Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 3, 103 - 108

The Group Congruences on E-Inversive Semigroups

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Abstract

Let S be an E-inversive semigroup, γ and ρ are the arbitrary group congruence and regular congruence on S, respectively, then the join $\gamma \vee \rho$ is also a group congruence on S. In particular, when γ is equal to the least group congruence σ, the results are also true. As a consequence, when S is an E-inversive E-semigroup the mapping $\rho \longmapsto \sigma \vee \rho$ is a $\cap$-homomorphism from the partially ordered set of all regular congruences on S onto the partially ordered set of all group congruences on S. Some results extend the results of Lotorre in paper[6].

Mathematics Subject Classification: 20M10

Keywords: E-inversive semigroup; group congruences; the least group congruences

1 Introduction and preliminaries

Let S be a semigroup and $E(S)$ is the set of all idempotents of S. An element a in S is called E-inversive if there exists $x \in S$ such that $ax \in E(S)$. A semigroup S is called E-inversive if each element of S is E-inversive. A semigroup S is called an E-semigroup if $E(S)$ forms a subsemigroup of S.

The class of E-inversive semigroups contains both of the class of all regular semigroups and the class of all periodic semigroups. It also contains all eventually regular semigroups[1], every Bruck semigroup over a monoid[2] as well as semigroups with zero. The research strategy of E-inversive semigroups was to generalize known results for regular semigroups and for periodic semigroups to E-inversive semigroups. Mitsch[2] studied the subdirect product of E-inversive semigroups. Some basic properties of E-inversive semigroups were given by Mitsch and Petrich[3]. Weipoltshammer[4] described some special congruences on E-inversive E-semigroups, such as the least group congruence, certain semilattice congruences, some regular congruences and a certain idempotent-separating congruence. Luo[5] investigated the regular congruences on an E-inversive semigroup by means of the regular congruence pair and their kernel normal systems.

The aim of this paper is to establish analogues to the results on group congruences on regular semigroups true for E-inversive semigroups. After some preliminaries in Section 1, the join $\gamma \vee \rho$ of a group congruence γ and an arbitrary regular congruence ρ on an E-inversive semigroup is described in Section 2. For such congruences, we will show $\gamma \vee \rho$ itself is a group congruence and obtain the expression for its kernel. As a consequence, when S is E-inversive E-semigroup the mapping $\rho \longmapsto \sigma \vee \rho$ is a $\cap$-homomorphism from the partially ordered set of all regular congruences on S onto the partially ordered set of all group congruences on S, this extends a result of Lotorre[6] for orthodox semigroups.

We shall use the standard terminology and notation of semigroup theory as in Howie[7]. Let S be a semigroup and $a \in S$, an element $x \in S$ is called a weak inverse of a if $x = xax$. Denote by $W(a)$ the set of all weak inverses of a in S, that is $W(a) = \{x \in S|\ x = xax\}$. A congruence ρ on semigroup S is said to be a group congruence if S/ρ is a group.

Recall that a congruence ρ on a semigroup S is said to be regular if S/ρ is a regular semigroup. We have known in [5] that a congruence ρ on an eventually regular semigroup S is regular if and only if ρ satisfies the following property(P)

$$(P) \quad \text{for any } a \in S, \text{ there exist } \ a' \in W(a) \text{ such that } \ a\rho aa'a.$$

In general, the regular congruence on an E-inversive semigroup do not satisfy the property (P). As in [5], in this paper we shall be interested in the regular congruence which possesses the property (P) and we call such congruences are regular congruence again. That is, a congruence ρ on an E-inversive semigroup S is called to be regular if for any $a \in S$, there exists $a' \in W(a)$ such that $a\rho aa'a$. Clearly, in this case, S/ρ is a regular semigroup.

Lemma 1. [5] *Let S be an E-inversive semigroup and ρ be a regular congruence on S. If $a\rho$ is an idempotent of S/ρ, then an idempotent e can be found in $a\rho$ such that $H_e \leq H_a$.*

Let ρ be a congruence on a semigroup S. The subset $\{a \in S|a\rho a^2\}$ of S is called the kernel of ρ and is denoted by $\ker \rho$. From Lemma 1, if ρ is a regular congruence on an E-inversive semigroup S, then

$$\ker \rho = \{a \in S|a\rho e \text{ for some } e \in E(S)\}$$

A subset H of a semigroup S is full if $E(S) \subseteq H$. A subsemigroup H of a semigroup S is called weakly self-conjugate if for all $a \in S, a' \in W(a), x \in H$, $axa', a'xa \in H$. For any subset H of a semigroup S, let

$$H_w = \{x \in S | hx \in H \text{ for some } h \in H\}$$

which is called the closure of H. If H is a subsemigroup of S, then $H \subseteq H_w$. A subsemigroup H of a semigroup S is closed if $H = H_w$.

For an E-inversive semigroup S, we shall use the following notations: $\mathcal{C}$ is the class of all full and weakly self-conjugate subsemigroups of S, $\overline{\mathcal{C}}$ is the class of all closed subsemigroups of S in $\mathcal{C}$. Let $U = \cap_{H \in \mathcal{C}} H$. Then clearly, U is full and weakly self-conjugate. Note that U is the smallest element in $\mathcal{C}$.

Lemma 2. [8] *If S is an E-inversive semigroup and $H \in \mathcal{C}$, then*

$$\beta_H = \{(a, b) \in S \times S | xa = by \text{ for some } x, y \in H\}$$

is a group congruence on S.

Lemma 3. [8] *Let S be an E-inversive semigroup.*

(i) If $H \in \mathcal{C}$, then $ker\beta_H = H_w$.

(ii) If $H \in \overline{\mathcal{C}}$, then $ker\beta_H = H = H_w$.

(iii) If ρ is a group congruence on S, then $ker\rho \in \overline{\mathcal{C}} \subseteq \mathcal{C}$, and $\rho = \beta_{\ker \rho}$.

Lemma 4. [8] *Let S be an E-inversive semigroup. If U is the smallest element in $\mathcal{C}$, then β_U is the least group congruence on S.*

Lemma 5. [8] *Let S be an E-inversive semigroup and $H \in \overline{\mathcal{C}}$. The mapping $\phi : H \longmapsto \beta_H$ and $\varphi : \rho \longmapsto \ker \rho$ are mutually-inverse inclusion-preserving mapping from $\overline{\mathcal{C}}$ onto the set of all group congruences on S.*

2 Main results

Proposition 1. *Let S be an E-inversive semigroup, for any group congruence γ, and any regular congruence ρ on S, $\gamma \vee \rho = \gamma \circ \rho \circ \gamma$.*

Proof. The result immediately follows from[9,Lemma 5.1].

Theorem 1. *Let S be an E-inversive semigroup, for any regular congruence ρ, and any group congruence γ on S,*

$$(a, b) \in \gamma \vee \rho \Leftrightarrow (xa, by) \in \rho \text{ for some } x, y \in \ker \gamma.$$

Moreover, if σ is the least group congruence on S,

$$(a, b) \in \sigma \vee \rho \Leftrightarrow (ua, bv) \in \rho \text{ for some } u, v \in U_w.$$

Proof. We have that $ker\sigma = ker\rho_U = U_w$ from Lemma 4 and Lemma 3(i). So to complete this proof it is sufficient to prove that for any group congruence β_H

with $H \in \overline{\mathcal{C}}$, we have $(a,b) \in \beta_H \vee \rho$ if and only if $(xa, by) \in \rho$ for some $x, y \in H$.

From Proposition 1, $\beta_H \vee \rho = \beta_H \circ \rho \circ \beta_H$. Suppose $(a,b) \in \beta_H \vee \rho$. Then for some c, d in S we have

$$(a,c) \in \beta_H, \quad (c,d) \in \rho, \quad (d,b) \in \beta_H.$$

Since $(a,c) \in \beta_H$,$(d,b) \in \beta_H$, then by Lemma 2, so $ha = ck$ for some $h, k \in H$ and $h_1 d = bk_1$ for some h_1,$k_1 \in H$. Now let $x = h_1 h, y = k_1 k$, then $x, y \in H$ and we have

$$xa = h_1 ha = h_1 ck \rho h_1 dk = bk_1 k = by.$$

Conversely, suppose $(xa, by) \in \rho$ for some $x, y \in H$. Since $H = H_w = \ker \beta_H$, so $x, y \in \ker \beta_H$, and $x\beta_H (= y\beta_H)$ is identity in the group S/β_H, thus we have

$$(a, xa) \in \beta_H, \quad (by, b) \in \beta_H.$$

From $(a, xa) \in \beta_H, (xa, by) \in \rho, (by, b) \in \beta_H$ we have $(a,b) \in \beta_H \circ \rho \circ \beta_H = \beta_H \vee \rho$.

For any subset H of S and any congruence ρ on S, let $H_\rho = \{x \in S | x\rho h \text{ for some } h \in H\}$.

Theorem 2. *Let S be an E-inversive semigroup, for any regular congruence ρ, and any group congruence γ on S, $\ker(\gamma \vee \rho) = [(\ker\gamma)_\rho]_w$, and is closed, weakly self-conjugate, full, subsemigroup of S. Moreover, $\ker(\sigma \vee \rho) = [(U_w)_\rho]_w$.*

Proof. It is enough to prove that for any group congruence β_H, where $H \in \overline{\mathcal{C}}$, $\ker(\beta_H \vee \rho) = (H_\rho)_w$, and $(H_\rho)_w \in \overline{\mathcal{C}}$.

Let $a \in \ker(\beta_H \vee \rho)$, then $(a, e) \in \beta_H \vee \rho$ for some $e \in E(S)$. By Theorem 1 it follows that there exist $x, y \in \ker \beta_H = H$ such that $(xa, ey) \in \rho$. Since $ey \in EH \subseteq H$, we have $xa \in H_\rho$. But $x \in H \subseteq H_\rho$, so $a \in (H_\rho)_w$. Thus $\ker(\beta_H \vee \rho) \subseteq (H_\rho)_w$.

Conversely, let $a \in (H_\rho)_w$, then $xa \in H_\rho$ for some $x \in H_\rho$. Since $x \in H_\rho$, we get $x\rho h$ for some $h \in H$, and since $xa \in H_\rho$, we get $xa\rho k$ for some $k \in H$. From $x\rho h$ comes $xa\rho ha$, so $ha\rho xa\rho k$, that is, $ha\rho k$. Since ρ is a regular congruence on S, so there exists $k' \in W(k)$ such that $k\rho kk'k$. thus $ha\rho kk'k$. So by the proof of Theorem 1, $(a, kk') \in \beta_H \vee \rho$, thus $a \in \ker(\beta_H \vee \rho)$, so $(H_\rho)_w \subseteq \ker(\beta_H \vee \rho)$. We have shown that $\ker(\beta_H \vee \rho) = (H_\rho)_w$.

Finally, note that $(H_\rho)_w$ is closed immediately follows from $((H_\rho)_w)_w = (H_\rho)_w$. So to show that $(H_\rho)_w \in \overline{\mathcal{C}}$, we only need to show $H_\rho \in \mathcal{C}$. That is, H_ρ is full, weakly self-conjugate subsemigroup of S.

Clearly, H_ρ is full because of $E \subseteq H \subseteq H_\rho$.

Now we shall show that H_ρ is a subsemigroup of S. Let $a, b \in H_\rho$, then $a\rho h, b\rho k$ for some $h, k \in H$, so $ab\rho hk$. Note that $H \in \mathcal{C}$, so $hk \in H$, thus $ab \in H_\rho$. That is, H_ρ is a subsemigroup of S.

Next, we shall show that H_ρ is weakly self-conjugate. For any $a \in S, a' \in W(a), x \in H_\rho$, then there exists $h \in H$ such that $x\rho h$. Since ρ is a regular congruence on S, so

$$axa'\rho aha', \quad a'xa\rho a'ha.$$

Note that H is weakly self-conjugate, then $aha', a'ha \in H$, therefore $axa', a'xa \in H_\rho$ and H_ρ is weakly self-conjugate, so $H_\rho \in \mathcal{C}$, whence $(H_\rho)_w \in \overline{\mathcal{C}}$.

Theorem 3. *Let S be an E-inversive semigroup, for any regular congruence ρ, and any group congruence γ on S, $\gamma \vee \rho$ is a group congruence. thus*

$$(a,b) \in \gamma \vee \rho \Leftrightarrow xa = by \text{ for some } x, y \in \ker(\gamma \vee \rho),$$

where $\ker(\gamma \vee \rho) = [(\ker \gamma)_\rho]_w$. *Moreover, if σ is the least group congruence on S,*

$$(a,b) \in \sigma \vee \rho \Leftrightarrow ua = bv \text{ for some } u, v \in [(U_w)_\rho]_w.$$

Proof. That $\ker(\gamma \vee \rho) = [(\ker \gamma)_\rho]_w \in \overline{\mathcal{C}}$ follows from the Theorem 2. By Lemma 3(ii), we have

$$\ker \beta_{\ker(\gamma \vee \rho)} = \ker(\gamma \vee \rho),$$

then $\beta_{\ker(\gamma\vee\rho)} = \gamma \vee \rho$ is a group congruence by Lemma 5. From alternative characterization of a group congruence on an E-inversive semigroup [8, Proposition 2.6] and Theorem 2, we get that

$$(a,b) \in \gamma \vee \rho \Leftrightarrow xa = by \text{ for some } x, y \in \ker(\gamma \vee \rho),$$

where $\ker(\gamma \vee \rho) = [(\ker \gamma)_\rho]_w$. Moreover,

$$(a,b) \in \sigma \vee \rho \Leftrightarrow ua = bv \text{ for some } u, v \in [(U_w)_\rho]_w.$$

When S is an orthodox semigroup, Lotorre[6; Theorem 11] has shown that the mapping $\phi : \rho \longmapsto \sigma \vee \rho$ is a homomorphism from the lattice Λ of all congruences on S onto the lattice Γ of all group congruences on S. Here is the analogue result for E-inversive E-semigroup.

Theorem 4. *Let S be an E-inversive E-semigroup, Λ and Γ are the the partially ordered set of all regular congruences on S and the partially ordered set of all group congruences on S, the mapping defined by $\phi : \rho \longmapsto \sigma \vee \rho$ is a $\cap-$ homomorphism of Λ onto Γ.*

Proof. By Theorem 3, ϕ maps Λ onto Γ. We will show that for any $\rho, \xi \in \Lambda$, $(\rho \cap \xi)\phi = \rho\phi \cap \xi\phi$, that is $\sigma \vee (\rho \cap \xi) = (\sigma \vee \rho) \cap (\sigma \vee \xi)$.

Since S is E-inversive E-semigroup, so $U_w = E(S)$. Then by Theorem 1 we have

$$\begin{aligned}(a,b) \in \sigma \vee (\rho \cap \xi) &\Leftrightarrow (\exists e, f \in E(S))(ea, bf) \in \rho \cap \xi \\ &\Leftrightarrow (\exists e, f \in E(S))(ea, bf) \in \rho, (ea, bf) \in \xi \\ &\Leftrightarrow (a,b) \in \sigma \vee \rho, (a,b) \in \sigma \vee \xi,\end{aligned}$$

hence $\sigma \vee (\rho \cap \xi) \subseteq (\sigma \vee \rho) \cap (\sigma \vee \xi)$.

To show the inverse inclusion, let $(a, b) \in (\sigma \vee \rho) \cap (\sigma \vee \xi)$. Since S is E-inversive E-semigroup and the hypothesis $(a, b) \in \sigma \vee \xi$, then from Theorem 1 there exist $e, f \in E(S)$ such that $(ea, bf) \in \xi$. Since $\sigma \vee \rho$ is a group congruence, then we have that

$$(ea, a) \in \sigma \vee \rho, \quad (b, bf) \in \sigma \vee \rho.$$

From the hypothesis $(a, b) \in \sigma \vee \rho$ it follows that $(ea, bf) \in \sigma \vee \rho$. By Theorem 1, there exist $e_1, f_1 \in E(S)$ such that $(e_1ea, bff_1) \in \rho$, then $(e_1^2eaf_1, e_1bff_1^2) \in \rho$, that is,

$$(e_1eaf_1, e_1bff_1) \in \rho.$$

From $(ea, bf) \in \xi$ comes $(e_1eaf_1, e_1bff_1) \in \xi$, thus $(e_1eaf_1, e_1bff_1) \in \rho \cap \xi$. But $e_1e, ff_1 \in E(S)$, so

$$(af_1, e_1b) \in \sigma \vee (\rho \cap \xi).$$

Since $\sigma \vee (\rho \cap \xi)$ is a group congruence by Theorem 1, so $(a, b) \in \sigma \vee (\rho \cap \xi)$.

We have already showed that $\sigma \vee (\rho \cap \xi) = (\sigma \vee \rho) \cap (\sigma \vee \xi)$, thus ϕ is an $\cap-$ homomorphism.

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Received: July, 2010