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Hereditarily Hypercyclicity of Weighted Composition Operators

Bahmann Yousefi

Department of Mathematics, Payame-Noor University
Shahrake Golestan, P.O.Box: 71955-1368, Shiraz, Iran
b_yousefi@pnu.ac.ir

"Dedicated to Mola Ali"

Abstract

In this paper we give some sufficient conditions for the adjoint of a weighted composition operator on some function spaces to be hereditarily hypercyclic.

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1 Introduction

The first example of a hypercyclic operator on a Hilbert space was constructed by Rolewicz in 1969 ([4]). He showed that if B is the backward shift on $\ell^2(N)$, then λB is hypercyclic if and only if $|\lambda| > 1$.

The holomorphic self maps of U are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in U. It is well known that this map is conjugate to a rotation $z \to \lambda z$ for some complex number λ with $|\lambda| = 1$. The maps of that are not elliptic are called of non-elliptic type. The iterate of a non-elliptic map can be characterized by the Denjoy-Wolff Iteration Theorem ([5]).

Let H be a Hilbert space of functions analytic on a plane domain G such that for each λ in G the linear functional of evaluation at λ given by $f \longrightarrow f(\lambda)$ is a bounded linear functional on H. By the Riesz representation theorem there is a vector K_λ in H such that $f(\lambda) =< f, K_\lambda >$. We call K_λ the reproducing kernel at λ.

Let T be a bounded linear operator on H. For $x \in H$, the orbit of x under T is the set of images of x under the successive iterates of T: $orb(T, x) = \{x, Tx, T^2x, \ldots\}$. The vector x is called hypercyclic for T if $orb(T, x)$ is dense in H. Also a hypercyclic operator is one that has a hypercyclic vector.

A complex-valued function ψ on G is called a multiplier of H if $\psi H \subset H$. The operator of multiplication by ψ is denoted by M_ψ and is given by $f \longrightarrow \psi f$. By the closed graph theorem M_ψ is bounded. The collection of all multipliers is denoted by $M(H)$. Each multiplier is a bounded analytic function on G. In fact $\|\varphi\|_G \leq \|M_\varphi\|$.

If w is a multiplier of H and φ is a mapping from G into G such that $f \circ \varphi \in H$ for all $f \in H$, then C_φ (defined on H by $C_\varphi f = f \circ \varphi$) and $M_w C_\varphi$ are called composition and weighted composition operator respectively. We define the iterates $\varphi_n = \varphi \circ \varphi \circ \ldots \circ \varphi$ (n times). Note that $C_{\varphi_n} = C_\varphi^n$ for all n. In this paper we investigate the hereditarily hypercyclicity of the adjoint of a weighted composition operator acting on a Hilbert space of analytic functions. For some sources on hypercyclic topics see [1–9].

2 Main Results

A nice criterion namely the Hypercyclicity Criterion is used in the proof of our main theorem. It was developed by Kitai ([3]). This criterion has been used to show that hypercyclic operators arise within the classes of composition operators, weighted shifts, adjoints of multiplication operators, and adjoints of subnormal and hyponormal operators.

The formulation of the Hypercyclicity Criterion in the following theorem was given in [1]).

Theorem 2.1 *(The Hypercyclicity Criterion) Suppose X is a separable Banach space and T is a continuous linear mapping on X i.e. $T \in B(X)$. If there exist two dense subsets Y and Z in X and a sequence $\{n_k\}$ such that:*

1. *$T^{n_k} y \to 0$ for every $y \in Y$, and*
2. *There exist functions $S_{n_k} : Z \to X$ such that for every $z \in Z, S_{n_k} z \to 0$, and $T^{n_k} S_{n_k} z \to z$,*

then T is hypercyclic.

Definition 2.2 *Let $T \in B(X)$ and $\{n_k\}$ be an increasing sequence of non-negative integers. We say that T is hereditarily hypercyclic with respect to $\{n_k\}$ provided for all subsequences $\{n_{k_j}\}$ of $\{n_k\}$, the sequence $\{T^{n_{k_j}}\}_{j \geq 1}$ is hypercyclic. Also, an operator T will be called hereditarily hypercyclic if it is hereditarily hypercyclic with respect to some sequence $\{n_k\}$.*

Throughout this section let H be a Hilbert space of analytic functions on the open unit disc D such that H contains constants and the functional of evaluation at λ is bounded for all λ in D. Also let $w : D \to C$ be a multiplier of H and φ be an analytic univalent map from D onto D. By φ_n^{-1} we mean the nth iterate of φ^{-1}.

Theorem 2.3 *Suppose that the composition operator C_φ is bounded on H and w is a nonconstant multiplier of H such that the sets $E_m = \{\lambda \in D : \prod_{i=1}^{\infty}(w(\varphi_i^m(\lambda)))^m = 0\}$ have limit points in D for $m = -1, 1$. If for each $\lambda \in E_m$ the sequence $\{K_{\varphi_i^m(\lambda)}\}_i$ is bounded for $m = -1, 1$, then the adjoint of the weighted composition operator $M_w C_\varphi$ is hereditarily hypercyclic.*

Proof. Put $A = M_w C_\varphi$ and $\varphi_0 = I$ where I is the identity mapping on D. Then for all $n \in N$ and all λ in D we get $(A^*)^n K_\lambda = \left(\prod_{i=0}^{n-1} \overline{w(\varphi_i(\lambda))}\right) K_{\varphi_n(\lambda)}$. Put $H_{E_m} = span\{K_\lambda : \lambda \in E_m\}$ for $m = -1, 1$. The set H_{E_m} is dense in H, because: if $f \in H$ and $< f, K_\lambda >= 0$ for all λ in E_m, then $f(\lambda) = 0$ for all λ in E_m. So by using the hypothesis of the theorem, the zeros of f has limit point in D which implies that $f \equiv 0$ on D. Thus H_{E_m} is dense in H. Note that if $\lambda \in E_1$, then we have $\prod_{i=0}^{\infty} w(\varphi_i(\lambda)) = 0$ and so we have $\lim_n (A^*)^n K_\lambda = 0$. Thus $(A^*)^n \longrightarrow 0$ pointwise on H_{E_1} that is dense in H. By Theorem 2.3 in [2], it is sufficient to show that A^* satisfies the Hypercyclicity Criterion. Now to find the right inverse of A^*, first consider the special case where the collection of linear functionals of point evaluations $\{K_\lambda : \lambda \in E_{-1}\}$ is linearly independent. Note that in the following definition there is no possibility of dividing by zero. Define $B : \mathrm{H}_{E_{-1}} \longrightarrow H$ by extending the definition

$$BK_\lambda = (\overline{w(\varphi^{-1}(\lambda))})^{-1} K_{\varphi^{-1}(\lambda)} \qquad (\lambda \in E_{-1})$$

linearly to $H_{E_{-1}}$. Now clearly we get $B^n K_\lambda = (\prod_{i=1}^{n} (\overline{w(\varphi_i^{-1}(\lambda))})^{-1}).K_{\varphi_n^{-1}(\lambda)}$, where φ_i^{-1} is the ith iterate of φ^{-1} and $n \in N$. By the definition of B we have $A^* B K_\lambda = A^*((\overline{w(\varphi^{-1}(\lambda))})^{-1} K_{\varphi^{-1}(\lambda)}) = K_{\varphi(\varphi^{-1}(\lambda))} = K_\lambda$ for all λ in E_{-1}. Thus $A^* B$ is identity on the dense subset $H_{E_{-1}}$ of H. Note that if $\lambda \in E_{-1}$, then $\lim_n (\prod_{i=1}^{n} |w(\varphi_i^{-1}(\lambda))|^{-1}).K_{\varphi_n^{-1}(\lambda)} = 0$. This implies that $B^n \longrightarrow 0$ pointwise on $H_{E_{-1}}$ that is dense in H. Thus $A^* = (M_w C_\varphi)^*$ satisfies the Hypercyclicity Criterion.

In the case that linear functionals of point evaluations are not linearly independent, by the same way we can use a standard method as in Theorem 4.5 in [2] to complete the proof: consider a countable dense subset $F_1 = \{\lambda_n : n \geq 1\}$ of E_{-1} and inductively choose a subsequence $\{z_n\}$ as follows. Let $z_1 = \lambda_1$. Now define $F_2 = F_1 \backslash \{\lambda \in F_1 : K_\lambda \in span\{K_{z_1}\}\}$. Denote the first element of F_2 by z_2 and define $F_3 = F_2 \backslash \{\lambda \in F_2 : K_\lambda \in span\{K_{z_1}, K_{z_2}\}\}$. By continuing this manner, we obtain a subset $G = \{z_n\}_n$ of E_{-1} for which the set $H_G = span\{K_\lambda : \lambda \in G\}$ is dense in H with linearly independent linear functionals of point evaluations $\{K_\lambda : \lambda \in G\}$. Now for each $n \in N$, define the mappings $S_n : H_G \longrightarrow H$ by extending the definition

$$S_n K_\lambda = (\prod_{i=1}^{n} (\overline{w(\varphi_i^{-1}(\lambda))})^{-1}).K_{\varphi_n^{-1}(\lambda)} \qquad (\lambda \in G)$$

linearly to H_G. Note that if we substitute $\varphi_n^{-1}(\lambda)$ instead of λ in the formula obtained earlier for $(A^*)^n K_\lambda$, we get

$$\begin{aligned} (A^*)^n K_{\varphi_n^{-1}(\lambda)} &= \left(\prod_{i=0}^{n-1} \overline{w(\varphi_i(\varphi_n^{-1}(\lambda)))}\right) K_{\varphi_n(\varphi_n^{-1}(\lambda))} \\ &= \left(\prod_{i=0}^{n-1} \overline{w(\varphi_{n-i}^{-1}(\lambda))}\right) K_{\varphi_n o \varphi_n^{-1}(\lambda)} \\ &= \prod_{i=1}^{n} (\overline{w(\varphi_i^{-1}(\lambda))}) K_\lambda \end{aligned}$$

for all λ in G. Now by the definition of S_n we have

$$\begin{aligned} (A^*)^n S_n K_\lambda &= (A^*)^n ((\prod_{i=1}^{n} (\overline{w(\varphi_i^{-1}(\lambda))})^{-1}).K_{\varphi_n^{-1}(\lambda)}) \\ &= (\prod_{i=1}^{n} (\overline{w(\varphi_i^{-1}(\lambda))})^{-1})(A^*)^n K_{\varphi_n^{-1}(\lambda)} = K_\lambda \end{aligned}$$

for all λ in G. Thus for all $n \in N$, $(A^*)^n S_n$ is identity on the dense subset H_G of H. Also, exactly as before it is proved that $B^n \longrightarrow 0$ pointwise on $H_{E_{-1}}$, we can see that $S_n \longrightarrow 0$ pointwise on H_G that is dense in H. Thus the conditions of the Hypercyclicity Criterion are satisfied and so it is hereditarily hypercyclic. This completes the proof.

Corollary 2.4 *Let $\varphi(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi]$ and every $z \in D$. Also, let $w : D \to C$ be such that the sets $E_m = \{\lambda \in D : \prod_{i=1}^{\infty} (w(\varphi_i^m(\lambda)))^m = 0\}$ have limit points in D for $m = -1, 1$. Then the adjoint of the weighted composition operator $M_w C_\varphi$ is hereditarily hypercyclic.*

Proof. Let $\varphi(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi]$ and every $z \in D$. Also, let $w : D \to C$ be such that the sets $E_m = \{\lambda \in D : \prod_{i=1}^{\infty} (w(\varphi_i^m(\lambda)))^m = 0\}$ have limit points in D for $m = -1, 1$. Then the adjoint of the weighted composition operator $M_w C_\varphi$ is hereditarily hypercyclic.

Corollary 2.5 *Let φ be an elliptic automorphism with interior fixed point p and $w : D \to C$ satisfies the inequality: $|w(p)| < 1 < \lim_{|z| \to 1^-} \inf |w(z)|$. Then the adjoint of the weighted composition operator $M_w C_\varphi$ is hereditarily hypercyclic.*

Proof. Put $\Phi = \alpha_p \circ \varphi \circ \alpha_p$ and $W = w \circ \alpha_p$ where $\alpha_p(z) = \frac{p-z}{1-\overline{p}z}$. Since Φ is an automorphism with $\Phi(0) = 0$, thus Φ is a rotation. By substituting W and Φ instead of w and φ in Corollary 2.4, we can conclude that the operator $T = (M_W C_\Phi)^*$ is hereditarily hypercyclic. But $M_w C_\varphi = C_{\alpha_p} T^* C_{\alpha_p}^{-1}$, thus T is similar to $(M_w C_\varphi)^*$ which implies that $(M_w C_\varphi)^*$ is hereditarily hypercyclic. This completes the proof.

Corollary 2.6 *Let φ be an elliptic automorphism with interior fixed point p and $w : D \to C$ satisfies the inequality: $|w(p)| < 1 < \lim_{|z|\to 1^-} \inf |\varphi(z)|$. Then for $A = (M_w C_\varphi)^*$ we have:*
1) the operator A satisfies the Hypercyclicity Criterion.
2) the operator $A \oplus A$ is hypercyclic.

Proof. By Theorem 2.3 in [1] and Corollary 2.5, it is clear.

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