Generalised $L^* - (p, q)$th Order of the Derivative of a Meromorphic Function

Sanjib Kumar Datta

Department of Mathematics, University of Kalyani
P.O.-Kalyani, Dist-Nadia, Pin-741235
West Bengal, India
Former Address:
(Department of Mathematics, University of North Bengal
P.O.-North Bengal University, Raja Rammohunpur
Dist-Darjeeling, Pin-734013
West Bengal, India)
sanjib.kr.datta@yahoo.co.in
s.kr.datta.ku@yahoo.co.in
sk.datta.nbu@yahoo.co.in

Meghlal Mallik

Panighata U.D.M. High School
P.O.-Paglachandi, Dist Nadia, Pin-741181
West Bengal, India
meghlal1982@yahoo.com
meghlal_mallik@yahoo.com

Abstract

In this paper we generalise the results of Datta and Mondal [3].

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1 Introduction, Definitions and Notations.

We know \{cf.[10], p.36\} that the order of the derivative of an entire function is equal to the order of the function. The same result is proved for a meromorphic function in \{cf.[1], [9], [11]\}. In [6] and [7] Lahiri proved that
the generalised order (generalised lower order) of a meromorphic function \( f \) is equal to the generalised order of its derivative \( f' \). Using the notion of \((p, q)\)th order \((p, q)\)th lower order\) for any two positive integers with \( p > q \) of an entire function introduced by Juneja, Kapoor and Bajpai [5] and the notion of slowly changing functions investigated by Somasundaram and Thamizharasi [8], Datta and Mondal [3] established a relationship between the \( L - (p, q) \)th order of the derivative of a meromorphic function and that of the original function where \( L \equiv L(r) \) is a positive continuous function increasing slowly i.e., \( L(ar) \sim L(r) \) as \( r \to \infty \) for every constant ‘a’ and \( p, q \) are any two positive integers with \( p > q \). In this paper we generalise the results of Datta and Mondal [3] and for this we introduce the following definition:

**Definition 1.** The generalised \( L^* - (p, q) \)th order with rate \( t \), \( (t) \rho_{L^*}^f(p, q) \) and generalised \( L^* - (p, q) \)th lower order with rate \( t \), \( (t) \lambda_{L^*}^f(p, q) \) of an entire function \( f \) are defined as

\[
(t) \rho_{L^*}^f(p, q) = \limsup_{r \to \infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q]} [r \exp^t L(r)]}
\]

and

\[
(t) \lambda_{L^*}^f(p, q) = \liminf_{r \to \infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q]} [r \exp^t L(r)]}
\]

where \( \log^{[k]} x = \log(\log^{[k-1]} x) \) for \( k = 1, 2, 3, \ldots \) and \( \log^{[0]} x = x \) and \( \exp^t x = \exp(\exp^{t-1} x) \) for \( t = 1, 2, 3, \ldots \) and \( \exp^{[0]} x = x \) and also \( p, q \) are any two positive integers with \( p > q \). When \( f \) is meromorphic, one can easily verify that

\[
(t) \rho_{L^*}^f(p, q) = \limsup_{r \to \infty} \frac{\log^{[p]} T(r, f)}{\log^{[q]} [r \exp^t L(r)]}
\]

and

\[
(t) \lambda_{L^*}^f(p, q) = \liminf_{r \to \infty} \frac{\log^{[p]} T(r, f)}{\log^{[q]} [r \exp^t L(r)]}.
\]

In the paper we do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [10] and [4].

### 2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1.** [7] Let \( f \) be a transcendental meromorphic function. Then

\[
T(r, f') \leq 2T(2r, f) + o\{T(2r, f)\}
\]

for all large values of \( r \).
Lemma 2. {Theorem 4.1, [12]; see also Lemma C, [2]} Let $f$ be a meromorphic function. Then for all large $r$,

$$T(r, f) < C \{T(2r, f') + \log r\}$$

where $C$ is a constant which is only dependent on $f(0)$.

3 Theorems.

In this section we present the main results of the paper.

Theorem 1. The generalised $L^* - (p, q)$th order with rate $t$ of a meromorphic function $f$ is equal to the generalised $L^* - (p, q)$th order of its derivative $f'$ where $p, q$ are positive integers and $p > q$ with $t = 1, 2, 3, ...$

Proof. We suppose that $f$ is a transcendental meromorphic function because otherwise the theorem follows easily.

From Lemma 1 we get by taking logarithms ($p - 1$)times

$$\log^{[p-1]} T(r, f') \leq \log^{[p-1]} T(2r, f) + O(1)$$

which gives that

$$(^t) \rho_f^{L^*}(p, q) \leq \limsup_{r \to \infty} \left\{ \frac{\log^{[p-1]} T(r, f)}{\log^{[q]} [r \exp[t] L(r)]} \cdot \lim_{r \to \infty} \frac{1}{1 - \frac{\log 2}{\log^{[2]} [r \exp[t] L(r)]}} \right\}$$

$$= \limsup_{r \to \infty} \frac{\log^{[p-1]} T(r, f)}{\log^{[q]} [r \exp[t] L(r)]} \cdot \lim_{r \to \infty} \frac{1}{1 - \frac{\log 2}{\log^{[2]} [r \exp[t] L(r)]}}$$

$$= (^t) \rho_f^{L^*}(p, q). \quad (1)$$

Since $f$ is transcendental, we have

$$\log r = o\{T(r, f)\}.$$

From Lemma 2 we obtain by taking repeated logarithms

$$\log^{[p-1]} T(r, f) + O(1) \leq \log^{[p-1]} T(2r, f')$$

which gives that

$$(^t) \rho_f^{L^*}(p, q) \leq \limsup_{r \to \infty} \frac{\log^{[p-1]} T(r, f')}{\log^{[q]} [r \exp[t] L(r)]} \cdot \lim_{r \to \infty} \frac{1}{1 - \frac{\log 2}{\log^{[2]} [r \exp[t] L(r)]}}$$
Thus the theorem follows from (1) and (2).

**Remark 1.** Theorem 1 is a generalisation of Theorem 1 [3].

**Theorem 2.** The generalised $L^*(p, q)$th lower order with rate $t$ of a meromorphic function $f$ is equal to the generalised $L^*(p, q)$th lower order of its derivative $f'$ where $p, q$ are positive integers and $p > q$ with $t = 1, 2, 3, ...$

We omit the proof of Theorem 2 as it is similar to that of Theorem 1.

**Remark 2.** Theorem 2 is a generalisation of Theorem 2 [3].

**Theorem 3.** If $f$ is a transcendental meromorphic function having a finite number of zeros with $f(0) \neq 0, \infty, f'(0) \neq 0$ and $(t)\rho_L^*(2, 1) < \infty$ then $(t)\rho_{L'}^*(p, q) = (t)\rho_L^*(p, q)$

and $(t)\lambda_{L'}^*(p, q) = (t)\lambda_L^*(p, q)$

where $p, q$ are positive integers and $p > q$ with $t = 1, 2, 3, ...$

**Proof.** From {Theorem 2.2, [4], p.40} we know that

$$m(r, \frac{f'}{f}) = O(\log r).$$

Also by Theorem {2.3, [4], p.41} we obtain in the present case,

$$\log r = o\{T(r, f)\} \text{ as } r \to \infty.$$  

So combining the two we get that

$$m(r, \frac{f'}{f}) = o\{T(r, f)\} \text{ as } r \to \infty.$$  

Since $f$ has a finite number of zeros, it is clear that

$$N(r, \frac{1}{f}) = O(\log r).$$

Hence $N(r, \frac{1}{f}) = o\{T(r, f)\} \text{ as } r \to \infty.$

Now $m(r, f') \leq m(r, \frac{f'}{f}) + m(r, f)$

i.e., $m(r, f') \leq m(r, f) + o\{T(r, f)\} \text{ as } r \to \infty.$

Also if $f$ has a pole of order $p$ at $z_0$, $f'(z)$ has a pole of order $p + 1 \leq 2p$, so that

$$N(r, f') \leq 2N(r, f) \{p.56,[4]\}.$$
Thus by addition we deduce that
\[ T(r, f') \leq m(r, f) + 2N(r, f) + o\{T(r, f)\} \]
i.e., \( T(r, f') \leq 2T(r, f) + o\{T(r, f)\} \)
i.e., \( T(r, f') \leq \{2 + o(1)\}T(r, f) \) as \( r \to \infty \). \( (3) \)

This gives that
\[ (t) \rho_{f'}^L (p, q) \leq (t) \rho_f^L (p, q). \] \( (4) \)

Again we have
\[ T(r, f) = m(r, \frac{1}{f}) + N(r, \frac{1}{f}) + O(1) \]
i.e., \( T(r, f) \leq m(r, \frac{1}{f}) + m(r, \frac{f'}{f}) + N(r, \frac{1}{f}) + O(1) \)
i.e., \( T(r, f) \leq m(r, \frac{1}{f}) + o\{T(r, f)\} \)
i.e., \( T(r, f) \leq T(r, \frac{1}{f}) + o\{T(r, f)\} \)
i.e., \( T(r, f) \leq T(r, f') + o\{T(r, f)\} \) as \( r \to \infty \)
i.e., \( \{1 + o(1)\}T(r, f) \leq T(r, f') \) as \( r \to \infty \). \( (5) \)

This gives that
\[ (t) \rho_{f'}^L (p, q) \leq (t) \rho_{f'}^L (p, q). \] \( (6) \)

Thus the first part of the theorem follows from (4) and (6).

Similarly, \( (t) \lambda_{f'}^L (p, q) = (t) \lambda_f^L (p, q). \)

This proves the theorem.

Remark 3. Theorem 3 is a generalisation of Theorem 3 [3].

Remark 4. Theorem 3 can also be proved with a lesser hypothesis
\[ N(r, \frac{1}{f}) = O(\log r) \]

than ‘having a finite number of zeros.’
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