Generalized \( W \)-Distance and a Fixed Point Theorem

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Abstract

In this paper we prove a fixed point theorem by employing notion of generalized \( w \)-distance in a metric space that includes Kannan’s fixed point theorem [4].

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1 Introduction

In 1996, W. Takahashi et.al.[5] had introduced the concept of \( w \)-distance in a metric space. In this paper our attempt is to introduce the concept of generalized \( w \)-distance in a metric space where the triangular inequality of a \( w \)-distance is replaced by a more general inequality involving four points instead of three. As a result, any \( w \)-distance becomes a generalized \( w \)-distance but the converse is not true. Finally using the concept of generalized \( w \)-distance, we prove a fixed point theorem in a complete metric space. This theorem is a generalization of Kannan’s fixed point theorem [4].

2 Definitions and Examples

Definition 2.1 [5] Let \((X, d)\) be a metric space. Then a function \( p : X \times X \rightarrow [0, \infty) \) is called a \( w \)-distance on \( X \) if the following conditions are satisfied:

(i) \( p(x, z) \leq p(x, y) + p(y, z) \) for any \( x, y, z \in X \);
(ii) for any \( x \in X, p(x, .) : X \rightarrow [0, \infty) \) is lower semi continuous;
(iii) for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( p(z, x) \leq \delta \) and \( p(z, y) \leq \delta \) imply \( d(x, y) \leq \epsilon \).
Clearly every metric is a $w$-distance but the converse is not true. The following example supports our contention.

**Example 2.2** [5] Let $(X, d)$ be a metric space. A function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = c$ for every $x, y \in X$ is a $w$-distance on $X$, where $c$ is a positive real number. But $p$ is not a metric since $p(x, x) = c \neq 0$ for any $x \in X$.

**Definition 2.3** Let $(X, d)$ be a metric space. A function $p : X \times X \to [0, \infty)$ is called a generalized $w$-distance on $X$ if for all $x, z \in X$ and for all distinct points $\xi, \eta \in X$, each of them different from $x$ and $z$, the following conditions are satisfied:

(i) $p(x, z) \leq p(x, \xi) + p(\xi, \eta) + p(\eta, z)$;

(ii) for any $x \in X$, $p(x, .) : X \to [0, \infty)$ is lower semi continuous;

(iii) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

From Definition 2.3 it follows that every $w$-distance is a generalized $w$-distance. Now we consider the following example to show that a generalized $w$-distance may not be a $w$-distance.

**Example 2.4** Let $X = \{1, 2, 3, 4\}$ be a metric space with metric

$$d(x, y) = |x - y| \text{ for all } x, y \in X.$$ 

Let $p : X \times X \to [0, \infty)$ be defined by

$$p(1, 2) = p(2, 1) = 3, \quad p(1, 3) = p(3, 1) = p(2, 3) = p(3, 2) = 1,$$

$$p(1, 4) = p(4, 1) = p(2, 4) = p(4, 2) = p(3, 4) = p(4, 3) = 2$$

and $p(x, x) = 0.6$ for every $x \in X$.

Then (i) and (ii) of Definition 2.3 are obvious. To show (iii), for any $\epsilon > 0$, put $\delta = \frac{1}{2}$. Then

$$p(z, x) \leq \delta \text{ and } p(z, y) \leq \delta \text{ imply } d(x, y) \leq \epsilon.$$ 

Thus $p$ is a generalized $w$-distance on $X$ but it is not a $w$-distance on $X$ since it lacks the triangular property:

$$p(1, 2) = 3 > 1 + 1 = p(1, 3) + p(3, 2).$$
3 Main Result

In this section we prove a fixed point theorem in a complete metric space by employing notion of generalized \( w \)-distance. The following Lemma is crucial in the proof of the theorem.

**Lemma 3.1** Let \((X, d)\) be a metric space and let \( p \) be a generalized \( w \)-distance on \( X \). Let \( \{x_n\} \) and \( \{y_n\} \) be sequences in \( X \), let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be sequences in \([0, \infty)\) converging to 0, and let \( x, y, z \in X \). Then the following hold:

(i) If \( p(x_n, y) \leq \alpha_n \) and \( p(x_n, z) \leq \beta_n \) for any \( n \in \mathbb{N} \), then \( y = z \). In particular, if \( p(x, y) = 0 \) and \( p(x, z) = 0 \), then \( y = z \);

(ii) if \( p(x_n, y_n) \leq \alpha_n \) and \( p(x_n, z) \leq \beta_n \) for any \( n \in \mathbb{N} \), then \( \{y_n\} \) converges to \( z \);

(iii) if \( p(x_n, x_m) \leq \alpha_n \) for any \( n, m \in \mathbb{N} \) with \( m > n \), then \( \{x_n\} \) is a Cauchy sequence;

(iv) if \( p(y, x_n) \leq \alpha_n \) for any \( n \in \mathbb{N} \), then \( \{x_n\} \) is a Cauchy sequence.

Proof is similar to that of Lemma 1 [5] and we left it.

**Theorem 3.2** Let \((X, d)\) be a complete metric space with generalized \( w \)-distance \( p \) and \( T : X \rightarrow X \) be a mapping such that

\[
p(T^n x, T^n y) \leq a_n \left[ p(x, Tx) + p(y, Ty) \right]
\]

for all \( x, y \in X \) where \( a_n(>0) \) is independent of \( x, y \) such that \( 0 < a_1 < 1 \) and \( a_1 + a_n < 1 \) for \( n \geq 2 \). Suppose that

\[
\inf \{ p(x, y) + p(x, Tx) : x \in X \} > 0
\]

for every \( y \in X \) with \( y \neq Ty \). Then if the series \( \sum_{n=1}^{\infty} a_n \) is convergent, \( T \) has a fixed point in \( X \). Moreover, if \( v = Tv \), then \( p(v, v) = 0 \).

Proof. Let \( u \in X \) and consider the sequence \( \{u_n\} \) where \( u_n = T^n u \) for any \( n \in \mathbb{N} \). We can suppose that \( T^n u \neq T^m u \) for all distinct \( n, m \in \mathbb{N} \). In fact, if \( T^n u = T^m u \) for some \( m, n \in \mathbb{N} \), \( m \neq n \), then assuming \( m > n \), we have

\[
T^{m-n}(T^n u) = T^m u
\]

i.e., \( T^k y = y \) where \( k = m - n > 0 \) and \( y = T^m u \).

If \( k = 1 \), then \( Ty = y \) and \( y \) is a fixed point of \( T \).

Again if \( k > 1 \), then

\[
p(y, Ty) = p(T^k y, T^{k+1} y) \leq a_k \left[ p(y, Ty) + p(Ty, T^2 y) \right].
\]
Now \( p(Ty, T^2y) \leq a_1 \left[ p(y, Ty) + p(Ty, T^2y) \right] \) gives
\[
p(Ty, T^2y) \leq \frac{a_1}{1 - a_1} p(y, Ty).
\]

Hence
\[
p(y, Ty) \leq a_k \left[ 1 + \frac{a_1}{1 - a_1} \right] p(y, Ty) = \frac{a_k}{1 - a_1} p(y, Ty).
\]

As \( k > 1 \), by our supposition \( a_k < 1 - a_1 \) i.e., \( \frac{a_k}{1 - a_1} < 1 \), and so \( p(y, Ty) = 0 \). Also,
\[
p(y, y) = p(T^k y, T^k y) \leq a_k \left[ p(y, Ty) + p(y, Ty) \right] = 0
\]
which gives \( p(y, y) = 0 \).

Since \( p(y, Ty) = 0 \) and \( p(y, y) = 0 \), by using Lemma 3.1(i), we get \( Ty = y \) i.e., \( y \) is a fixed point of \( T \).

Thus in the sequel of the proof we can suppose that \( T^n u \neq T^m u \) for all distinct \( n, m \in N \).

For \( n \in N \), we have
\[
p(T^n u, T^{n+1} u) \leq a_n \left[ p(u, Tu) + p(Tu, T^2u) \right].
\]

Now
\[
p(Tu, T^2u) \leq a_1 \left[ p(u, Tu) + p(Tu, T^2u) \right].
\]
i.e., \( p(Tu, T^2u) \leq \frac{a_1}{1 - a_1} p(u, Tu). \quad (3)\)

Therefore, we obtain
\[
p(T^n u, T^{n+1} u) \leq a_n \left[ 1 + \frac{a_1}{1 - a_1} \right] p(u, Tu) = \frac{a_n}{1 - a_1} p(u, Tu). \quad (4)\)

Also,
\[
p(T^n u, T^{n+2} u) \leq a_n \left[ p(u, Tu) + p(T^2u, T^3u) \right].
\]

But
\[
p(T^2u, T^3u) \leq a_1 \left[ p(Tu, T^2u) + p(T^2u, T^3u) \right]
\]
i.e., \( p(T^2u, T^3u) \leq \frac{a_1}{1 - a_1} p(Tu, T^2u) \leq \left( \frac{a_1}{1 - a_1} \right)^2 p(u, Tu) \), by using (3).

Thus
\[
p(T^n u, T^{n+2} u) \leq a_n \left[ 1 + \left( \frac{a_1}{1 - a_1} \right)^2 \right] p(u, Tu). \quad (5)\)

Since \( \sum_{n=1}^{\infty} a_n \) is convergent, \( a_n \to 0 \) as \( n \to \infty \). So it follows from (4) and (5) that
\[
p(u_n, u_{n+m}) \to 0 \text{ as } n \to \infty \text{ for each } m = 1, 2. \quad (6)\)
Since (6), continuing in this way, we obtain
\[ p(T^n u, T^{n+3} u) \leq p(T^n u, T^{n+1} u) + p(T^{n+1} u, T^{n+2} u) + p(T^{n+2} u, T^{n+3} u), \]  
(7)

\[ p(T^n u, T^{n+5} u) \leq p(T^n u, T^{n+3} u) + p(T^{n+3} u, T^{n+4} u) + p(T^{n+4} u, T^{n+5} u) \]
\[ \leq p(T^n u, T^{n+1} u) + p(T^{n+1} u, T^{n+2} u) + p(T^{n+2} u, T^{n+3} u) \]
\[ + p(T^{n+3} u, T^{n+4} u) + p(T^{n+4} u, T^{n+5} u), \text{ using (7).} \]

Continuing in this way, we obtain
\[ p(u_n, u_{n+m}) \leq p(T^n u, T^{n+1} u) + p(T^{n+1} u, T^{n+2} u) + \ldots \]
\[ + p(T^{n+m-1} u, T^{n+m} u) \text{ if } m \text{ is odd} \]
\[ \leq \frac{p(u, Tu)}{1 - a_1} [a_n + a_{n+1} + \ldots + a_{n+m-1}], \text{ by using (4)} \]
\[ \leq p(u_n, u_{n+2}) + \frac{p(u, Tu)}{1 - a_1} \sum_{r=n}^{\infty} a_r \text{ for } m = 1, 3, 5, \ldots \]

Again,
\[ p(T^n u, T^{n+4} u) \leq p(T^n u, T^{n+2} u) + p(T^{n+2} u, T^{n+3} u) + p(T^{n+3} u, T^{n+4} u), \]  
(8)

\[ p(T^n u, T^{n+6} u) \leq p(T^n u, T^{n+4} u) + p(T^{n+4} u, T^{n+5} u) + p(T^{n+5} u, T^{n+6} u) \]
\[ \leq p(T^n u, T^{n+2} u) + p(T^{n+2} u, T^{n+3} u) + p(T^{n+3} u, T^{n+4} u) \]
\[ + p(T^{n+4} u, T^{n+5} u) + p(T^{n+5} u, T^{n+6} u), \text{ using (8).} \]

Continuing in this way, we obtain
\[ p(u_n, u_{n+m}) \leq p(T^n u, T^{n+2} u) + p(T^{n+2} u, T^{n+3} u) + \ldots \]
\[ + p(T^{n+m-1} u, T^{n+m} u) \text{ for } m = 4, 6, \ldots \]
\[ \leq p(u_n, u_{n+2}) + \frac{p(u, Tu)}{1 - a_1} [a_{n+2} + a_{n+3} + \ldots + a_{n+m-1}] \]
\[ \leq p(u_n, u_{n+2}) + \frac{p(u, Tu)}{1 - a_1} \sum_{r=n}^{\infty} a_r \text{ for } m = 4, 6, \ldots \]

Thus for all \( n, m \in \mathbb{N} \) one has
\[ p(u_n, u_{n+m}) \leq p(u_n, u_{n+2}) + \frac{p(u, Tu)}{1 - a_1} \sum_{r=n}^{\infty} a_r. \]  
(9)

Since \( \sum_{n=1}^{\infty} a_n \) is convergent, each of \( a_n, a_{n+1}, \ldots, a_{n+m-1} \) tends to zero as \( n \to \infty \) and also by (6), \( p(u_n, u_{n+2}) \to 0 \) as \( n \to \infty \), it follows from Lemma 3.1(iii)
that \( \{u_n\} \) is a Cauchy sequence in \( X \). Since \((X,d)\) is complete, \( \{u_n\} \) converges to some point \( z \in X \).

Let \( n \in N \) be fixed. Then since \( \{u_m\} \) converges to \( z \) and \( p(u_n,.) \) is lower semicontinuous, we have

\[
p(u_n, z) \leq \lim_{m \to \infty} \inf p(u_n, u_m) \\
\leq p(u_n, u_{n+2}) + \frac{p(u, Tu)}{1 - a_1} \sum_{r=n}^{\infty} a_r.
\]

Assume that \( z \neq Tz \). Then by hypothesis, one has

\[
0 < \inf \{p(x, z) + p(x, Tx) : x \in X\} \\
\leq \inf \{p(u_n, z) + p(u_n, u_{n+1}) : n \in N\} \\
\leq \inf \left\{ p(u_n, u_{n+2}) + \frac{p(u, Tu)}{1 - a_1} \sum_{r=n}^{\infty} a_r + \frac{a_n}{1 - a_1} p(u, Tu) : n \in N \right\} \\
= 0.
\]

This is a contradiction. Therefore, \( z = Tz \).

If \( v = Tv \), then

\[
p(v, v) = p(T^n v, T^n v) \leq a_n [p(v, Tv) + p(v, Tv)] = 2a_n p(v, v).
\]

Since \( \sum_{n=1}^{\infty} a_n \) is convergent, \( a_n \to 0 \) as \( n \to \infty \) and so \( p(v, v) = 0 \).

We now show that Kannan’s fixed point theorem [4] follows from our theorem.

**Corollary 3.3 (Kannan’s Fixed Point Theorem):** Let \((X,d)\) be a complete metric space and \( T : X \to X \) be a mapping such that

\[
d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]
\]

where \( 0 < \beta < \frac{1}{2} \) and for all \( x, y \in X \). Then \( T \) has a unique fixed point in \( X \).

**Proof.** The metric \( d \) is a generalized \( w \)-distance. We put \( \alpha = \frac{\beta}{1 - \beta} \). Then \( 0 < \alpha < 1 \) and

\[
d(Tx, Ty) \leq \left( \frac{\beta}{\alpha} \right) \alpha[d(x, Tx) + d(y, Ty)].
\]

Again,

\[
d(T^2x, T^2y) \leq \beta[d(Tx, T^2x) + d(Ty, T^2y)].
\]

But

\[
d(Tx, T^2x) \leq \beta[d(x, Tx) + d(Tx, T^2x)]
\]
i.e., \( d(Tx, T^2x) \leq \frac{\beta}{1-\beta} d(x, Tx) = \alpha \, d(x, Tx). \)

Similarly, \( d(Ty, T^2y) \leq \alpha \, d(y, Ty). \)

So,

\[
d(T^2x, T^2y) \leq \beta \alpha [d(x, Tx) + d(y, Ty)]
= \left( \frac{\beta}{\alpha} \right)^2 [d(x, Tx) + d(y, Ty)].
\]

Proceeding in this way at the \( n \)-th step, we obtain

\[
d(T^n x, T^n y) \leq \left( \frac{\beta}{\alpha} \right) \alpha^n [d(x, Tx) + d(y, Ty)]
= a_n [d(x, Tx) + d(y, Ty)] \text{ where } a_n = \left( \frac{\beta}{\alpha} \right) \alpha^n.
\]

Now

\[
a_1 = \left( \frac{\beta}{\alpha} \right) \alpha = \beta < \frac{1}{2}.
\]

For \( n \geq 2 \), \( 0 < a_n = \left( \frac{d}{a_n} \right) \alpha^n = \beta \alpha^{n-1} < \beta < \frac{1}{2} \).

Thus for \( n \geq 2 \), \( a_1 + a_n < \frac{1}{2} + \frac{1}{2} = 1 \).

Also,

\[
\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left( \frac{\beta}{\alpha} \right) \alpha^n = \frac{\beta}{\alpha} \sum_{n=1}^{\infty} \alpha^n
\]

which is convergent.

Assume that there exists \( y \in X \) with \( y \neq Ty \) and

\[
\inf \{ d(x, y) + d(x, Tx) : x \in X \} = 0.
\]

Then there is a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} \{ d(x_n, y) + d(x_n, Tx_n) \} = 0 \). Thus \( d(x_n, Ty) \to 0 \) and \( d(x_n, Tx_n) \to 0 \) as \( n \to \infty \); So from Lemma 3.1(ii), we have \( d(Tx_n, y) \to 0 \) as \( n \to \infty \).

On the other,

\[
d(Tx_n, Ty) \leq \beta [d(x_n, Tx_n) + d(y, Ty)]
\]

for any \( n \in N \) and hence

\[
d(y, Ty) \leq \beta \, d(y, Ty).
\]

This is a contradiction. Hence, if \( y \neq Ty \), then

\[
\inf \{ d(x, y) + d(x, Tx) : x \in X \} > 0.
\]

By Theorem 3.2, there exists \( z \in X \) such that \( z = Tz \).

Clearly, a fixed point of \( T \) is unique.
References


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