The Group Congruences on E-Inversive Semigroups

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Abstract

Let \( S \) be an \( E \)-inversive semigroup, \( \gamma \) and \( \rho \) are the arbitrary group congruence and regular congruence on \( S \), respectively, then the join \( \gamma \vee \rho \) is also a group congruence on \( S \). In particular, when \( \gamma \) is equal to the least group congruence \( \sigma \), the results are also true. As a consequence, when \( S \) is an \( E \)-inversive \( E \)-semigroup the mapping \( \rho \mapsto \sigma \vee \rho \) is a \( \cap \)-homomorphism from the partially ordered set of all regular congruences on \( S \) onto the partially ordered set of all group congruences on \( S \). Some results extend the results of Lotorre in paper[6].

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1 Introduction and preliminaries

Let \( S \) be a semigroup and \( E(S) \) is the set of all idempotents of \( S \). An element \( a \) in \( S \) is called \( E \)-inversive if there exists \( x \in S \) such that \( ax \in E(S) \). A semigroup \( S \) is called \( E \)-inversive if each element of \( S \) is \( E \)-inversive. A semigroup \( S \) is called an \( E \)-semigroup if \( E(S) \) forms a subsemigroup of \( S \).

The class of \( E \)-inversive semigroups contains both of the class of all regular semigroups and the class of all periodic semigroups. It also contains all eventually regular semigroups[1], every Bruck semigroup over a monoid[2] as well as semigroups with zero. The research strategy of \( E \)-inversive semigroups was to generalize known results for regular semigroups and for periodic semigroups to \( E \)-inversive semigroups. Mitsch[2] studied the subdirect product of \( E \)-inversive semigroups. Some basic properties of \( E \)-inversive semigroups were
given by Mitsch and Petrich[3]. Weipoltshammer[4] described some special congruences on $E$-inversive $E$-semigroups, such as the least group congruence, certain semilattice congruences, some regular congruences and a certain idempotent-separating congruence. Luo[5] investigated the regular congruences on an $E$-inversive semigroup by means of the regular congruence pair and their kernel normal systems.

The aim of this paper is to establish analogues to the results on group congruences on regular semigroups true for $E$-inversive semigroups. After some preliminaries in Section 1, the join $\gamma \vee \rho$ of a group congruence $\gamma$ and an arbitrary regular congruence $\rho$ on an $E$-inversive semigroup is described in Section 2. For such congruences, we will show $\gamma \vee \rho$ itself is a group congruence and obtain the expression for its kernel. As a consequence, when $S$ is $E$-inversive $E$-semigroup the mapping $\rho \mapsto \sigma \vee \rho$ is a $\cap$-homomorphism from the partially ordered set of all regular congruences on $S$ onto the partially ordered set of all group congruences on $S$, this extends a result of Lotorre[6] for orthodox semigroups.

We shall use the standard terminology and notation of semigroup theory as in Howie[7]. Let $S$ be a semigroup and $a \in S$, an element $x \in S$ is called a weak inverse of $a$ if $x = xax$. Denote by $W(a)$ the set of all weak inverses of $a$ in $S$, that is $W(a) = \{x \in S \mid x = xax\}$. A congruence $\rho$ on semigroup $S$ is said to be a group congruence if $S/\rho$ is a group.

Recall that a congruence $\rho$ on a semigroup $S$ is said to be regular if $S/\rho$ is a regular semigroup. We have known in [5] that a congruence $\rho$ on an eventually regular semigroup $S$ is regular if and only if $\rho$ satisfies the following property $(P)$

$$(P) \text{ for any } a \in S, \text{ there exist } a' \in W(a) \text{ such that } apad'a.$$ 

In general, the regular congruence on an $E$-inversive semigroup do not satisfy the property $(P)$. As in [5], in this paper we shall be interested in the regular congruence which possesses the property $(P)$ and we call such congruences are regular congruence again. That is, a congruence $\rho$ on an $E$-inversive semigroup $S$ is called to be regular if for any $a \in S$, there exists $a' \in W(a)$ such that $apad'a$. Clearly, in this case, $S/\rho$ is a regular semigroup.

**Lemma 1.** [5] Let $S$ be an $E$-inversive semigroup and $\rho$ be a regular congruence on $S$. If $a\rho$ is an idempotent of $S/\rho$, then an idempotent $e$ can be found in $a\rho$ such that $H_e \leq H_a$.

Let $\rho$ be a congruence on a semigroup $S$. The subset $\{a \in S \mid a\rho a^2\}$ of $S$ is called the kernel of $\rho$ and is denoted by $\ker \rho$. From Lemma 1, if $\rho$ is a regular congruence on an $E$-inversive semigroup $S$, then

$$\ker \rho = \{a \in S \mid a\rho e \text{ for some } e \in E(S)\}$$
A subset $H$ of a semigroup $S$ is full if $E(S) \subseteq H$. A subsemigroup $H$ of a semigroup $S$ is called weakly self-conjugate if for all $a \in S, a' \in W(a), x \in H, axa', a'xa \in H$. For any subset $H$ of a semigroup $S$, let

$$H_w = \{ x \in S | hx \in H \text{ for some } h \in H \}$$

which is called the closure of $H$. If $H$ is a subsemigroup of $S$, then $H \subseteq H_w$.

A subsemigroup $H$ of a semigroup $S$ is closed if $H = H_w$.

For an $E$-inversive semigroup $S$, we shall use the following notations: $C$ is the class of all full and weakly self-conjugate subsemigroups of $S$, $\overline{C}$ is the class of all closed subsemigroups of $S$ in $C$. Let $U = \cap_{H \in C} H$. Then clearly, $U$ is full and weakly self-conjugate. Note that $U$ is the smallest element in $C$.

Lemma 2. [8] If $S$ is an $E$-inversive semigroup and $H \in C$, then

$$\beta_H = \{(a, b) \in S \times S | xa = by \text{ for some } x, y \in H \}$$

is a group congruence on $S$.


(i) If $H \in C$, then $\ker \beta_H = H_w$.

(ii) If $H \in \overline{C}$, then $\ker \beta_H = H = H_w$.

(iii) If $\rho$ is a group congruence on $S$, then $\ker \rho \subseteq C$, and $\rho = \beta_{\ker \rho}$.

Lemma 4. [8] Let $S$ be an $E$-inversive semigroup. If $U$ is the smallest element in $C$, then $\beta_U$ is the least group congruence on $S$.

Lemma 5. [8] Let $S$ be an $E$-inversive semigroup and $H \in \overline{C}$. The mapping $\phi : H \mapsto \beta_H$ and $\varphi : \rho \mapsto \ker \rho$ are mutually-inverse inclusion-preserving mapping from $\overline{C}$ onto the set of all group congruences on $S$.

2 Main results

Proposition 1. Let $S$ be an $E$-inversive semigroup, for any group congruence $\gamma$, and any regular congruence $\rho$ on $S$, $\gamma \lor \rho = \gamma \circ \rho \circ \gamma$.

Proof. The result immediately follows from[9,Lemma 5.1].

Theorem 1. Let $S$ be an $E$-inversive semigroup, for any regular congruence $\rho$, and any group congruence $\gamma$ on $S$,

$$(a, b) \in \gamma \lor \rho \iff (xa, by) \in \rho \text{ for some } x, y \in \ker \gamma.$$  

Moreover, if $\sigma$ is the least group congruence on $S$,

$$(a, b) \in \sigma \lor \rho \iff (ua, bv) \in \rho \text{ for some } u, v \in U_w.$$  

Proof. We have that $\ker \sigma = \ker \rho_U = U_w$ from Lemma 4 and Lemma 3(i). So to complete this proof it is sufficient to prove that for any group congruence $\beta_H$
with $H \in \mathcal{C}$, we have $(a, b) \in \beta_H \lor \rho$ if and only if $(xa, by) \in \rho$ for some $x, y \in H$.

From Proposition 1, $\beta_H \lor \rho = \beta_H \circ \rho \circ \beta_H$. Suppose $(a, b) \in \beta_H \lor \rho$. Then for some $c, d \in S$ we have

$$(a, c) \in \beta_H, \quad (c, d) \in \rho, \quad (d, b) \in \beta_H.$$ 
Since $(a, c) \in \beta_H, (d, b) \in \beta_H$, then by Lemma 2, so $ha = ck$ for some $h, k \in H$ and $h_1d = bk_1$ for some $h_1, k_1 \in H$. Now let $x = h_1h, y = k_1k$, then $x, y \in H$ and we have

$$xa = h_1ha = h_1ck\rho_hd = bk_1k = by.$$ 

Conversely, suppose $(xa, by) \in \rho$ for some $x, y \in H$. Since $H = H_w = \ker \beta_H$, so $x, y \in \ker \beta_H$, and $x\beta_H(= y\beta_H)$ is identity in the group $S/\beta_H$, thus we have

$$(a, xa) \in \beta_H, \quad (by, b) \in \beta_H.$$ 
From $(a, xa) \in \beta_H, (xa, by) \in \rho, (by, b) \in \beta_H$ we have $(a, b) \in \beta_H \circ \rho \circ \beta_H = \beta_H \lor \rho$.

For any subset $H$ of $S$ and any congruence $\rho$ on $S$, let $H_\rho = \{x \in S|xph\text{ for some }h \in H\}$.

**Theorem 2.** Let $S$ be an $E$-inversive semigroup, for any regular congruence $\rho$, and any group congruence $\gamma$ on $S$, $\ker(\gamma \lor \rho) = [(\ker \gamma)_\rho]_w$, and is closed, weakly self-conjugate, full, subsemigroup of $S$. Moreover, $\ker(\sigma \lor \rho) = [(U_\rho)_\rho]_w$.

**Proof.** It is enough to prove that for any group congruence $\beta_H$, where $H \in \mathcal{C}$, $\ker(\beta_H \lor \rho) = (H_\rho)_w$, and $(H_\rho)_w \in \mathcal{C}$.

Let $a \in \ker(\beta_H \lor \rho)$, then $(a, e) \in \beta_H \lor \rho$ for some $e \in E(S)$. By Theorem 1 it follows that there exist $x, y \in \ker \beta_H = H$ such that $(xa, ey) \in \rho$. Since $ey \in EH \subseteq H$, we have $xa \in H_\rho$. But $x \in H \subseteq H_\rho$, so $a \in (H_\rho)_w$. Thus $\ker(\beta_H \lor \rho) \subseteq (H_\rho)_w$.

Conversely, let $a \in (H_\rho)_w$, then $xa \in H_\rho$ for some $x \in H_\rho$. Since $x \in H_\rho$, we get $xph$ for some $h \in H$, and since $xa \in H_\rho$, we get $xapk$ for some $k \in H$. From $xph$ comes $xhap\gamma$, that is, $h\rho pk$. Since $\rho$ is a regular congruence on $S$, so there exists $k' \in W(k)$ such that $k\rho pk'k$. Thus $h\rho pk'k$. So by the proof of Theorem 1, $(a, kk') \in \beta_H \lor \rho$, thus $a \in \ker(\beta_H \lor \rho)$, so $(H_\rho)_w \subseteq \ker(\beta_H \lor \rho)$. We have shown that $\ker(\beta_H \lor \rho) = (H_\rho)_w$.

Finally, note that $(H_\rho)_w$ is closed immediately follows from $((H_\rho)_w)_w = (H_\rho)_w$. So to show that $(H_\rho)_w \in \mathcal{C}$, we only need to show $H_\rho \in \mathcal{C}$. That is, $H_\rho$ is full, weakly self-conjugate subsemigroup of $S$.

Clearly, $H_\rho$ is full because of $E \subseteq H \subseteq H_\rho$.

Now we shall show that $H_\rho$ is a subsemigroup of $S$. Let $a, b \in H_\rho$, then $aph, bpk$ for some $h, k \in H$, so $abphk$. Note that $H \in \mathcal{C}$, so $hk \in H$, thus $ab \in H_\rho$. That is, $H_\rho$ is a subsemigroup of $S$. 


Next, we shall show that $H_\rho$ is weakly self-conjugate. For any $a \in S$, $a' \in W(a), x \in H_\rho$, then there exists $h \in H$ such that $x h \rho a$. Since $\rho$ is a regular congruence on $S$, so

$$axa' \rho a' h x a,$$

Note that $H$ is weakly self-conjugate, then $aha', a'xa \in H$, therefore $axa', a'xa \in H_\rho$ and $H_\rho$ is weakly self-conjugate, so $H_\rho \in \mathcal{C}$, whence $(H_\rho)_w \in \overline{\mathcal{C}}$.

**Theorem 3.** Let $S$ be an E-inversive semigroup, for any regular congruence $\rho$, and any group congruence $\gamma$ on $S$, $\gamma \lor \rho$ is a group congruence. Thus

$$(a, b) \in \gamma \lor \rho \iff xa = by \text{ for some } x, y \in \ker(\gamma \lor \rho),$$

where $\ker(\gamma \lor \rho) = [(\ker \gamma)_\rho]_w$. Moreover, if $\sigma$ is the least group congruence on $S$,

$$(a, b) \in \sigma \lor \rho \iff ua = bv \text{ for some } u, v \in [(U_w)\rho]_w.$$
hence \( \sigma \vee (\rho \cap \xi) \subseteq (\sigma \vee \rho) \cap (\sigma \vee \xi) \).

To show the inverse inclusion, let \((a, b) \in (\sigma \vee \rho) \cap (\sigma \vee \xi)\). Since \(S\) is \(E\)-inversive \(E\)-semigroup and the hypothesis \((a, b) \in \sigma \vee \xi\), then from Theorem 1 there exist \(e, f \in E(S)\) such that \((ea, bf) \in \xi\). Since \(\sigma \vee \rho\) is a group congruence, then we have that

\[
(ea, a) \in \sigma \vee \rho, \quad (b, bf) \in \sigma \vee \rho.
\]

From the hypothesis \((a, b) \in \sigma \vee \rho\) it follows that \((ea, bf) \in \sigma \vee \rho\). By Theorem 1, there exist \(e_1, f_1 \in E(S)\) such that \((e_1ea, bff_1) \in \rho\), then \((e_1^2eaf_1, e_1bffe_1^2) \in \rho\), that is,

\[
(e_1eaf_1, e_1bffe_1) \in \rho.
\]

From \((ea, bf) \in \xi\) comes \((e_1eaf_1, e_1bffe_1) \in \xi\), thus \((e_1eaf_1, e_1bffe_1) \in \rho \cap \xi\). But \(e_1e, fffe_1 \in E(S)\), so

\[
(af_1, e_1b) \in \sigma \vee (\rho \cap \xi).
\]

Since \(\sigma \vee (\rho \cap \xi)\) is a group congruence by Theorem 1, so \((a, b) \in \sigma \vee (\rho \cap \xi)\).

We have already showed that \(\sigma \vee (\rho \cap \xi) = (\sigma \vee \rho) \cap (\sigma \vee \xi)\), thus \(\phi\) is an \(\cap-\)homomorphism.

References


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