

Compatibility of Type (A) and Fixed Point Theorem in Fuzzy Metric Space

Bijendra Singh*, Arihant Jain** and Amit Kumar Govery*

*School of Studies in Mathematics, Vikram University, Ujjain - 456010 (M.P.)

** Department of Applied Mathematics, Shri Guru Sandipani Institute of Technology
and Science, Ujjain - 456550 (M.P.) India
arihant2412@gmail.com

Abstract

In this paper, the concept of compatible maps of type (A) in Fuzzy metric space has been applied to prove common fixed point theorem. A fixed point theorem for six self maps has been established using the concept of compatible maps of type (A), which generalizes the result of Cho [1].

Mathematics Subject Classification: Primary 47H10, Secondary 54H25

Keywords: Common fixed points, fuzzy metric space, compatible maps, compatible maps of type (A).

1. Introduction

The concept of Fuzzy sets was initially investigated by Zadeh [11] as a new way to represent vagueness in everyday life. Subsequently, it was developed by many authors and used in various fields. To use this concept in Topology and Analysis, several researchers have defined Fuzzy metric space in various ways. In this paper we deal with the Fuzzy metric space defined by Kramosil and Michalek [8] and modified by George and Veeramani [3]. Recently, Grebiec [4] has proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan [10] introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem. Jungck et. al. [6] introduced the concept of

compatible maps of type (A) in metric space and proved fixed point theorems. Using the concept of compatible maps of type (A), Jain et. al. [5] proved a fixed point theorem for six self maps in a fuzzy metric space.

In this paper, a fixed point theorem for six self maps has been established using the concept of compatible maps of type (A), which generalizes the result of Cho [1].

For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

2. Preliminaries

Definition 2.1. [9] A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a *t-norm* if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for $a, b, c, d \in [0, 1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2. [9] The 3-tuple $(X, M, *)$ is said to be a *Fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions :

for all $x, y, z \in X$ and $s, t > 0$.

$$(FM-1) \quad M(x, y, 0) = 0,$$

$$(FM-2) \quad M(x, y, t) = 1 \text{ for all } t > 0 \text{ if and only if } x = y,$$

$$(FM-3) \quad M(x, y, t) = M(y, x, t),$$

$$(FM-4) \quad M(x, y, t) * M(y, z, s) \leq M(x, z, t + s),$$

$$(FM-5) \quad M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1] \text{ is left continuous,}$$

$$(FM-6) \quad \lim_{t \rightarrow \infty} M(x, y, t) = 1.$$

Note that $M(x, y, t)$ can be considered as the degree of nearness between x and y with respect to t . We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. [9] Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$ and all $t > 0$. Then $(X, M, *)$ is a Fuzzy metric space. It is called the Fuzzy metric space induced by d .

Definition 2.3. [9] A sequence $\{x_n\}$ in a Fuzzy metric space $(X, M, *)$ is said to be a *Cauchy sequence* if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

The sequence $\{x_n\}$ is said to *converge* to a point x in X if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

A Fuzzy metric space $(X, M, *)$ is said to be *complete* if every Cauchy sequence in it converges to a point in it.

Definition 2.4. [10] Self mappings A and S of a Fuzzy metric space $(X, M, *)$ are said to be *compatible* if and only if $M(ASx_n, SAx_n, t) \rightarrow 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \rightarrow p$ for some p in X as $n \rightarrow \infty$.

Definition 2.5. [5] Self maps A and S of a Fuzzy metric space $(X, M, *)$ are said to be compatible maps of type (A) if $M(ASx_n, SSx_n, t) \rightarrow 1$ and $M(SAx_n, AAx_n, t) \rightarrow 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \rightarrow p$ for some p in X as $n \rightarrow \infty$.

Remark 2.1. [5] The concept of compatible maps of type (A) is more general than the concept of compatible maps in a fuzzy metric space.

Proposition 2.1. [5] In a fuzzy metric space $(X, M, *)$ limit of a sequence is unique.

Lemma 2.1. [4] Let $(X, M, *)$ be a fuzzy metric space. Then for all $x, y \in X$, $M(x, y, \cdot)$ is a non-decreasing function.

Lemma 2.2. [1] Let $(X, M, *)$ be a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$, $M(x, y, kt) \geq M(x, y, t) \forall t > 0$, then $x = y$.

Lemma 2.3. [5] Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. If there exists a number $k \in (0, 1)$ such that

$$M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t) \quad \forall t > 0 \quad \text{and } n \in \mathbb{N}.$$

Then $\{x_n\}$ is a Cauchy sequence in X .

Proposition 2.2. [5] Let S and T be compatible self mappings of type (A) of a complete fuzzy metric space $(X, M, *)$ with continuous t norm $*$ defined by $a * b = \min\{a, b\}$, $a, b \in [0, 1]$ and $Su = Tu$ for some u in X . Then $STu = TSu = SSu = TTu$.

Lemma 2.4.[7] The only t -norm $*$ satisfying $r * r \geq r$ for all $r \in [0, 1]$ is the minimum t -norm, that is

$$a * b = \min \{a, b\} \text{ for all } a, b \in [0, 1].$$

3. Main Result

Theorem 3.1. Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied :

- (a) $P(X) \subset ST(X), Q(X) \subset AB(X)$;
- (b) $AB = BA, ST = TS, PB = BP, QT = TQ$;
- (c) either P or AB is continuous;
- (d) (P, AB) and (Q, ST) are pairs of compatible maps of type (A);
- (e) there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$

$$M(Px, Qy, qt) \geq M(ABx, STy, t) * M(Px, ABx, t) * M(Qy, STy, t) * M(Px, STy, t).$$

Then A, B, S, T, P and Q have a unique common fixed point in X .

Proof : Let $x_0 \in X$. From (a) there exist $x_1, x_2 \in X$ such that

$$Px_0 = STx_1 \quad \text{and} \quad Qx_1 = ABx_2.$$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned} Px_{2n-2} &= STx_{2n-1} = y_{2n-1} \quad \text{and} \\ Qx_{2n-1} &= ABx_{2n} = y_{2n} \quad \text{for } n = 1, 2, 3, \dots \end{aligned}$$

Step 1. Put $x = x_{2n}$ and $y = x_{2n+1}$ in (e), we get

$$\begin{aligned} M(Px_{2n}, Qx_{2n+1}, qt) &\geq M(ABx_{2n}, STx_{2n+1}, t) * M(Px_{2n}, ABx_{2n}, t) \\ &\quad * M(Qx_{2n+1}, STx_{2n+1}, t) * M(Px_{2n}, STx_{2n+1}, t). \\ &= M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) \\ &\quad * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+1}, t) \\ &\geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t). \end{aligned}$$

From lemma 2.1 and 2.2, we have

$$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t).$$

Similarly, we have

$$M(y_{2n+2}, y_{2n+3}, qt) \geq M(y_{2n+1}, y_{2n+2}, t).$$

Thus, we have

$$M(y_{n+1}, y_{n+2}, qt) \geq M(y_n, y_{n+1}, t) \text{ for } n = 1, 2, \dots$$

$$\begin{aligned}
 M(y_n, y_{n+1}, t) &\geq M(y_n, y_{n+1}, t/q) \\
 &\geq M(y_{n-2}, y_{n-1}, t/q^2) \\
 &\quad \dots \quad \dots \quad \dots \quad \dots \\
 &\geq M(y_1, y_2, t/q^n) \rightarrow 1 \text{ as } n \rightarrow \infty,
 \end{aligned}$$

and hence $M(y_n, y_{n+1}, t) \rightarrow 1$ as $n \rightarrow \infty$ for any $t > 0$.

For each $\varepsilon > 0$ and $t > 0$, we can choose $n_0 \in \mathbb{N}$ such that

$$M(y_n, y_{n+1}, t) > 1 - \varepsilon \text{ for all } n > n_0.$$

For $m, n \in \mathbb{N}$, we suppose $m \geq n$. Then we have

$$\begin{aligned}
 M(y_n, y_m, t) &\geq M(y_n, y_{n+1}, t/m-n) * M(y_{n+1}, y_{n+2}, t/m-n) \\
 &\quad * \dots * M(y_{m-1}, y_m, t/m-n) \\
 &\geq (1 - \varepsilon) * (1 - \varepsilon) * \dots * (1 - \varepsilon) \text{ (m - n) times} \\
 &\geq (1 - \varepsilon)
 \end{aligned}$$

and hence $\{y_n\}$ is a Cauchy sequence in X .

Since $(X, M, *)$ is complete, $\{y_n\}$ converges to some point $z \in X$. Also its subsequences converges to the same point i.e. $z \in X$

$$\text{i.e., } \{Qx_{2n+1}\} \rightarrow z \quad \text{and} \quad \{STx_{2n+1}\} \rightarrow z \tag{1}$$

$$\{Px_{2n}\} \rightarrow z \quad \text{and} \quad \{ABx_{2n}\} \rightarrow z. \tag{2}$$

Case I. Suppose AB is continuous.

Since AB is continuous, we have

$$(AB)^2x_{2n} \rightarrow ABz \text{ and}$$

$$ABPx_{2n} \rightarrow ABz.$$

As (P, AB) is compatible pair of type (A), we have

$$PABx_{2n} \rightarrow ABz.$$

Step 2. Put $x = ABx_{2n}$ and $y = x_{2n+1}$ in (e), we get

$$\begin{aligned}
 M(PABx_{2n}, Qx_{2n+1}, qt) &\geq M(ABABx_{2n}, STx_{2n+1}, t) * M(PABx_{2n}, ABABx_{2n}, t) \\
 &\quad * M(Qx_{2n+1}, STx_{2n+1}, t) * M(PABx_{2n}, STx_{2n+1}, t).
 \end{aligned}$$

Taking $n \rightarrow \infty$, we get

$$\begin{aligned} M(ABz, z, qt) &\geq M(ABz, z, t) * M(ABz, ABz, t) * M(z, z, t) * M(ABz, z, t) \\ &\geq M(ABz, z, t) * M(ABz, z, t) \end{aligned}$$

i.e. $M(ABz, z, qt) \geq M(ABz, z, t)$.

Therefore, by using lemma 2.2, we get

$$ABz = z. \quad (3)$$

Step 3. Put $x = z$ and $y = x_{2n+1}$ in (e), we have

$$\begin{aligned} M(Pz, Qx_{2n+1}, qt) &\geq M(ABz, STx_{2n+1}, t) * M(Pz, ABz, t) \\ &\quad * M(Qx_{2n+1}, STx_{2n+1}, t) * M(Pz, STx_{2n+1}, t). \end{aligned}$$

Taking $n \rightarrow \infty$ and using equation (1), we get

$$\begin{aligned} M(Pz, z, qt) &\geq M(z, z, t) * M(Pz, z, t) * M(z, z, t) * M(Pz, z, t) \\ &\geq M(Pz, z, t) * M(Pz, z, t) \end{aligned}$$

i.e. $M(Pz, z, qt) \geq M(Pz, z, t)$.

Therefore, by using lemma 2.2, we get

$$Pz = z.$$

Therefore, $ABz = Pz = z$.

Step 4. Putting $x = Bz$ and $y = x_{2n+1}$ in condition (e), we get

$$\begin{aligned} M(PBz, Qx_{2n+1}, qt) &\geq M(ABBz, STx_{2n+1}, t) * M(PBz, ABBz, t) \\ &\quad * M(Qx_{2n+1}, STx_{2n+1}, t) * M(PBz, STx_{2n+1}, t). \end{aligned}$$

As $BP = PB$, $AB = BA$, so we have

$$\begin{aligned} P(Bz) &= B(Pz) = Bz \quad \text{and} \\ (AB)(Bz) &= (BA)(Bz) = B(ABz) = Bz. \end{aligned}$$

Taking $n \rightarrow \infty$ and using (1), we get

$$\begin{aligned} M(Bz, z, qt) &\geq M(Bz, z, t) * M(Bz, Bz, t) * M(z, z, t) * M(Bz, z, t) \\ &\geq M(Bz, z, t) * M(Bz, z, t) \end{aligned}$$

i.e. $M(Bz, z, qt) \geq M(Bz, z, t)$.

Therefore, by using lemma 2.2, we get

$$Bz = z$$

and also we have

$$ABz = z$$

$$\Rightarrow Az = z.$$

Therefore, $Az = Bz = Pz = z.$ (4)

Step 5. As $P(X) \subset ST(X)$, there exists $u \in X$ such that

$$z = Pz = STu.$$

Putting $x = x_{2n}$ and $y = u$ in (e), we get

$$\begin{aligned} M(Px_{2n}, Qu, qt) &\geq M(ABx_{2n}, STu, t) * M(Px_{2n}, ABx_{2n}, t) \\ &\quad * M(Qu, STu, t) * M(Px_{2n}, STu, t). \end{aligned}$$

Taking $n \rightarrow \infty$ and using (1) and (2), we get

$$\begin{aligned} M(z, Qu, qt) &\geq M(z, z, t) * M(z, z, t) * M(Qu, z, t) * M(z, z, t) \\ &\geq M(Qu, z, t) \end{aligned}$$

i.e. $M(z, Qu, qt) \geq M(z, Qu, t).$

Therefore, by using lemma 2.2, we get

$$Qu = z.$$

Hence $STu = z = Qu.$

Since (Q, ST) is compatible pair of type (A), therefore, by proposition (2.2), we have

$$QSTu = STQu.$$

Thus $Qz = STz.$

Step 6. Putting $x = x_{2n}$ and $y = z$ in (e), we get

$$\begin{aligned} M(Px_{2n}, Qz, qt) &\geq M(ABx_{2n}, STz, t) * M(Px_{2n}, ABx_{2n}, t) \\ &\quad * M(Qz, STz, t) * M(Px_{2n}, STz, t). \end{aligned}$$

Taking $n \rightarrow \infty$ and using (2) and step 5, we get

$$M(z, Qz, qt) \geq M(z, Qz, t) * M(z, z, t) * M(Qz, Qz, t) * M(z, Qz, t)$$

$$\geq M(z, Qz, t) * M(z, Qz, t)$$

i.e. $M(z, Qz, qt) \geq M(z, Qz, t)$.

Therefore, by using lemma 2.2, we get

$$Qz = z.$$

Step 7. Putting $x = x_{2n}$ and $y = Tz$ in (e), we get

$$\begin{aligned} M(Px_{2n}, QTz, qt) &\geq M(ABx_{2n}, STTz, t) * M(Px_{2n}, ABx_{2n}, t) \\ &\quad * M(QTz, STTz, t) * M(Px_{2n}, STTz, t). \end{aligned}$$

As $QT = TQ$ and $ST = TS$, we have

$$QTz = TQz = Tz \quad \text{and}$$

$$ST(Tz) = T(STz) = TQz = Tz.$$

Taking $n \rightarrow \infty$, we get

$$\begin{aligned} M(z, Tz, qt) &\geq M(z, Tz, t) * M(z, z, t) * M(Tz, Tz, t) * M(z, Tz, t) \\ &\geq M(z, Tz, t) * M(z, Tz, t) \end{aligned}$$

i.e. $M(z, Tz, qt) \geq M(z, Tz, t)$.

Therefore, by using lemma 2.2, we get

$$Tz = z.$$

Now $STz = Tz = z$ implies $Sz = z$.

Hence $Sz = Tz = Qz = z$. (5)

Combining (4) and (5), we get

$$Az = Bz = Pz = Qz = Tz = Sz = z.$$

Hence, z is the common fixed point of A, B, S, T, P and Q .

Case II. Suppose P is continuous.

As P is continuous,

$$P^2x_{2n} \rightarrow Pz \quad \text{and} \quad P(AB)x_{2n} \rightarrow Pz.$$

As (P, AB) is compatible pair of type (A),

$$(AB)Px_{2n} \rightarrow Pz.$$

Step 8. Putting $x = Px_{2n}$ and $y = x_{2n+1}$ in condition (e), we have

$$M(PPx_{2n}, Qx_{2n+1}, qt) \geq M(ABPx_{2n}, STx_{2n+1}, t) * M(PPx_{2n}, ABPx_{2n}, t) \\ * M(Qx_{2n+1}, STx_{2n+1}, t) * M(PPx_{2n}, STx_{2n+1}, t).$$

Taking $n \rightarrow \infty$, we get

$$M(Pz, z, qt) \geq M(Pz, z, t) * M(Pz, Pz, t) * M(z, z, t) * M(Pz, z, t) \\ \geq M(Pz, z, t) * M(Pz, z, t)$$

i.e. $M(Pz, z, qt) \geq M(Pz, z, t)$.

Therefore by using lemma 2.2, we have

$$Pz = z.$$

Further, using steps 5, 6, 7, we get

$$Qz = STz = Sz = Tz = z.$$

Step 9. As $Q(X) \subset AB(X)$, there exists $w \in X$ such that

$$z = Qz = ABw.$$

Put $x = w$ and $y = x_{2n+1}$ in (e), we have

$$M(Pw, Qx_{2n+1}, qt) \geq M(ABw, STx_{2n+1}, t) * M(Pw, ABw, t) \\ * M(Qx_{2n+1}, STx_{2n+1}, t) * M(Pw, STx_{2n+1}, t).$$

Taking $n \rightarrow \infty$, we get

$$M(Pw, z, qt) \geq M(z, z, t) * M(Pw, z, t) * M(z, z, t) * M(Pw, z, t) \\ \geq M(Pw, z, t) * M(Pw, z, t)$$

i.e. $M(Pw, z, qt) \geq M(Pw, z, t)$.

Therefore, by using lemma 2.2, we get

$$Pw = z.$$

Therefore, $ABw = Pw = z$.

As (P, AB) is compatible pair of type (A), then by proposition (2.2), we have

$$Pz = ABz.$$

Also, from step 4, we get $Bz = z$.

Thus, $Az = Bz = Pz = z$ and we see that z is the common fixed point of the six maps in this case also.

Uniqueness : Let u be another common fixed point of A, B, S, T, P and Q .

$$\text{Then } Au = Bu = Pu = Qu = Su = Tu = u.$$

Put $x = z$ and $y = u$ in (e), we get

$$\begin{aligned} M(Pz, Qu, qt) &\geq M(ABz, STu, t) * M(Pz, ABz, t) * M(Qu, STu, t) \\ &\quad * M(Pz, STu, t). \end{aligned}$$

Taking $n \rightarrow \infty$, we get

$$\begin{aligned} M(z, u, qt) &\geq M(z, u, t) * M(z, z, t) * M(u, u, t) * M(z, u, t) \\ &\geq M(z, u, t) * M(z, u, t) \end{aligned}$$

i.e. $M(z, u, qt) \geq M(z, u, t)$.

Therefore by using lemma (2.2), we get

$$z = u.$$

Therefore z is the unique common fixed point of self maps A, B, S, T, P and Q .

Remark 3.1. If we take $B = T = I$, the identity map on X in theorem 3.1, then condition (b) is satisfied trivially and we get

Corollary 3.1. Let $(X, M, *)$ be a complete fuzzy metric space and let A, S, P and Q be mappings from X into itself such that the following conditions are satisfied :

- (a) $P(X) \subset S(X), Q(X) \subset A(X)$;
- (b) either A or P is continuous;
- (c) (P, A) and (Q, S) are compatible maps of type (A);
- (d) there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$

$$M(Px, Qy, qt) \geq M(Ax, Sy, t) * M(Px, Ax, t) * M(Qy, Sy, t) * M(Px, Sy, t).$$

Then A, S, P and Q have a unique common fixed point in X .

Remark 3.2. In view of remark 3.1, corollary 3.1 is a generalization of the result of Cho [1] in the sense that condition of compatibility of the pairs of self maps has been restricted to compatibility of type (A) and only one map of the first pair is needed to be continuous.

ACKNOWLEDGEMENT : Authors are thankful to the referee for his valuable comments.

REFERENCES

1. S.H., Cho, *On common fixed point theorems in fuzzy metric spaces*, J. Appl. Math. & Computing Vol. 20 (2006), No. 1 -2, 523-533.
2. Y.J. Cho, *Fixed point in Fuzzy metric space*, J. Fuzzy Math. 5(1997), 949-962.
3. A. George and P. Veeramani, *On some results in Fuzzy metric spaces*, Fuzzy Sets and Systems 64 (1994), 395-399.
4. M. Grebiec, *Fixed points in Fuzzy metric space*, Fuzzy sets and systems, 27(1998), 385-389.
5. A. Jain and B. Singh, *A fixed point theorem for compatible mappings of type (A) in fuzzy metric space*, Acta Ciencia Indica, Vol. XXXIII M, No. 2 (2007), 339-346.
6. G. Jungck, P.P. Murthy and Y.J. Cho, *Compatible mappings of type (A) and common fixed points*, Math. Japonica, 38 (1993), 381-390.
7. E.P. Klement, R. Mesiar and E. Pap, *Triangular Norms*, Kluwer Academic Publishers.
8. I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetika 11 (1975), 336-344.
9. S.N. Mishra, N. Mishra and S.L. Singh, *Common fixed point of maps in fuzzy metric space*, Int. J. Math. Math. Sci. 17(1994), 253-258.
10. B. Singh and M.S. Chouhan, *Common fixed points of compatible maps in Fuzzy metric spaces*, Fuzzy sets and systems, 115 (2000), 471-475.
11. L. A. Zadeh, *Fuzzy sets*, Inform and control 89 (1965), 338-353.

Received: October, 2010