2.3) \[
\nabla^\perp_X \phi Y = \phi Q\nabla_X Y + Ch(X, Y) - \beta \epsilon \eta(Y)\phi Q X, \quad \text{for} \quad X, Y \in D^\perp.
\]

Now we see that

\[
\tilde{\nabla}_X \phi Y = (\tilde{\nabla}_X \phi)Y + \phi \tilde{\nabla}_X Y
= \alpha [g(X, Y)\xi - \epsilon \eta(Y)X] + \beta [g(\phi X, Y)\xi - \epsilon \eta(Y)\phi X]
+ \phi P\nabla_X Y + \phi Q\nabla_X Y + Bh(X, Y) + Ch(X, Y)
\]

and from (1.7)

\[
\tilde{\nabla}_X \phi Y = -A_{\phi Y}X + \nabla^\perp_X \phi Y.
\]

Then we obtain

(2.4) \[
\nabla^\perp_X \phi Y = \alpha [g(X, Y)\xi - \epsilon \eta(Y)X] + \beta [g(\phi X, Y)\xi - \epsilon \eta(Y)\phi X]
+ \phi P\nabla_X Y + \phi Q\nabla_X Y + Bh(X, Y)
+ Ch(X, Y) + A_{\phi Y}X, \quad \forall \ X, Y \in D^\perp.
\]

Therefore from (2.3) and (2.4), it can be written as

\[
\phi P\nabla_X Y = -\alpha [g(X, Y)\xi - \epsilon \eta(Y)X] - \beta [g(\phi X, Y)\xi - \epsilon \eta(Y)\phi X]
- A_{\phi Y}X - Bh(X, Y) - \beta \epsilon \eta(Y)\phi Q X,
\]

Therefore we obtain

\[
\phi P[X, Y] = -A_{\phi Y}X + A_{\phi X}Y + \alpha \epsilon[\eta(Y)X - \eta(X)Y]
- \beta [g(\phi X, Y)\xi - \epsilon \eta(Y)\phi X] + \beta [g(\phi Y, X)\xi - \epsilon \eta(X)\phi Y]
- \beta \epsilon \eta(Y)\phi Q X + \beta \epsilon \eta(X)\phi Q Y.
\]

Now since \( M \) is \( \xi \)-vertical, so \([X, Y] \in D^\perp\) iff
\[ A_{\phi X}Y - A_{\phi Y}X = \alpha \epsilon[\eta(X)Y - \eta(Y)X]. \]

This proves the theorem.

Now we define $D$-umbilic (resp. $D^\perp$-umbilic) contact CR-submanifold of indefinite trans-Sasakian manifold.

**Definition 2.2.** A contact CR-submanifold $M$ of an indefinite trans-Sasakian manifold is said to be $D$-umbilic (resp. $D^\perp$-umbilic) if $h(X,Y) = g(X,Y)L$ holds for all $X,Y \in D$ (resp. $X,Y \in D^\perp$); $L$ being some normal vector field.

Then we give proposition 2.3 according to [10].

**Proposition 2.3.** Let $M$ be a $D$-umbilic (resp. $D^\perp$-umbilic) contact CR-submanifold of an indefinite trans-Sasakian manifold $\tilde{M}$. If $M$ is $\xi$-horizontal (resp. $\xi$-vertical) then $M$ is $D$-totally geodesic (resp. $D^\perp$-totally geodesic).

**Proof:** Let $M$ be $D$-umbilic $\xi$-horizontal contact CR-submanifold, then

\[ h(X,Y) = g(X,Y)L \quad \forall \quad X,Y \in D, \]

where $L$ being some normal vector field on $M$.

Now by putting $X = Y = \xi$ and taking (1.19) we have $L = 0$ and consequently we get $h(X,Y) = 0$, which implies that $M$ is $D$-totally geodesic.

Similarly, it can be easily proved that if $M$ is $D^\perp$-umbilic $\xi$-vertical contact CR-submanifold, then it is $D^\perp$-totally geodesic.

3. Mixed totally geodesic contact CR-submanifold of an indefinite trans-Sasakian manifold

Here we state the definition of mixed totally geodesic contact CR-submanifold of indefinite trans-Sasakian manifold.

**Definition 3.1.** A contact CR-submanifold $M$ of an indefinite trans-Sasakian manifold $\tilde{M}$ is said to be mixed totaly geodesic if $h(X,Y) = 0$ \forall \quad X \in D \text{ and } Y \in D^\perp.$

Then we obtain
**Lemma 3.1.** Let $M$ be a contact CR-submanifold of an indefinite trans-Sasakian manifold. Then $M$ is mixed totally geodesic iff

\[(3.1) \quad A_N X \in D, \quad \forall \ X \in D, \text{ and } \forall \text{ normal vector field } N \]

\[(3.2) \quad A_N X \in D^\perp, \quad \forall \ X \in D^\perp \text{ and normal vector field } N. \]

**Proof:** If $M$ is mixed totally geodesic contact CR-submanifold, then we get from (1.8)

\[h(X, Y) = 0\]

i.e., iff \( A_N X \in D, \quad \forall \ X \in D \text{ and } \forall \text{ normal vector field } N. \)

Similarly, if $M$ is mixed totally geodesic, then using (1.8) we easily obtain that \( A_N X \in D^\perp, \quad \forall \ X \in D^\perp \text{ and normal vector field } N. \)

This completes the proof. \(\square\)

Now using the above Lemma 3.1 we get

**Theorem 3.1.** If $M$ is a mixed totally geodesic contact CR-submanifold of an indefinite trans-Sasakian manifold, then

\[(3.3) \quad A_\phi N X = \phi A_N X, \]

\[(3.4) \quad \nabla_X^\perp \phi N = \phi \nabla_X^\perp N, \quad \forall \ X \in D \text{ and normal vector field } N. \]

**Proof:** Taking Lemma 3.1. and using Gauss and Weingarten formulas (1.7) and (1.8), we simply calculate (3.3) and (3.4). \(\square\)

Now we define foliate $\xi$-horizontal contact CR-submanifold of an indefinite trans-Sasakian manifold.

**Definition 3.2.** A contact CR-submanifold $M$ of an indefinite trans-Sasakian manifold $\tilde{M}$ is said to be foliate contact CR-submanifold of $\tilde{M}$ if $D$ is involutive. Then if $M$ is a foliate $\xi$-horizontal contact CR-submanifold, we have

\[h(\phi X, \phi Y) = h(\phi^2 X, Y) = -h(X, Y)\]

Then we get Proposition 3.1.
**Proposition 3.1.** If $M$ is a foliate $\xi$-horizontal mixed totally geodesic contact CR-submanifold of an indefinite trans-Sasakian manifold, then

\[(3.5) \quad \phi A_N X = A_N \phi X, \quad \forall \ X \in D \text{ and normal vector field } N.\]

*Proof:* Using (2.1) in (1.8), we simply derive the required equation (3.5). \[\square\]

### 4. Sectional curvatures of a contact CR-submanifold of an indefinite trans-Sasakian space form

Let $\tilde{M}(c)$ be an indefinite trans-Sasakian space form, then the curvature tensor $R$ of $\tilde{M}(c)$ satisfies by [12]

\[(4.1) \quad g(R(X,Y)Z,W) = \frac{(c+3(\alpha^2-\beta^2))}{4}\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\}
\]

\[\quad + \frac{(c-\alpha^2+\beta^2)}{4}\{g(X,\phi W)g(Y,\phi Z)-g(X,\phi Z)g(Y,\phi W)
\]

\[\quad - 2g(X,\phi Y)g(Z,\phi W)\}
\]

\[\forall \ X, Y, Z, W \in \Gamma(ker\ \eta).\]

Then the equation of Gauss is given by

\[(4.2) \quad g(R(X,Y)Z,W) = \frac{(c+3(\alpha^2-\beta^2))}{4}\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\}
\]

\[\quad + \frac{(c-\alpha^2+\beta^2)}{4}\{g(X,\phi PW)g(Y,\phi PZ)-g(X,\phi PZ)g(Y,\phi PW)
\]

\[\quad - 2g(X,\phi PY)g(Z,\phi PW)\} + g(h(X,Z),h(Y,W))
\]

\[\quad - g(h(X,W),h(Y,Z))\]

Now by orthonormal vectors $X, Y$, the sectional curvature $K_M(X \wedge Y)$ is determined from (4.2):

\[(4.3) \quad K_M(X \wedge Y) = -\frac{(c+3(\alpha^2-\beta^2))}{4} - \frac{(c-\alpha^2+\beta^2)}{4}g(\phi PX, PY)^2
\]

\[\quad + g(h(X,X),h(Y,Y))- \| h(X,Y) \|^2\]

When $M$ is $\xi$-horizontal, we easily obtain by (4.3)

**Proposition 4.1.** If $M$ is a $\xi$-horizontal contact CR-submanifold of $\tilde{M}(c)$. Then for $X, Y \in D^\perp$, the sectional curvature is given by
Moreover if $M$ is a $\xi$-vertical, then by using (4.3) we obtain

**Proposition 4.2.** If $M$ is a $\xi$-vertical CR-submanifold of $\tilde{M}(c)$. Then the sectional curvature determined by $X, Y \in D$ is given by

\[(4.5) \quad K_M(X \wedge Y) = -\frac{(c+3)(\alpha^2 - \beta^2)}{4} - \frac{(c-\alpha^2 + \beta^2)}{4}g(\phi X, Y)^2 + g(h(X, X), h(Y, Y)) - \| h(X, Y) \|^2.\]

A $\xi$-horizontal contact CR-submanifold $M$ is said to be $D^\perp$-minimal if $h(E_i, E_i) = 0$, $i = 1, 2, \ldots, m$ and $\{E_i\}$ is local field of orthonormal frames on $D$.

And the definition of $D$-minimal $\xi$-vertical contact CR-submanifold $M$ is similar as above.

Now we get

**Theorem 4.1.** If $M$ is a $D$-minimal $\xi$-vertical contact CR-submanifold of an indefinite trans-Sasakian space form $\tilde{M}(c)$, then $M$ is $D$-totally geodesic iff

\[(4.6) \quad K_M(X \wedge Y) = -\frac{(c+3)(\alpha^2 - \beta^2)}{4} \quad \text{for} \quad X, Y \in D.\]

**Proof:** If $M$ is $D$-totally geodesic, then using (4.3), we obtain

\[K_M(X \wedge Y) = -\frac{(c+3)(\alpha^2 - \beta^2)}{4}.\]

Conversely, if $M$ is $D$-minimal $\xi$-vertical contact CR-submanifold and (4.6) holds, then we get

\[h(X, Y) = 0. \quad \text{for} \quad X, Y \in D\]

i.e., $M$ is $D$-totally geodesic contact CR-submanifold.

Hence the theorem is proved. \(\square\)

Similarly we obtain

**Theorem 4.2.** If $M$ is $D^\perp$-minimal $\xi$-horizontal contact CR-submanifold of an indefinite trans-Sasakian space form $\tilde{M}(c)$, then $M$ is $D^\perp$-totally geodesic iff
\( K_M(X \wedge Y) = -\frac{(c+3(\alpha^2-\beta^2))}{4} \) for \( X, Y \in D^\perp \).

\textit{Proof:} If \( M \) is \( D^\perp \)-totally geodesic, then using (4.4) we obtain

\[ K_M(X \wedge Y) = -\frac{(c+3(\alpha^2-\beta^2))}{4} \]

Conversely, if \( M \) is \( D^\perp \)-minimal \( \xi \)-horizontal contact CR-submanifold and (4.7) holds, we get

\[ h(X,Y) = 0 \text{ for } X, Y \in D^\perp \]

i.e., \( M \) is \( D^\perp \)-totally geodesic CR-submanifold.

Hence the theorem is proved. \( \square \)

\textbf{References}


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