Abstract

For $\Omega$ an open subset of the Euclidean space $\mathbb{R}^n$, $T : \Omega \to \Omega$ a measurable non-singular transformation and $u$ a real-valued measurable function on $\mathbb{R}^n$, we study boundedness of the weighted composition operator $uC_T : f \mapsto u \cdot (f \circ T)$ on the Sobolev-Lorentz space $W^{1,n,q}(\Omega)$, consisting of those functions of the Lorentz space $L(n,q)$, whose distributional derivatives of the first order belong to $L(n,q)$, $1 \leq q \leq n$.

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1 Introduction

Suppose $(\Omega, \mathcal{A}, \mu)$ is a measure space where $\Omega$ is an open subset of the Euclidean space $\mathbb{R}^n$, $\mathcal{A}$ the $\sigma$-algebra of Lebesgue measurable subsets of $\Omega$ and $\mu$ the Lebesgue measure. Let $f$ be a complex-valued Lebesgue measurable function defined on $\Omega$. For $s \geq 0$, define $\mu_f$ the distribution function of $f$ as

$$
\mu_f(s) = \mu\{x \in \Omega : |f(x)| > s\}.
$$
By \( f^* \) we mean the non-increasing rearrangement of \( f \) given as
\[
    f^*(t) = \inf\{s > 0 : \mu_f(s) \leq t\}, \quad t \geq 0.
\]

For \( 1 \leq q \leq n \), the Lorentz norm of \( f \) is given by
\[
    \|f\|_{n,q} = \left( \int_0^\infty \left( t^{1/n} f^*(t)^q \right) \frac{dt}{t} \right)^{1/q}.
\]
The Lorentz space \( L(n,q) \) is the set of equivalence classes of complex-valued Lebesgue measurable functions \( f \) on \( \Omega \) with \( \|f\|_{n,q} < \infty \). \( L(n,q) \) is a Banach space [11] with respect to above norm.

The Sobolev-Lorentz space \( W^{1,n,q}(\Omega) \) is defined as the set of all complex-valued functions \( f \) in \( L(n,q) \) whose weak partial derivatives \( \partial f / \partial x_i \) (in the distributional sense) belong to \( L(n,q) \), \( i = 1, 2, \ldots, n \). It is a Banach space [25] with respect to the norm:
\[
    \|f\|_{1,n,q} = \|f\|_{n,q} + \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{n,q}.
\]

On \( \sigma \)-finite measure space \( (\Omega, \mathcal{A}, \mu) \), let \( T : \Omega \rightarrow \Omega \) be a measurable \( (T^{-1}(E) \in \mathcal{A} \text{ for every } E \in \mathcal{A}) \) non-singular transformation \( (\mu(T^{-1}(E))=0, \text{ whenever } \mu(E) = 0) \). Let the function \( f_T = d(\mu \circ T^{-1})/d\mu \) be the Radon-Nikodym derivative. Suppose \( u \) is a complex-valued measurable function defined on \( \mathbb{R}^n \). Then \( T \) induces a well-defined linear transformation \( uC_T \) on \( W^{1,n,q}(\Omega) \) defined by
\[
    (uC_Tf)(x) = u(x)f(T(x)), \quad x \in \Omega, \; f \in W^{1,n,q}(\Omega).
\]
If \( uC_T \) maps \( W^{1,n,q}(\Omega) \) into itself and is bounded, then we call \( uC_T \) a weighted composition operator on \( W^{1,n,q}(\Omega) \) induced by \( T \) with weight \( u \). If \( u \equiv 1 \), then \( C_T \) is called a composition operator induced by \( T \).

Our study here, of weighted composition operators, on Sobolev-Lorentz space \( W^{1,n,q}(\Omega) \) is motivated by the work of Herbert Kamowitz and Dennis Wortman[12]. Other similar references include [2–7, and 15–17]. The paper contains two sections. In the first section we define the composition operator on \( W^{1,n,q}(\Omega) \), and the second section is devoted to the study of weighted composition operator on \( W^{1,n,q}(\Omega) \).

## 2 Composition operator on \( W^{1,n,q}(\Omega) \)

**Lemma 2.1** Let \( f_T, \frac{\partial T_k}{\partial x_i} \in L^\infty(\mu) \) with \( \left\| \frac{\partial T_k}{\partial x_i} \right\|_\infty \leq M, (M > 0) \) \( i, k = 1, 2, \ldots, n \), where \( T = (T_1, T_2, \ldots, T_n) \) and \( \frac{\partial T_k}{\partial x_i} \) denotes the first order partial derivative (in the classical sense). Then for each \( f \) in \( W^{1,n,q}(\Omega) \) we have
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\( f \circ T \in L(n,q), 1 \leq q \leq n, \) and the first order distributional derivatives of 
\((f \circ T), \) given by,

\[
\frac{\partial}{\partial x_i} (f \circ T) = \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i}
\]

(1)

for \( i = 1, 2, \ldots, n, \) are in \( L(n,q). \)

**Proof.** The Radon-Nikodym derivative \( f_T = d(\mu \circ T^{-1})/d\mu \in L^\infty(\mu) \) implies that for each \( E \in \mathcal{A}, \)

\[
(\mu \circ T^{-1})(E) = \int_E f_T d\mu \leq \|f_T\|_\infty \mu(E).
\]

For \( f \) in \( W^{1,n,q}(\Omega), \) the distribution of \( f \circ T \) satisfies

\[
\mu_{f \circ T}(s) = \mu\{x \in \Omega : |f(T(x))| > s\}
\]

\[
= (\mu \circ T^{-1})\{x \in \Omega : |f(x)| > s\}
\]

\[
\leq \|f_T\|_\infty \mu\{x \in \Omega : |f(x)| > s\} = \|f_T\|_\infty \mu_f(s).
\]

Therefore

\[
\{s > 0 : \mu_f(s) \leq t\} \subseteq \{s > 0 : \mu_{f \circ T}(s) \leq \|f_T\|_\infty t\}.
\]

This gives

\[
(f \circ T)^*(\|f_T\|_\infty t) \leq f^*(t), \text{ i.e., } (f \circ T)^*(t) \leq f^*(t/\|f_T\|_\infty).
\]

Now \( f \in L(n,q), 1 \leq q \leq n, \) gives

\[
\|f \circ T\|_{n,q}^q = \int_0^\infty (t^{1/n}(f \circ T)^*(t))^q dt \leq \int_0^\infty (t^{1/n} f^*(t/b))^q dt = \|f_T\|_\infty \|f\|_{n,q}^q.
\]

Thus \( \|f \circ T\|_{n,q} \leq \|f_T\|_\infty^{1/n} \|f\|_{n,q}, \) and hence \( f \circ T \in L(n,q). \)

By the same arguments, as each weak partial derivative \( \partial f/\partial x_k \in L(n,q), \)

it follows that \( \partial f/\partial x_k \circ T \in L(n,q), \) for each \( k = 1, 2, \ldots, n. \)

Also \( \partial T_k/\partial x_i \in L^\infty(\mu), \) therefore

\[
\left( \frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} \in L(n,q), \text{ for each } i, k = 1, 2, \ldots, n.
\]

Hence, using triangle inequality, it follows that the function in right hand side
of (1) belongs to \( L(n,q), \) for each \( i = 1, 2, \ldots, n. \)

Since \( f \in W^{1,n,q}(\Omega), 1 \leq q \leq n, \) following the same computation as in
Friedrich’s Theorem [14, Theorem 2.2.1, p. 57], there exists a sequence \( < f_m >\)
in $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$ such that $f_m \to f$ in $L(n, q)(\Omega)$ and $\frac{\partial f_m}{\partial x_i} \to \frac{\partial f}{\partial x_i}$ in $L(n, q)(\Omega')$ for every $1 \leq i \leq n$ and for every relatively compact set $\Omega'$ in $\Omega$.

Let $g \in \mathcal{D}(\mathbb{R}^n)$. We choose relatively compact set $\Omega'$ in $\Omega$ with $\text{supp}(g) \subset \Omega'$. Then by the ordinary chain rule for smooth function $f_m$, we have for each $i = 1, 2, \ldots, n$

$$\int_{\Omega} (f_m \circ T) \frac{\partial g}{\partial x_i} d\mu = \int_{\Omega'} (f_m \circ T) \frac{\partial g}{\partial x_i} d\mu = -\int_{\Omega'} \frac{\partial}{\partial x_i} (f_m \circ T) gd\mu = -\int_{\Omega'} \sum_{k=1}^{n} \left( \frac{\partial f_m}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} g d\mu. \quad (2)$$

Now $g \in \mathcal{D}(\mathbb{R}^n)$ implies that for each $i$, $g \frac{\partial g}{\partial x_i} \in X'$, where $X'$ is the associate space of the Banach function space $X = L(n, q)(\Omega)$, $1 \leq q \leq n$. Therefore by using the Hölder’s inequality in Banach function spaces, we have

$$\int_{\Omega} | f_m \circ T - f \circ T | \left| \frac{\partial g}{\partial x_i} \right| d\mu \leq \left\| (f_m - f) \circ T \right\|_X \left\| \frac{\partial g}{\partial x_i} \right\|_{X'}$$

Therefore as $m \to \infty$, for each $i = 1, 2, \ldots, n$

$$\int_{\Omega} (f_m \circ T) \frac{\partial g}{\partial x_i} d\mu \to \int_{\Omega} (f \circ T) \frac{\partial g}{\partial x_i} d\mu.$$

By the similar arguments, using $\frac{\partial f_m}{\partial x_k} \to \frac{\partial f}{\partial x_k}$ in $L(n, q)(\Omega')$ and $\frac{\partial f}{\partial x_k} \in L^\infty(\mu)$, we obtain that in $L(n, q)(\Omega')$,

$$\sum_{k=1}^{n} \left( \frac{\partial f_m}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} \to \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} \quad \text{as } m \to \infty, \text{ for each } i = 1, 2, \ldots, n$$

So by the Hölder’s inequality in Banach function spaces again, we obtain as $m \to \infty$, for each $i = 1, 2, \ldots, n$

$$\int_{\Omega'} \sum_{k=1}^{n} \left( \frac{\partial f_m}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} g d\mu \to \int_{\Omega'} \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} g d\mu.$$

Hence by taking limits on both the sides of (2) as $m \to \infty$, we obtain

$$\int_{\Omega} (f \circ T) \frac{\partial g}{\partial x_i} d\mu = -\int_{\Omega'} \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} g d\mu$$

$$= -\int_{\Omega} \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} g d\mu.$$
Therefore for all $i = 1, 2, \ldots, n$

$$- \int_\Omega \frac{\partial}{\partial x_i} (f \circ T) g d\mu = - \int_\Omega \sum_{k=1}^n \left( \frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} g d\mu.$$ 

As $g$ was chosen arbitrarily, the equation (1) follows.

**Theorem 2.2** Let $\Omega \subset \mathbb{R}^n$ be an open set and $T : \Omega \to \Omega$ a measurable non-singular transformation with Radon-Nikodym derivative $f_T = d(\mu \circ T^{-1})/d\mu$, $
frac{\partial T_k}{\partial x_i}$ in $L^\infty(\mu)$, and $\left\| \frac{\partial T_k}{\partial x_i} \right\|_\infty \leq M, M > 0$, for $i, k = 1, 2, \ldots, n$, where $T = (T_1, T_2, \ldots, T_n)$ and $\frac{\partial T_k}{\partial x_i}$ denotes the first order partial derivatives (in the classical sense). Then the mapping $C_T$ defined by $C_T(f) = f \circ T$ is a composition operator on the Sobolev-Lorentz space $W^{1,n,q}(\Omega), 1 \leq q \leq n$.

**Proof.** By the Lemma 2.1, we have $f \circ T \in W^{1,n,q}(\Omega)$ and its norm satisfies the following:

$$\left\| f \circ T \right\|_{1,n,q} = \left\| f \circ T \right\|_{n,q} + \sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} (f \circ T) \right\|_{n,q}$$

$$\leq \left\| f_T \right\|_{1/n} \left\| f \right\|_{n,q} + \sum_{i=1}^n \sum_{k=1}^n \left\| \frac{\partial f}{\partial x_k} \right\|_{1/n} \left\| \frac{\partial f}{\partial x_k} \right\|_{n,q} \left\| \frac{\partial T_k}{\partial x_i} \right\|_\infty$$

$$\leq \left\| f_T \right\|_{1/n} \left\| f \right\|_{n,q} + \left\| f_T \right\|_{1/n} Mn \sum_{k=1}^n \left\| \frac{\partial f}{\partial x_k} \right\|_{n,q}$$

The result follows.

### 3 Weighted composition operator on $W^{1,n,q}(\Omega)$

Suppose $u$ is a real-valued measurable function defined on $\mathbb{R}^n$. Also suppose that $T : \Omega \to \Omega$ is a measurable non-singular transformation and $(\Omega, \mathcal{A}, \mu)$ is the $\sigma$-finite measure space, where $\Omega$ an open subset of $\mathbb{R}^n$. On the same lines as in Lemma 2.1, we have the following, for $i = 1, 2, \ldots, n$.

$$\frac{\partial}{\partial x_i} (u \cdot (f \circ T)) = \frac{\partial u}{\partial x_i} (f \circ T) + u \sum_{k=1}^n \left( \frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i}.$$
Theorem 3.1 If all the conditions stated in the Theorem 2.2 are satisfied and, in addition, \( u \in L^\infty(\mu) \) such that the first order classical partial derivatives \( \partial u/\partial x_i \) satisfy \( \| \partial u/\partial x_i \|_\infty \leq M_1, M_1 > 0, \) for \( i = 1, 2, \ldots, n, \) then the mapping \( uC_T \) defined by \( (uC_T)f = u \cdot (f \circ T) \) is a weighted composition operator on the Sobolev-Lorentz space \( W^{1,n,q}(\Omega), 1 \leq q \leq n. \)

Proof. By the same arguments as in Lemma 2.1, we find
\[
\|u \cdot (f \circ T)\|_{n,q} \leq \|u\|_\infty \|f_T\|_\infty^{1/n} \|f\|_{n,q}
\]
\[
\left\| \frac{\partial u}{\partial x_i} (f \circ T) \right\|_{n,q} \leq M_1 \|f_T\|_\infty^{1/n} \|f\|_{n,q},
\]
and
\[
\left\| u \sum_{k=1}^n \left( \frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} \right\|_{n,q} \leq \|u\|_\infty M \|f_T\|_\infty \sum_{k=1}^n \left\| \frac{\partial f}{\partial x_k} \right\|_{n,q}.
\]
Hence it follows that
\[
\| (uC_T)f \|_{1,n,q} = \|u \cdot (f \circ T)\|_{1,n,q} = \|u \cdot (f \circ T)\|_{n,q} + \sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} (u \cdot (f \circ T)) \right\|_{n,q}
\]
\[
\leq \|u\|_\infty \|f_T\|_\infty^{1/n} \|f\|_{n,q} + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} (f \circ T) \right\|_{n,q} + \sum_{i=1}^n \left\| u \sum_{k=1}^n \left( \frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} \right\|_{n,q}
\]
\[
\leq \|u\|_\infty \|f_T\|_\infty \|f\|_{1,n,q} + nM_1 \|f_T\|_\infty^{1/n} \|f\|_{1,n,q} + nM \|u\|_\infty \|f_T\|_\infty^{1/n} \|f\|_{1,n,q}
\]
Thus \( \| (uC_T)f \|_{1,n,q} \leq K \|f\|_{1,n,q}, \) for some \( K > 0. \)

References


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