g*-Closed Sets in Topological Spaces

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Abstract

G.B.Navalagi [2] introduced a new class of set called g*-closed sets in topological space. In this paper, we study the properties of g*-closed sets.

Key words: g*-closed sets, g*-open sets, g*-continuous functions, g*-closed maps and g*-open maps.

1. Introduction

2. Preliminaries

**Definition 2.1** A subset of a topological space \((x, \tau)\) is called

(i) Generalised closed (briefly g-closed) \([1]\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).

(ii) Generalised semiclosed (briefly gs-closed) \([11]\) if \(\text{scl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).

(iii) Semi-generalised closed (briefly sg-closed) \([9]\) if \(A \subseteq U\) and \(U\) is semiopen in \(X\). Every semi closed set is sg-closed.

(iv) Weakly closed (briefly w-closed) \([10]\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open in \(X\).

(v) Weakly generalized closed (briefly wg-closed) \([7]\) if \(\text{cl}(\text{int} A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).

(vi) Generalised \(\alpha\)-closed (briefly g\(\alpha\)-closed) \([3]\) if \(\alpha\)-cl \((A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\alpha\)-open in \(X\).

(vii) \(\alpha\)-generalised closed (briefly \(\alpha\)g-closed) \([4]\) if \(\alpha\)-cl \((A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).

(viii) Regular w-closed (briefly rw-closed) \([12]\) if \(\alpha\)-cl \((A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular semiopen in \(X\).

(xi) Strongly g-closed \([1]\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is g-open in \(X\).

(Xii) \(g^*\)s-closed set \([2]\) if \(\text{scl}(A) \subseteq U\) whenever \(A \subseteq U\), \(U\) is gs-open.

The class of all \(g^*\)s-closed set \(s\) in a topological space \((x, \tau)\) is denoted by \(g^*\text{-c} (x, \tau)\).

The complements of the above mentioned closed sets are their respective open sets.

**Definition 2.2** A function \(f: X \rightarrow Y\) is called

(i) Strongly continuous if \(f^{-1}(V)\) is both open and closed in \(X\) for each subset \(V\) in \(Y\) \([5]\)

(ii) Perfectly continuous if \(f^{-1}(V)\) is both open and closed in \(X\) for each open set \(V\) in \(Y\) \([8]\)

(iii) Generalized continuous (g-continuous) if \(f^{-1}(V)\) is g-open in \(X\) for each open set \(V\) in \(Y\) \([13]\)
(iv) Strongly $g$-continuous if $f^{-1}(V)$ is open in $X$ for each $g$-open set $V$ in $Y$
(v) Semi-generalized continuous (sg-continuous) if $f^{-1}(V)$ is sg-open in $X$ for each open set $V$ in $Y$ [13]
(vi) Generalized semi-continuous (gs-continuous) if $f^{-1}(V)$ is gs-open in $X$ for each open set $V$ in $Y$

3. $g^*$s-closed sets in topological spaces

**Definition 3.1**

A subset $A$ of $X$ is called a $g^*$s-closed set if $scl(A) \subseteq U$ whenever $A \subseteq U$, $U$ is gs-open.

The class of all $g^*$s-closed sets in a topological space $(X, \tau)$ is denoted by $g^*$s-c $(X, \tau)$.

**Remark:3.2**

The complement of $g^*$s-closed set is $g^*$s-open set.

**Theorem:3.3**

Every closed set in $X$ is $g^*$s-closed in $X$ but not conversely.

**Proof:-**

Let $A$ be a closed set in $X$. Let $U$ be a gs-open set such that $A \subseteq U$. Since $A$ is closed, that is $cl(A) = A$, $cl(A) \subseteq U$. But $scl(A) \subseteq cl(A) \subseteq U$. Therefore $scl(A) \subseteq U$. Hence $A$ is $g^*$s-closed set in $X$.

The converse of the above theorem need not be true as seen from the following example.

**Example:3.4**

Consider the topological space $X = \{a, b, c\}$ with the topology $\tau = \emptyset, X, \{a\}$. The sets $\{b\}$ and $\{c\}$ are $g^*$s-closed sets but not closed.

**Theorem:3.5**

Union of two $g^*$s-closed sets is $g^*$s-closed.
Proof:

Let $A$ and $B$ be $g^*s$-closed sets in $X$. Let $U$ be a $gs$-open in $X$ such that $A \cup B \subseteq U$. Then $A \subseteq U$ and $B \subseteq U$. Since $A$ and $B$ are $g^*s$-closed sets, $\text{scl}(A) \subseteq U$ and $\text{scl}(B) \subseteq U$. Hence $\text{scl}(A \cup B) = \text{scl}(A) \cup \text{scl}(B) \subseteq U$. Therefore $A \cup B$ is $g^*s$-closed.

Theorem 3.6

Every $g^*s$-closed set is $gs$-closed but not conversely.

Proof:

Let $A$ be a $g^*s$-closed set in $X$. Let $U$ be an open set such that $A \subseteq U$. Since every open set is $gs$-open and $A$ is $g^*s$-closed, we have $\text{scl}(A) \subseteq U$. Therefore $A$ is $gs$-closed in $X$.

The converse of the above theorem need not be true as seen from the following example.

Example 3.7

Consider the topological space $X = \{a, b, c\}$ with topology $\tau = \emptyset, X, \{a\}, \{a, c\}$. Then the set $\{a, b\}$ is $gs$-closed but not $g^*s$-closed.

Theorem 3.8

Every $g^*s$-closed set set in $X$ is a $sg$-closed set in $X$ but not conversely.

Proof:

Let $A$ be a $g^*s$-closed in $X$. Let $U$ be a semi-open set in $X$ such that $A \subseteq U$. Since every semiopen set is $gs$-open and $A$ is $g^*s$-closed, we have $\text{scl}(A) \subseteq U$. Therefore $A$ is $sg$-closed in $X$.

The converse of the above theorem need not be true as seen from the following example.

Example 3.9

Consider the topological space $X = \{a, b, c\}$ with the topology $\tau = \emptyset, X, \{a, b\}$. The sets $\{a, c\}, \{b, c\}$ are $sg$-closed sets but not $g^*s$-closed sets.

From the above theorem 3.3, 3.6, 3.8 and examples 3.4, 3.7, 3.9. We get the following diagram.
The concept of $g^s$-closed set is independent of the following classes of sets namely $g$-closed set, $g^*$-closed set, $w$-closed set, pre-closed set, $g\alpha$-closed set, $\alpha g$-closed set and strongly generalized closed sets.

Example:3.11

Consider the topological space $X=\{a, b, c\}$ with topology $\tau=\emptyset, X, \{a\}, \{a, c\}$. In this space, the set $\{a, b\}$ is $g$-closed set but not $g^s$-closed set and the set $\{c\}$ is $g^s$-closed set but not $g$-closed set.

Example:3.12

Consider the topological space $X=\{a, b, c\}$ with topology $\tau=\emptyset, X, \{a\}, \{a, b\}$. In this space the set $\{a, c\}$ is $g^*$-closed set but not $g^s$-closed set and the set $\{b\}$ is $g^s$-closed set but not $g^s$-closed set.

Example:3.13

Consider the topological space $X=\{a, b, c\}$ with topology $\tau=\emptyset, X, \{c\}$. In this space $\{a\} & \{b\}$ are $g^s$-closed sets but not $w$-closed sets. In topology $\tau=\emptyset, X, \{a\}, \{b, c\}$, the sets $\{b\}, \{c\}, \{a, b\}, \{a, c\}$ are $w$-closed sets but not $g^s$-closed sets.

Example:3.14

Consider the topological space $X=\{a, b, c\}$ with topology $\tau=\emptyset, X, \{a\}, \{b\}, \{a, b\}$. In this space $\{a\}, \{b\}$ are $g^s$-closed but not pre-closed sets. In topology $\tau=\emptyset, X, \{a\}, \{b, c\}$, the sets $\{b\}, \{c\}, \{a, b\}, \{a, c\}$ are pre-closed sets but not $g^s$-closed sets.

Example:3.15

Consider the topological space $X=\{a, b, c\}$ with topology $\tau=\emptyset, X, \{a\}, \{a, c\}$. In this space, the set $\{a, b\}$ is strongly generalized closed set, $g\alpha$-closed set and $\alpha g$-closed set but not $g^s$-closed set. Also the set $\{c\}$ is $g^s$-closed but any of the sets mentioned above.
Theorem: 3.16

A subset A of X is g*s-closed set in X iff scl(A) - A contains no non empty gs-closed set in X.

Proof:

Suppose that F is a non empty gs-closed subset of scl(A) - A. Now F is scl(A) - A. Then F is scl(A) - A. Since F is gs-open set and A is g*s-closed, scl(A) - F is contained in scl(A) - A. Which is a contradiction. Therefore F = ∅.

Thus scl(A) - A contains no non empty gs-closed set.

Conversely, assume that scl(A) - A contains no non empty gs-closed set. Let A is g*s-closed set. Let A ⊆ U, U is g*s-closed. Suppose that scl(A) is not contained in U. Then scl(A)∩U is a non empty gs-closed set contained in scl(A) - A. Which is a contradiction. Therefore scl(A) ⊆ U and hence A is g*s-closed set.

Theorem: 3.17

Let (X, τ) be a compact topological space. If A is g*s-closed subset of X, then A is compact.

Proof:

Let {U_i} be a open cover of A. Since every open set is gs-open and A is g*s-closed. We get scl(A) ⊆ U_i. Since a closed subset of a compact space is compact, scl(A) is
compact. Therefore there exist a finite subover say \{U_1 \cup U_2 \cup \ldots \cup U_n\} of \{U_i\} for \text{scl}(A). So, A \subseteq \text{scl}(A) \subseteq U_1 \cup U_2 \cup \ldots \cup U_n. Therefore A is compact.

**Theorem 3.18**

Let \((X,\tau)\) be Lindelof [countably compact] and suppose that A is \(g^s\)-closed subset of X. Then A is Lindelof of [countably compact].

**Proof:**

Let \{U_i\} be a countable open cover of A. Since every open set is \(g^s\)-open \{U_i\} is a countable \(g^s\)-open cover of A. U_i is \(g^s\)-open. Then \text{scl}(A) \subseteq \bigcup U_i because A is \(g^s\)-closed. Since a closed subset of a Lindelof space is Lindelof, \text{scl}(A) is Lindelof. There \text{scl}(A) has countable subover, say \{U_1, U_2, \ldots \cup U_n\} and it follows that A \subseteq \text{scl}(A) \subseteq U_1 \cup U_2 \cup \ldots \cup U_n. Hence A is Lindelof.

**Definition 3.19**

A subset A of a topological space X is called \(g^s\)-open set if \(A^c\) is \(g^s\)-closed. The class of all \(g^s\)-closed sets is denoted as \(g^s\)-O \((X,\tau)\).

**Theorem 3.20**

If A and B are \(g^s\)-open sets in X than A \(\cap\) B also \(g^s\)-open set in X.

**Proof:**

Let A and B be two \(g^s\)-open sets in X. Then \(A^c\) and \(B^c\) are \(g^s\)-closed sets in X. By theorem 3.5, \(A^c \cup B^c\) is a \(g^s\)-closed set in X. That is \((A \cap B)^c\) is a \(g^s\)-closed set in X. Therefore \((A \cap B)^c\) is \(g^s\)-open set in X.

**Theorem 3.21**

For each \(x \in X\), either \{x\} is gs-closed or \([x]^c\) is \(g^s\)-closed in X.

**Proof:**

If \{x\} is not gs-closed, then the only gs-open set containing \([x]^c\) in X. Thus semi closure of \([x]^c\) is contained in X and hence \([x]^c\) is \(g^s\)-closed in X.
4. $g^s$-continuous functions in Topological spaces

Levine [5] introduced semi continuous functions using semi open sets. The study on the properties of semi-continuous functions is further carried out by Noiri[8], Crossely and Hildebrand and many others. Sundram [13] introduced the concept of generalized continuous functions includes the class of continuous functions and studies several properties related to it.

In this section, we introduce the concepts of $g^s$–continuous functions, $g^s$–closed maps, $g^s$–open maps, $g^s$–irresolute maps and $g^s$–homeomorphisms in Topological spaces and study their properties.

Definition : 4.1

A map $f: x \rightarrow y$ from a topological space $X$ into a topological space $y$ is called $g^s$–continuous if the inverse image of every closed set in $y$ is $g^s$–closed in $X$.

Theorem: 4.2

If a map $f: x \rightarrow y$ is continuous, then it is $g^s$–continuous but not conversely.

Proof:

Let $f: x \rightarrow y$ be continuous Let $F$ be any closed set in $Y$. The the inverse image $f^{-1}(F)$ is closed in $Y$. Since every closed set is $g^s$–closed, $f^{-1}(F)$ is $g^s$–closed in $X$. Therefore $f$ is $g^s$–continuous.

The converse need not be true as seen from the following example.

Example: 4.3

Let $X=Y=\{a,b,c\}$, $\tau=\{\emptyset, x, \{b\}\}$ and $\sigma=\{\emptyset, y, \{a, b\}\}$. Let $f: x \rightarrow y$ be the identify map.

Then $f$ is not continuous, since for the closed $\{c\}$ in $y$, $f^{-1}(\{c\})=\{c\}$ is not closed in $X$. But $f$ is $g^s$–continuous.

Theorem: 4.4

If a map $f: x \rightarrow y$ is $g^s$–continuous then it is $g^s$–continuous but not conversely.

Proof:

Let $f: x \rightarrow y$ be $g^s$–continuous. Let $F$ be any closed in $Y$ (ie) $F$ is $g^s$–closed set is $g^s$–closed in $y$, $f^{-1}(F)$ is $g^s$–closed in $X$. Therefore $f$ is $g^s$–continuous.
The converse of the above theorem need not be true in the following example.

**Example: 4.5**

Let \( X = \{a, b, c\} \), \( T = \{\emptyset, X, \{a\}\} \), \( S = \{\emptyset, y, \{b\}\} \). Let \( f: (X, T) \rightarrow (y, S) \) be the identity map.

Then \( f \) is gs-continuous but not g*s-continuous. Since \( \{a, c\} \) is closed in \( y \) but \( f^{-1}(\{a, c\}) = \{a, c\} \) is not g*s-closed in \( X \).

**Theorem: 4.6**

Let \( f: x \rightarrow y \) be a map. Then the following statements are equivalent (a) \( f \) is g*s-continuous, (b) the inverse image of each open set in \( y \) is g*s-open in \( x \).

**Proof:**

Assume that \( f: x \rightarrow y \) is g*s-continuous. Let \( G \) be open in \( y \). The \( G^c \) is closed in \( y \). Since \( f \) is g*s-continuous, \( f^{-1}(G^c) \) is g*s-closed in \( X \). But \( f^{-1}(G^c) = x - f^{-1}(G) \). Thus \( f^{-1}(G) \) is g*s-open in \( X \).

Conversely assume that the inverse image of each open set in \( Y \) is g*s-open in \( X \). Let \( F \) be any closed set in \( Y \). By assumption \( F \) is g*s-open in \( X \). But \( f^{-1}(F) = X - f^{-1}(F) \). Thus \( X - f^{-1}(F) \) is g*s-open in \( X \) and so \( f^{-1}(F) \) is g*s-closed in \( X \). Therefore \( f \) is g*s-continuous. Hence (a) & (b) are equivalent.

**Theorem: 4.7**

Let \( X \) and \( Z \) be any topological spaces and \( Y \) be a \( T_{g^{**}} \)-space and \( y \) be a \( T_{g^{**}} \)-space. Then the composition \( g \circ f : x \rightarrow z \) of the g*s-continuous map, \( f: x \rightarrow z \) and \( g: y \rightarrow z \) is also g*s-continuous.

**Proof:**

Let \( F \) be closed in \( Z \). Since \( g \) is g*s-continuous, \( g^{-1}(F) \) is g*s-closed in \( y \). But \( y \) is a \( T_{g^{**}} \)-space and so \( g^{-1}(F) \) is closed. Since \( f \) is g*s-continuous, \( f^{-1}(g^{-1}(F)) \) is g*s-closed in \( X \). But \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \). Therefore \( g \circ f \) is g*s-continuous.

We illustrate the relations between various generalizations of continuous functions in the following diagram:

[Diagram showing relations between continuity, g*s-continuity, gs-continuity]
None of the implications in the above diagram can be reversed.

5 g*s –closed maps and g*s-open maps in Topological spaces

In this section, we introduce the concepts of g*s –closed maps and g*s-open maps in Topological spaces.

**Definition : 5.1**

A map \( f: x \to y \) is called g*s –closed map if for each closed set \( F \) in \( X \), \( f(F) \) is a g*s –closed in \( y \).

**Theorem: 5.2**

If \( f: x \to y \) is a closed map then it is g*s –closed but not conversely.

**Proof:**

Since every closed set is g*s –closed, the result follows.

The converse of the above theorem need not be true as seen from the following example.

**Example: 5.3**

Consider the topological spaces \( X = y = \{a, b, c\} \) with topologies \( \tau = \{\emptyset, X, \{a\}\} \) and \( \sigma = \{\emptyset, y, \{b, c\}\} \). Hence \( ((x, \tau), (y, \sigma)) \). Let \( f \) be the identify map from \( X \) onto \( y \). Then \( f \) is gs –continuous but not a closed map. Since the closed set \( \{b, c\} \) in \( (x, \tau) \), \( f(\{b, c\}) = \{b, c\} \) is not closed set in \( Y \).

**Definition : 5.4**

A map \( f: x \to y \) is called a gs –open map if \( f(U) \) is g*s –open in \( Y \) for every open set \( U \) in \( X \).

**Theorem: 5.5**

If \( f: x \to y \) is an open map then it is g*s –open but not conversely.
Proof:

Let \( f: x \rightarrow y \) be an open map. Let \( U \) be any open set in \( X \). Then \( f(U) \) is an open set in \( Y \). Therefore \( f(U) \) is \( g^*s \) –open. Since every open set is \( g^*s \) –open.

The converse of the above theorem need not be true as seen from the following examples.

**Example:5.6**

Let \( X = y = \{a, b, c\} \) with topologies \( \tau = \{\emptyset, X, \{a, b\}\} \) and \( \sigma = \{\emptyset, y, \{b\}\} \). Here \( g^*s \) –

\( o(y, \sigma) = \{\emptyset, X, \{b, c\}, \{a, b\}, \{b\}\} \). Then the identify function \( f: x \rightarrow y \) is \( g^*s \) –open but not open.

Since for the open set \( \{a, b\} \) in \( x, f(\{a, b\}) = \{a, b\} \) is \( g^*s \) –open but not open in \( y, \sigma \).

Therefore \( f \) is not an open map.

**Theorem:5.7**

A map \( f: x \rightarrow y \) is \( g^*s \) –closed if and only if for each subset \( S \) of \( y \) and for each open set \( U \) containing \( f^{-1}(s) \) there is a \( g^*s \) –open set \( V \) of \( y \) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

**Proof:**

Suppose \( f \) is \( g^*s \) –closed. Let \( S \) be a sub set of \( Y \) and \( U \) is an open set of \( X \) such that \( f^{-1}(V) \subseteq U \).

Conversely suppose that \( F \) is a closed set in \( X \). Then \( f^{-1}(y - f(F)) = X - f \) and \( X - F \) is open. By hypothesis, there is a \( g^*s \) –open set \( V \) of \( Y \) such that \( f^{-1}(F) \subseteq U \) and \( f^{-1}(V) \subseteq X - F \). Therefore \( y - v \subseteq f(F) \subseteq f(x - y - v) \) which implies \( f(F) = y - v \). Since \( y - v \) is \( g^*s \) –

closed, \( f(F) \) is \( g^*s \) –closed and thus \( f \) is \( g^*s \) –closed map.

**Theorem:5.8**

If \( f: x \rightarrow y \) is closed and \( h: y \rightarrow z \) is \( g^*s \) –closed then \( h \circ f: x \rightarrow z \) is \( g^*s \) –closed.

**Proof:**

Let \( f: x \rightarrow y \) is a closed map and \( h: y \rightarrow z \) is a \( g^*s \) –closed map. Let \( V \) be any closed set in \( X \).

Since \( h \) is \( g^*s \) –closed, \( h(f(V)) \) is closed in \( y \) and since \( h \circ f: y \rightarrow z \) is \( g^*s \) –closed, \( h(f(V)) \) is a \( g^*s \) –closed set in \( Z \). Therefore \( h \circ f: x \rightarrow z \) is a \( g^*s \) –closed.

**Theorem:5.9**

If \( f: x \rightarrow y \) is a continuous, \( g^*s \) –closed map from a normal space \( X \) onto a space \( y \), then \( y \) is normal.
Proof:

Let $A$, $B$ b disjoint closed set of $Y$. Then $f^{-1}(A)$, $f^{-1}(B)$ are dispoint closed sets of $X$. Since $X$ is normal, then are dispoint open sets $U$, $V$ in $X$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. By theorem 2.3.7 and since $f$ is $g^*$s –closed set $G,H$ in $Y$ such that $A \subseteq G$, $B \subseteq H$,and $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq U$. Since $U,V$ are dispoint int$(G)$ and int $(H)$ are disjoint open sets. Since $G$ is $g^*$s –open $A$ is closed and $A \subseteq G \Rightarrow \subseteq$ int$(G)$. Similarly $B \subseteq$ int$(H)$. Hence $Y$ is normal.

References


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