On the Norms of Circulant Matrices with the \((k,h)\)-Fibonacci and \((k,h)\)-Lucas Numbers\(^1\)

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Abstract

In this paper, we give upper and lower bounds for the spectral norms of circulant matrices \(A_n = \text{Circ}(F_{0}^{(k,h)}, F_{1}^{(k,h)}, \ldots, F_{n-1}^{(k,h)})\) and \(B_n = \text{Circ}(L_{0}^{(k,h)}, L_{1}^{(k,h)}, \ldots, L_{n-1}^{(k,h)})\), where \(F_n^{(k,h)}\) and \(L_n^{(k,h)}\) are the \((k,h)\)-Fibonacci and \((k,h)\)-Lucas numbers, then we obtain some bounds for the spectral norms of Kronecker and Hadamard products of these matrices.

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1 Introduction and Preliminaries

For \(n > 0\), let \(k, h\) be any real numbers, and \(k > 0, h \leq -1\), then the \((k,h)\)-Fibonacci sequence \(\{F_n^{(k,h)}\}_{n \in \mathbb{N}}\) and the \((k,h)\)-Lucas sequence \(\{L_n^{(k,h)}\}_{n \in \mathbb{N}}\) are defined respectively by the following equations:

\[
F_{n+1}^{(k,h)} = kF_n^{(k,h)} - hF_{n-1}^{(k,h)}, \quad F_0^{(k,h)} = 0, \quad F_1^{(k,h)} = 1
\]

\[
L_{n+1}^{(k,h)} = kL_n^{(k,h)} - hL_{n-1}^{(k,h)}, \quad L_0^{(k,h)} = 2, \quad L_1^{(k,h)} = k
\]

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Obviously, when \( h = -1 \), these two sequences reduce to the well-known \( k \)-Fibonacci sequence \( \{F_n^{(k,-1)}\}_{n \in \mathbb{N}} \) and \( k \)-Lucas sequence \( \{L_n^{(k,-1)}\}_{n \in \mathbb{N}} \) [1-3], respectively.

Further, there have been several papers on the norms of some special matrices [4-10]. For example, Solak and Bozkurt [4] have established upper and lower bounds for the spectral norms of Cauchy-Toeplitz and Cauchy-Hankel matrices in the forms \( T_n = \left[ \frac{1}{a+(i-j)b} \right]_{i,j=1}^n \), \( H_n = \left[ \frac{1}{a+(i+j)b} \right]_{i,j=1}^n \). Solak [6,7] has defined \( A = [a_{ij}] \) and \( B = [b_{ij}] \) as \( n \times n \) circulant matrices, where \( a_{ij} \equiv F_{(\text{mod}(j-i,n))} \) and \( b_{ij} \equiv L_{(\text{mod}(j-i,n))} \), then he has given some bounds for the \( A \) and \( B \) matrices concerned with the spectral and Euclidean norms. Shen and Cen [10] have given upper and lower bounds for the spectral norms of \( r \)-circulant matrices \( A = C_r(F_0^{(k,-1)}, F_1^{(k,-1)}, \ldots, F_{n-1}^{(k,-1)}) \) and \( B = C_r(L_0^{(k,-1)}, L_1^{(k,-1)}, \ldots, L_{n-1}^{(k,-1)}) \).

In addition, they also have obtained some bounds for the spectral norms of Hadamard and Kronecker products of these matrices.

In this paper, let circulant matrices \( A_n = Circ(F_0^{(k,h)}, F_1^{(k,h)}, \ldots, F_{n-1}^{(k,h)}) \) and \( B_n = Circ(L_0^{(k,h)}, L_1^{(k,h)}, \ldots, L_{n-1}^{(k,h)}) \) be given. Afterwards, we give upper and lower bounds for the spectral norms of matrices \( A_n \) and \( B_n \). In the partial case \( h = -1 \), we obtain lower and upper bounds for the spectral norms of circulant matrices with the \( k \)-Fibonacci and \( k \)-Lucas numbers. In addition, we also obtain some bounds for the spectral norms of Hadamard and Kronecker products of these matrices.

Now we give some preliminaries related to our study. Let \( \alpha \) and \( \beta \) be the roots of the characteristic equation \( x^2 - kx + h = 0 \), where \( \alpha > \beta \), then the Binet formulas of the sequences \( \{F_n^{(k,h)}\}_{n \in \mathbb{N}} \) and \( \{L_n^{(k,h)}\}_{n \in \mathbb{N}} \) have the form

\[
F_n^{(k,h)} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n^{(k,h)} = \alpha^n + \beta^n
\]

A matrix \( C = [c_{ij}] \in M_{n,n}(\mathbb{C}) \) is called a circulant matrix if it is of the form

\[
c_{ij} = \begin{cases} 
c_{j-i}, & j \geq i \\
_{c_{n+j-i}}, & j < i
\end{cases}
\]

Obviously, the circulant matrix \( C \) is determined by its first row elements \( c_0, c_1, \ldots, c_{n-1} \), thus we denote \( C = Circ(c_0, c_1, \ldots, c_{n-1}) \).

For any \( A = [a_{ij}] \in M_{m,n}(\mathbb{C}) \). The well-known Frobenius (or Euclidean) norm of matrix \( A \) is

\[
\|A\|_F = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}
\]

and also the spectral norm of matrix \( A \) is

\[
\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(A^HA)}
\]
where $\lambda_i(A^H A)$ is eigenvalue of $A^H A$ and $A^H$ is conjugate transpose of matrix $A$. Then the following inequality holds:

$$
\frac{1}{\sqrt{n}}\|A\|_F \leq \|A\|_2 \leq \|A\|_F
$$

(1)

**Lemma 1**\textsuperscript{[11]} For any $A, B \in M_{m,n}(\mathbb{C})$, we have

$$
\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2
$$

(2)

where $A \circ B$ is the Hadamard product of $A$ and $B$.

**Lemma 2**\textsuperscript{[11]} Let $A \in M_{m,n}(\mathbb{C}), B \in M_{p,q}(\mathbb{C})$ be given, then we have

$$
\|A \otimes B\|_2 = \|A\|_2 \|B\|_2
$$

(3)

where $A \otimes B$ is the Kronecker product of $A$ and $B$.

**Lemma 3** For the sequences $\{F^{(k,h)}_n\}_{n \in \mathbb{N}}$ and $\{L^{(k,h)}_n\}_{n \in \mathbb{N}}$, then we have the following formulas

(i) $L^{(k,h)}_n = kF^{(k,h)}_n - 2hF^{(k,h)}_{n-1}$

(4)

(ii) $\sum_{i=0}^{n-1} [F^{(k,h)}_i]^2 = \begin{cases} 
\frac{1}{(1-h)^2} \left[n + \frac{(h^n-1)(h^n-2h-1)}{h^2-1}\right], & k + h = -1 \\
\frac{2 - (k^2 - 2h) + h^2 L^{(k,h)}_{2n} - 2(1-h^n)}{(k^2 - 4h)(1-h)}, & k + h \neq -1
\end{cases}$

(5)

(iii) $\sum_{i=0}^{n-1} [L^{(k,h)}_i]^2 = \begin{cases} 
\frac{n + (h^n-1)(h^n+2h+3)}{(h+1)^2 - k^2}, & k + h = -1 \\
\frac{2 - (k^2 - 2h) + h^2 L^{(k,h)}_{2n} + 2(1-h^n)}{1-h}, & k + h \neq -1
\end{cases}$

(6)

**Proof:** (i) Applying Binet formula $F^{(k,h)}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and taking into account $\alpha + \beta = k$, then we obtain

$$
kF^{(k,h)}_n - 2hF^{(k,h)}_{n-1} = 2F^{(k,h)}_{n+1} - kF^{(k,h)}_n = 2F^{(k,h)}_{n+1} - kF^{(k,h)}_n
$$

$$
= 2 \frac{\alpha - \beta}{\alpha - \beta} (\alpha^{n+1} - \beta^{n+1}) - \frac{k}{\alpha - \beta} (\alpha^n - \beta^n)
$$

$$
= \frac{1}{\alpha - \beta} [\alpha^n (2\alpha - k) - \beta^n (2\beta - k)]
$$

$$
= \alpha^n + \beta^n = L^{(k,h)}_n
$$

(ii) Taking into account $\alpha \beta = h$, then we have

$$
\sum_{i=0}^{n-1} [F^{(k,h)}_i]^2 = \sum_{i=0}^{n-1} \left(\frac{\alpha^i - \beta^i}{\alpha - \beta}\right)^2 = \frac{1}{(\alpha - \beta)^2} \left(\sum_{i=0}^{n-1} \alpha^{2i} + \sum_{i=0}^{n-1} \beta^{2i} - 2(1-h^n)\right)
$$
If $k + h = -1$, then $\alpha = -h$, $\beta = -1$, so we obtain
\[
\sum_{i=0}^{n-1} [F_i^{(k,h)}]^2 = \frac{1}{(1-h)^2} \left( n + \frac{1-h^{2n}}{1-h} - \frac{2(1-h^n)}{1-h} \right)
\]
\[
= \frac{1}{(1-h)^2} \left( n + \frac{(h^n - 1)(h^n - 2h - 1)}{h^2 - 1} \right).
\]

If $k + h \neq -1$, then $\beta \neq -1$, applying Binet formula $L_n^{(k,h)} = \alpha^n + \beta^n$, we also obtain
\[
\sum_{i=0}^{n-1} [F_i^{(k,h)}]^2 = \frac{1}{(\alpha - \beta)^2} \left( \frac{1-\alpha^{2n}}{1-\alpha^2} + \frac{1-\beta^{2n}}{1-\beta^2} - \frac{2(1-h^n)}{1-h} \right)
\]
\[
= \frac{1}{k^2 - 4h} \left( \frac{2 - (k^2 - 2h) - L_{2n}^{(k,h)} + h^2 L_{2n-2}^{(k,h)}}{(h+1)^2 - k^2} - \frac{2(1-h^n)}{1-h} \right).
\]

Similarly, we can verify formula (6). Thus, the proof is completed.

2 Main Results

**Theorem 1** Let $A_n = \text{Circ}(F_0^{(k,h)}, F_1^{(k,h)}, \ldots, F_{n-1}^{(k,h)})$ be circulant matrix, then we have
\[
\left\{ \begin{array}{ll}
\frac{1}{1-h} \sqrt{n + \frac{(h^n - 1)(h^n - 2h - 1)}{h^2 - 1}}, & k + h = -1 \\
\sqrt{\frac{2 - (k^2 - 2h) - L_{2n}^{(k,h)} + h^2 L_{2n-2}^{(k,h)}}{(h+1)^2 - k^2} - \frac{2(1-h^n)}{1-h}}, & k + h \neq -1
\end{array} \right.
\]
\[
\leq \frac{1 - F_n^{(k,h)} + hF_{n-1}^{(k,h)}}{1-k+h}.
\]

**Proof:** The matrix $A_n$ is of the form
\[
A_n = \begin{pmatrix}
F_0^{(k,h)} & F_1^{(k,h)} & F_2^{(k,h)} & \cdots & F_{n-1}^{(k,h)} \\
F_n^{(k,h)} & F_0^{(k,h)} & F_1^{(k,h)} & \cdots & F_{n-2}^{(k,h)} \\
F_{n-1}^{(k,h)} & F_{n-2}^{(k,h)} & F_0^{(k,h)} & \cdots & F_{n-3}^{(k,h)} \\
& \vdots & \vdots & \ddots & \vdots \\
F_1^{(k,h)} & F_2^{(k,h)} & F_3^{(k,h)} & \cdots & F_{n}^{(k,h)}
\end{pmatrix}
\]

From the define of Frobenius norm and formula (5), we have
\[
\|A_n\|_F^2 = n \sum_{i=0}^{n-1} [F_i^{(k,h)}]^2
\]
\[
= \left\{ \begin{array}{ll}
\frac{n}{(1-h)^2} [n + \frac{(h^n - 1)(h^n - 2h - 1)}{h^2 - 1}], & k + h = -1 \\
\frac{n}{k^2 - 4h} \left[ 2 - (k^2 - 2h) - L_{2n}^{(k,h)} + h^2 L_{2n-2}^{(k,h)} \right] - \frac{2(1-h^n)}{1-h}, & k + h \neq -1
\end{array} \right.,
\]
hence from (1), we obtain
\[
\|A_n\|_2 \geq \frac{1}{\sqrt{n}} \|A_n\|_F = \begin{cases}
\frac{1}{\sqrt{n}} \sqrt{n + \frac{(h^n-1)(h^n-2)}{h^2-1}}, & k + h = -1 \\
\frac{1}{\sqrt{n}} \sqrt{\frac{2-(k^2-2h)-L_{2n}^{(k,h)}+h^2 L_{2n-2}^{(k,h)}}{(k^2-4h)(h+1)^2-k^2)} - \frac{2(1-h^n)}{(1-h)(k^2-4h)}}, & k + h \neq -1
\end{cases}
\]

On the other hand, let \( f(x) = \sum_{i=0}^{n-1} F_i^{(k,h)} x^i \) be a scalar-valued polynomial, and \( \pi_n = Circ(0,1,0,\cdots,0) \) be an \( n \times n \) circulant matrix. then we have
\[
A_n = f(\pi_n) = \sum_{i=0}^{n-1} F_i^{(k,h)} \pi_n^i
\]
hence
\[
\|A_n\|_2 = \left\| \sum_{i=0}^{n-1} F_i^{(k,h)} \pi_n^i \right\| \leq \sum_{i=0}^{n-1} \| F_i^{(k,h)} \pi_n^i \| \leq \sum_{i=0}^{n-1} F_i^{(k,h)} \| \pi_n \|_2
\]
since \( \pi_n^H \pi_n = I_n \), where \( I_n \) is the \( n \times n \) identity matrix. hence
\[
\| \pi_n \|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(\pi_n^H \pi_n)} = 1
\]
while \( 1 - \alpha < 0 \) and \( 1 - \beta > 0 \), so we have
\[
\|A_n\|_2 \leq \sum_{i=0}^{n-1} F_i^{(k,h)} = \sum_{i=0}^{n-1} \frac{\alpha^i - \beta^i}{\alpha - \beta} = \frac{1}{\alpha - \beta} \left( \frac{1 - \alpha^n}{1 - \alpha} - \frac{1 - \beta^n}{1 - \beta} \right) = \frac{1}{\alpha - \beta} \left[ \alpha - \beta - (\alpha^n - \beta^n) + \alpha\beta(\alpha^{n-1} - \beta^{n-1}) \right] = \frac{1 - F_n^{(k,h)}}{1 - k + h}. \]
Thus, the proof is completed.

If we choose \( h = -1 \) in Theorem 1, then we have the following result, it gives lower and upper bounds for the spectral norms of circulant matrices with the \( k \)-Fibonacci numbers.

**Corollary 1** Let \( A_n = Circ(F_0^{(k,-1)}, F_1^{(k,-1)}, \cdots, F_{n-1}^{(k,-1)}) \) be circulant matrix, then we have
\[
\frac{1}{k} \sqrt{\frac{L_{2n}^{(k,-1)} - L_{2n-2}^{(k,-1)} + (-1)^n k^2}{k^2 + 4}} \leq \|A_n\|_2 \leq \frac{F_n^{(k,-1)} + F_{n-1}^{(k,-1)} - 1}{k}
\]
where \( F_n^{(k,-1)} \) and \( L_n^{(k,-1)} \) are the \( k \)-Fibonacci and \( k \)-Lucas numbers.
Theorem 2  Let $B_n = \text{Circ}(L_0^{(k,h)}, L_1^{(k,h)}, \cdots, L_{n-1}^{(k,h)})$ be circulant matrix, then we have

$$
\begin{align*}
\sqrt{n + \frac{(h^n-1)(h^n+2h+3)}{h^2+1}}, & \quad k + h = -1 \\
\sqrt{\frac{2-(k^2-2h) - L_0^{(k,h)} + h^2 L_0^{(k,h)}}{(h+1)^2-k^2} + \frac{2(1-h^n)}{1-h}}, & \quad k + h \neq -1
\end{align*}
$$

hence from (1), we obtain

$$
\|B_n\|_2 \leq 2 - k + (2h - k)F_n^{(k,h)} + h(2 - k)F_{n-1}^{(k,h)}.
$$

**Proof:** The matrix $B_n$ is of the form

$$
B_n = \begin{pmatrix}
L_0^{(k,h)} & L_1^{(k,h)} & L_2^{(k,h)} & \cdots & L_{n-1}^{(k,h)} \\
L_{n-1}^{(k,h)} & L_0^{(k,h)} & L_1^{(k,h)} & \cdots & L_{n-2}^{(k,h)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
L_1^{(k,h)} & L_2^{(k,h)} & L_3^{(k,h)} & \cdots & L_0^{(k,h)}
\end{pmatrix}
$$

then we have

$$
\|B_n\|_F^2 = n \sum_{i=0}^{n-1} |L_i^{(k,h)}|^2 = \begin{cases}
\frac{n[n + \frac{(h^n-1)(h^n+2h+3)}{h^2+1}]}{n \left[ \frac{2-(k^2-2h) - L_0^{(k,h)} + h^2 L_0^{(k,h)}}{(h+1)^2-k^2} + \frac{2(1-h^n)}{1-h} \right]}, & k + h = -1 \\
\frac{n[n + \frac{(h^n-1)(h^n+2h+3)}{h^2+1}]}{n \left[ \frac{2-(k^2-2h) - L_0^{(k,h)} + h^2 L_0^{(k,h)}}{(h+1)^2-k^2} + \frac{2(1-h^n)}{1-h} \right]}, & k + h \neq -1
\end{cases}
$$

hence from (1), we obtain

$$
\|B_n\|_2 \geq \frac{1}{\sqrt{n}}\|B_n\|_F = \begin{cases}
\frac{\sqrt{n + \frac{(h^n-1)(h^n+2h+3)}{h^2+1}}}{\sqrt{\frac{2-(k^2-2h) - L_0^{(k,h)} + h^2 L_0^{(k,h)}}{(h+1)^2-k^2} + \frac{2(1-h^n)}{1-h}}}, & k + h = -1 \\
\frac{\sqrt{n + \frac{(h^n-1)(h^n+2h+3)}{h^2+1}}}{\sqrt{\frac{2-(k^2-2h) - L_0^{(k,h)} + h^2 L_0^{(k,h)}}{(h+1)^2-k^2} + \frac{2(1-h^n)}{1-h}}}, & k + h \neq -1
\end{cases}
$$

On the other hand, let $g(x) = \sum_{i=0}^{n-1} L_i^{(k,h)} x^i$ be a scalar-valued polynomial, and $\pi_n = \text{Circ}(0, 1, 0, \cdots, 0)$ be an $n \times n$ circulant matrix. then we have

$$
B_n = g(\pi_n) = \sum_{i=0}^{n-1} L_i^{(k,h)} \pi_n^i
$$

hence

$$
\|B_n\|_2 = \|\sum_{i=0}^{n-1} L_i^{(k,h)} \pi_n^i\|_2 \leq \sum_{i=0}^{n-1} \|L_i^{(k,h)} \pi_n^i\|_2 \leq \sum_{i=0}^{n-1} L_i^{(k,h)} \|\pi_n\|_2
$$

while

$$
\|\pi_n\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(\pi_n^H \pi_n)} = 1
$$
and \(1 - \alpha < 0, \ 1 - \beta > 0\), hence from (4), we obtain
\[
\|B_n\|_2 \leq \sum_{i=0}^{n-1} L_i^{(k,h)} = \sum_{i=0}^{n-1} \alpha^i + \beta^i = \frac{1 - \alpha^n}{1 - \alpha} + \frac{1 - \beta^n}{1 - \beta}
\]
\[
= \frac{2 - (\alpha + \beta) - (\alpha^n + \beta^n) + \alpha \beta (\alpha^{n-1} + \beta^{n-1})}{1 - (\alpha + \beta) + \alpha \beta}
\]
\[
= \frac{2 - k - L_n^{(k,h)} + h L_{n-1}^{(k,h)}}{1 - k + h}
\]
\[
= \frac{2 - k + (2h - k) F_n^{(k,h)} + h(2 - k) F_{n-1}^{(k,h)}}{1 - k + h}
\].

Thus, the proof is completed.

When \(h = -1\) in Theorem 2, then we have the following result, it gives lower and upper bounds for the spectral norms of circulant matrices with the \(k\)-Lucas numbers.

**Corollary 2** Let \(B_n = \text{Circ}(L_0^{(k,-1)}, L_1^{(k,-1)}, \cdots, L_{n-1}^{(k,-1)})\) be circulant matrix, then we have
\[
\frac{1}{k} \sqrt{L_{2n}^{(k,-1)} - L_{2n-2}^{(k,-1)} + [2 - (-1)^n] k^2} \leq \|B_n\|_2 \leq \frac{(k + 2) F_n^{(k,-1)} + (2 - k) F_{n-1}^{(k,-1)} + k - 2}{k}
\]

where \(F_n^{(k,-1)}\) and \(L_n^{(k,-1)}\) are the \(k\)-Fibonacci and \(k\)-Lucas numbers.

Considering the results of Theorem 1 and Theorem 2, then we have the following important results.

**Corollary 3** Let circulant matrices \(A_n = \text{Circ}(F_0^{(k,h)}, F_1^{(k,h)}, \cdots, F_{n-1}^{(k,h)})\) and \(B_n = \text{Circ}(L_0^{(k,h)}, L_1^{(k,h)}, \cdots, L_{n-1}^{(k,h)})\) be given, then we have
\[
\|A_n \circ B_n\|_2 \leq \frac{1 - F_n^{(k,h)} + h F_{n-1}^{(k,h)} \times [2 - k + (2h - k) F_n^{(k,h)} + h(2 - k) F_{n-1}^{(k,h)}]}{(1 - k + h)^2}
\]

**Proof:** Since \(\|A_n \circ B_n\|_2 \leq \|A_n\|_2 \|B_n\|_2\), the proof is trivial by Theorems 1 and 2.

**Corollary 4** Let circulant matrices \(A_n = \text{Circ}(F_0^{(k,h)}, F_1^{(k,h)}, \cdots, F_{n-1}^{(k,h)})\) and \(B_n = \text{Circ}(L_0^{(k,h)}, L_1^{(k,h)}, \cdots, L_{n-1}^{(k,h)})\) be given, then we have
\[
\|A_n \otimes B_n\|_2 \leq \frac{1 - F_n^{(k,h)} + h F_{n-1}^{(k,h)} \times [2 - k + (2h - k) F_n^{(k,h)} + h(2 - k) F_{n-1}^{(k,h)}]}{(1 - k + h)^2}
\]

and
\[
\|A_n \otimes B_n\|_2 \geq \begin{cases} 
\frac{1}{1 - h} \sqrt{[n + (h^n - 1)(h^n - 2h - 1)]^{n} + \frac{(h^n - 1)(h^n + 2h + 3)}{h^2 - 1}}, & k + h = -1 \\
\frac{1}{\sqrt{k^2 - 4}} \sqrt{\frac{2 - (2k - 2h) - L_{n-2}^{(k,h)} + h^2 L_{n-2}^{(k,h)}}{(1 + h)(1 + h)^2 - k^2} - \frac{2(1 - h^n)^2}{(1 - h)^2}}, & k + h \neq -1
\end{cases}
\].
Proof: Since $\|A_n \otimes B_n\|_2 = \|A_n\|_2 \|B_n\|_2$, the proof is trivial by Theorems 1 and 2.

References


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