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On Pairwise H-Singular Maps

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Abstract

Introducing the notion of pairwise H-singular sets between pairwise locally H-closed sets in bitopological spaces and study various properties of these spaces.

Keywords: Pairwise H-Singular Sets, Pairwise H-Singular Maps, Pairwise H-Closed sets.

1. Introduction

The study of bitopological spaces was started by J.C. kelly [8] in 1963.If x is a set $\Im_1$ and $\Im_2$ are topologies on a set x then kelly [8] called triple (X, $\Im_1$, $\Im_2$) a bitopological space. kelly initiated the study of bitopological spaces by generalizing some of the classical result of topological spaces, such as urysohn's lemma Urysohn's Metrization theorem, Tietze's extentions theorem, Bair's category theorem. There after several other authors have contributed to the development of the theory of bitopological spaces, especially keeping an eye to the problem of how for classical results on topological spaces can be generalized to the bitopological setting. Pervin [15]studied on pairwise connected bitopological space and this has been extended by I.L.Reilly[17] Reilly studied bitopological compactness and pairwise locally compact spaces he studied other bitopological properties e.g., bitopological zero dimensionality, pairwise Lindelof bitopological spaces,. Raghavan and Reilly introduced the concept of minimal pairwise Housdorff bitopological spaces,the study of which has been extended by Kariofillis[10]. There are also many authors like Balsubrimanian ,Sen and Bhattacharya [20] and others worked in bitopological spaces. Here we study generalizations of standard topological properties in bitopological spaces.

A cover U of a Bitopological space $(X,\Im_1,\Im_2)$ is called pairwise open if $U \subseteq \Im_1 \cup \Im_2$ and U contains at least one non-empty member of $\Im_1$ and one non-empty member of $\Im_2$. A Bitopological space $(X,\Im_1,\Im_2)$ is called pairwise compact if every pairwise open cover of $(X,\Im_1,\Im_2)$ has a finite subcover [14]. According to I.L. Reilly [16] a Bitopological space $(X,\Im_1,\Im_2)$ is called a pairwise locally compact if $\Im_1$ is locally compact with respect to $\Im_2$ and $\Im_2$ is locally compact with respect to $\Im_1$. Recall that $\Im_1$ is locally compact with respect to $\Im_2$ if each point of X has a $\Im_1$ open neighborhood whose $\Im_2$ – Closure is pairwise compact.

A Bitopological space $(X,\Im_1,\Im_2)$ is called pairwise Housdroff if for two distinct points x and y there is a $\Im_1$ neighborhood U of x and $\Im_2$ neighborhood V of y such that $U \cap V = \phi$ [8]

A function f: $(X,\Im_1,\Im_2) \to (Y, L_1, L_2)$ is called pairwise continuous if the induced function f:$(X, \Im_1) \to (Y, L_1)$ and f:$(X, \Im_2) \to (Y, L_2)$ are continuous [16].

The study of singular sets , singular maps and singular compactifications in the context of H-closed spaces, as is remarked in the treatise by porter and woods (1988) that in many ways compact spaces are to Tychonff spaces as H-closed spaces are to Housdorff spaces. in this spirit Kavita srivastva has introduced H-sigular sets. The concept of H-singular set of a mapping introduced by kavita [19] to study the relationship of H-closed spaces to H-closed extensions. In this spirit we introduce the concept of pairwise H-singular sets in bitopological in section 2. A brief account of properties of pairwise H-singular maps in relation to continuous maps ,product, composition is given by in section 3. The last section 4 is devoted to see the behavior of pairwise singular maps in relation to various structures like G-spaces etc.

A space X is called an *H-closed space* if for an open cover U of X, there exists a finite subfamily of U, whose union is dense in X.

The idea of the pairwise singular map between Bitopological Spaces was introduced . Recall that a pairwise continuous map

$$f: (X, \Im_1, \Im_2) \to (Y, L_1, L_2)$$

is called a *pairwise singular map* if it is a L_1 singular with respect to $\Im_2$ and L_2 singular with respect to $\Im_1$.

f is L_1 singular with respect to $\Im_2$ if for each $U \in L_1$, $\Im_2$ cl $f^{-1}(U)$ is not compact and vice-versa.

Continuing our study in this area we have introduced pairwise H – singular sets and pairwise H – singular map between pairwise locally H – closed spaces to a pairwise locally H – closed space.

By a space we mean a Bitopological space and by a map we mean a pairwise continuous map between Bitopological spaces. Letters X,Y,Z are used for Bitopological spaces and f,g,h etc are used for maps between them.

2. Pairwise H- singular maps

Definition 2.1 A space X is called a *pairwise H-colsed* space if for a pairwise open cover U of X, their exists a finite pairwise subfamily of U whose union is dense in X.

Definition 2.2 If $(X, \Im_1, \Im_2)$ is a bitopological space then $\Im_1$ is *locally H-closed* with respect to $\Im_2$ if each point of x has a $\Im_1$ open neighborhood whose $\Im_2$ closure is $\Im_{2,}$ H-closed. And $\Im_2$ is *locally H-closed* with respect to $\Im_1$ if each point of x has a $\Im_2$ open neighborhood whose $\Im_1$ closure is $\Im_{1,}$ H-closed. $(X, \Im_1, \Im_2)$ is *pairwise locally H-closed* if $\Im_1$ is locally H-closed w.r.t. $\Im_2$ and $\Im_2$ is locally H-Closed w.r.t. $\Im_{1.}$

Definition 2.3 Let f be a map form $(X,\Im_1,\Im_2)$ to (Y, L_1, L_2). where X, Y be a pairwise locally H-closed spaces.
Then $\Im_1$ is H-singular set with respect to L_2 is
$S_{HB}(f, ,\Im_{1,} L_2)$ = {yε L_2 /for every open set U of Y containing y, $\Im_1$cl$f^{-1}(U)$ is not contained in an H-closed set of X }
Then $\Im_2$ is H-singular set with respect to $L_1$is
$S_{HB}(f, ,\Im_{2,} L_1)$ = {yε L_1 /for every open set U of Y containing y, $\Im_2$cl $f^{-1}(U)$ not contained in an H-closed set of X }
Then the pairwise H-Singular set $S_{HB}(f)$ of f is the set

$$S_{HB}(f) = S_{HB}(f, ,\Im_{2,} L_1) \cap S_{HB}(f, ,\Im_{1,} L_2)$$

3. Properties of pairwise H-Singular maps

In the present section we discuss a few properties of pairwise H-Singular maps in relation to the product, G-spaces and Equivalent maps.

Theorem 3.1: - Let f: $(X, \Im_1, \Im_2) \to (Y, L_1, L_2)$ and h: $(X', \Im_1', \Im_2') \to (Y', L_1', L_2')$ be pairwise continuous maps. Then the map

f x h: $(X \times X', \Im_1 \times \Im_1', \Im_2 \times \Im_2') \to (Y \times Y', L_1 \times L_1', L_2 \times L_2')$ defined by
(f x h) (x,x') = (f(x), h(x')),

is pairwise H-Singular if and only if either f or h is pairwise H-Singular.

Proof: - Let f x h: $X \times X' \to Y \times Y'$ be pairwise H-Singular.

If f is not pairwise H-Singular then there is a point $y \in Y$ which is not a pairwise H-Singular point. Then there are three cases.

Case I: - $y \in Y$ is not a $(L_1, \Im_2)$ H-Singular point of f.
Case II: - $y \in Y$ is not a $(L_2, \Im_1)$ H-Singular point of f.
Case III: - $y \in Y$ is neither $(L_1, \Im_2)$ nor $(L_2, \Im_1)$ H-Singular point of f.

Case I:- Since $y \in Y$ is not a $(L_1, \Im_2)$ H-Singular point of f, there exist $U \in L_1$ with $y \in U$ satisfying that $\Im_2$ cl $f^{-1}(U)$ is H-Closed. Let $y' \in Y'$ and $U' \in L_1'$ with $y' \in U'$. Since $(y, y') \in Y x Y'$ is a $(L_1 \text{ x } L_1', \Im_2 \text{ x } \Im_2')$ H-Singular point of f x h, we conclude that $(\Im_2 \text{ x } \Im_2')$ cl $(f \text{ x } h)^{-1} (U \text{ x } U')$ is not H-Closed. Since

$$(\Im_2 \text{ x } \Im_2') \text{ cl } (f \text{ x } h)^{-1} (U \text{ x } U') = \Im_2 \text{ cl } f^{-1}(U) \text{ x } \Im_2' \text{ cl } h^{-1}(U')$$

It follows that $\Im_2'$ cl $h^{-1}(U')$ is not H-Closed. Thus h is a $(L_1, \Im_2)$ H-Singular map.

The proof of other cases follows similarly. This implies that h is a pairwise H-Singular map.

Conversely, assume that f: $(X,\Im_1,\Im_2) \rightarrow (Y, L_1, L_2)$ is pairwise H-Singular. To prove that the product f x h: $(X \text{ x } X', \Im_1 \text{ x } \Im_1', L_1 \text{ x } L_1') \rightarrow (Y \text{ x } Y', \Im_2 \text{ x } \Im_2', L_2 \text{ x } L_2')$ is pairwise H-Singular. To prove this we show that f x h is $(L_2 \text{ x } L_2', \Im_1 \text{ x } \Im_1')$ H-Singular and $(L_1 \text{ x } L_1', \Im_2 \text{ x } \Im_2')$ H-Singular. Take a point $(y \text{ x } y') \in L_2 \text{ x } L_2'$ in Y x Y' and an open set V of Y x Y' in $L_2 \text{ x } L_2'$, satisfying $(y , y') \in (U \text{ x } U') \subseteq V$. where U and U' are open sets of y and y'

$$(\Im_1 \text{ x } \Im_1') \text{ cl } (f \text{ x } h)^{-1} (U\text{x}U') = \Im_1 \text{ cl } f^{-1}(U) \text{ x } \Im_1' \text{ cl } h^{-1}(U')$$ is not H-Closed, since $\Im_1'$ cl $h^{-1}(U')$ is not H-closed.

$$(\Im_1 \text{ x } \Im_1') \text{ cl } (f \text{ x } h)^{-1} (U\text{x}U') \subseteq (\Im_1 \text{ x } \Im_1') \text{ cl } (f \text{ x } h)^{-1}(V)$$ and $(\Im_1 \text{ x } \Im_1')$ cl $(f \text{ x } h)^{-1}$ (UxU') is a regular H-Closed set, $(\Im_1 \text{ x } \Im_1')$ cl $(f \text{ x } h)^{-1}(V)$ is not H-Closed. Thus f x h is $(L_2 \text{ x } L_2', \Im_1 \text{ x } \Im_1')$ H-Singular. Similarly for $(L_1 \text{ x } L_1', \Im_2 \text{ x } \Im_2')$ this implies that f x h is pairwise H-Singular.

Theorem 3.2: - let f : $(X, \Im_1, \Im_2) \rightarrow (Y, L_1, L_2)$ and g: $(Y, L_1, L_2) \rightarrow (Z, L_1', L_2')$ be pairwise singular maps. Then the composition gof : $(X, \Im_1, \Im_2) \rightarrow (Z, L_1', L_2')$ is a pairwise singular map.

Proof : To show that gof is pairwise singular map, we prove the following cases :

Case I : $y \in Z$ is a $(L_1', \Im_2)$ singular point of gof.
Case II : $y \in Z$ is a $(L_2', \Im_1)$ singular point of gof.

Case I : First we show that gof is $(L_1', \Im_2)$ singular. Take $p \in Z$ and $U' \in L_1'$ with $p \in U'$.

Consider, $\Im_2 Cl\, (g \circ f)^{-1}(U') = \Im_2 Cl\, (f^{-1}(g^{-1}(U'))$.

Since g : Y $\rightarrow$Z is pairwise continuous, $g^{-1}(U') \in L_1$. Since f is pairwise singular, by definition of $(L_1',\Im_2)$ singular map $\Im_2\, f^{-1}(g^{-1}(U'))$ is not compact in X. This implies that gof : X$\rightarrow$Z is a $(L_1', \Im_2)$ singular map. Similarly, it is proved that gof : X $\rightarrow$ Z is $(L_2', \Im_1)$ singular.

4. Pairwise H-Singular Maps and Various Structures

In this section the behavior of pairwise H-singular maps is seen in relation to the G-spaces, cone and twisted roduct etc.

Definition 4.1 A pairwise G-space consists of the following:

A Bitopological space $(X, \Im_1, \Im_2)$, a bitopological group (G, L_1, L_2) and a pairwise continuous map θ: $G \times X \rightarrow X$ satisfying

- $\theta(e, x) = x$
- $\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x) \quad \forall x \in X, g_1, g_2 \in G.$

Theorem 4.2: - Let X, Y be pairwise G-spaces with G -$\Im_1, \Im_2$ compact equivariant map. Then the induced map f_G: $(X/G, \Im_1', \Im_2') \rightarrow (Y/G, L_1', L_2')$ is pairwise H - singular. The orbit spaces are pairwise locally H-Closed and the inverse image of pairwise H-Closed sets of X/G are pairwise H-Closed under the map q_X: $(X, \Im_1, \Im_2) \rightarrow (X/G, \Im_1', \Im_2')$.

Proof: Consider the following commutative diagram:

$$\begin{array}{ccc} (X, \Im_1, \Im_2) & \xrightarrow{\quad f \quad} & (Y, \mathcal{L}_1, \mathcal{L}_2) \\ q_x \downarrow & & \downarrow q_Y \\ (X/G, \Im_1', \Im_2') & \xrightarrow[\quad f_G \quad]{} & (Y/G, \mathcal{L}_1', \mathcal{L}_2') \end{array}$$

Where q_X and q_Y are pairwise orbit maps.
To show that f_G is a pairwise H-Singular, we prove the following:

Case I: - $G(y) \in Y/G$ is a $(L_1', \Im_2')$ H-Singular point of f_G.
Case II: - $G(y)$ is a $(L_2', \Im_1')$ H-Singular point of f_G.

Case I: - Suppose $G(y) \in Y/G$ is not a $(L_1', \Im_2')$ H-Singular point of f_G. Then there exists an open set U of L_1' containing G(y) such that cl $f_G^{-1}(U)$ is H-Closed. Since q_X is a $\Im_2$ H-Closed map,

$q_X^{-1}((\Im_2' \text{ cl } (f_G^{-1}(U)))$ is H-Closed.

From continuity of q_x and commutativety of the above diagram it follows that

$q_x^{-1} (\Im_2' \text{ cl } (f_G^{-1}(U))) \supseteq \Im_2 \text{ cl } q_x^{-1} (f_G^{-1}(U)) = \Im_2 \text{ cl } (f^{-1}(q_y^{-1}(U)))$.

Since $f^{-1}(q_y^{-1}(U))$ is an open set, $\Im_2$ cl $(f^{-1}(q_y^{-1}(U)))$ is a regular closed set. Hence $\Im_2$ cl $(f^{-1}(q_y^{-1}(U)))$ is $\Im_2$, H-Closed, a contradiction to the hypothesis that f is H-Singular.

The proof of case II follows similarly. This implies that f_G is a pairwise H-singular map.

Theorem 4.3: - Let X, Y be pairwise G-spaces and f: X→Y be an equivariant map in Bitopological space. If f is pairwise H-Singular then the induced map f_H:G x_H X→G x_H Y is pairwise H-Singular provided the twisted products G x_H X, G x_H Y are pairwise locally H-Closed and the inverse image of H-Closed sets of G x_H X are pairwise H-Closed under the following commutative diagram.

Proof :- Consider the following commutative diagram

$$\begin{array}{ccc} (G \times X, \Im_1, \Im_2) & \xrightarrow{I_G \times f} & (G \times Y, L_1, L_2) \\ qx \downarrow & & \downarrow qy \\ (G \times HX, \Im'_1, \Im'_2) & \xrightarrow[f_H]{} & (G \times HY, L'_1, L'_2) \end{array}$$

where q_x and q_y are pairwise orbit maps.

We have to show that f_H is pairwise H-Singular map. i.e.

Case I: - f_H is $(L_1', \Im_2')$ H-Singular.

Case II: - f_H is $(L_2', \Im_1')$ H-Singular.

Case I: - Suppose f_H is not $(L_1', \Im_2')$ H-Singular. Then there is an open set $U \in L_1'$ in G x_H Y such that $\Im_2$ cl $f_H^{-1}(U)$ is H-Closed. Since q_X is a $\Im_2'$, H-Closed map, $q_X^{-1}(\Im_2'$ cl f_H^{-1} (U)) is also H-closed. From continuity of q_X and commutatively of the above diagram,

it follows that

q_x^{-1} ($\Im_2'$ cl (f_H^{-1}(U))) $\supseteq$ $\Im_2$ cl q_x^{-1} (f_H^{-1}(U)) = $\Im_2$ cl (q_{yo} (I_G x f)$^{-1}$)(U).

Since $\Im_2$ cl (q_{yo} (I_G x f)$^{-1}$)(U) is $\Im_2$, H-Closed, a contradiction to the hypothesis that (I_G x f) is pairwise H-Singular is obtained. This implies that f_H is $(L_1', \Im_2')$ H-Singular. The proof of Case II follows similarly. Thus f_H is pairwise H-Singular map.

4.4 Definition let (X, $\Im_1$, $\Im_2$) be a bitopological space. The pariwise cone (TX, L_1, L_2) is the quotient bitopological space X × I/A, where A = X × {1}.

4.5 Theorem let f : X →Y be a pairwise H-Singular map where X and Y are pairwise locally compact spaces. If the quotient map p : X × I →TX is a $\Im_1$, $\Im_2$ compact map, then the induced map T_f : T_X →T_Y is a pairwise H-Singular map.

Proof : Consider the following commutative diagram

$$\begin{array}{ccc} & f \times I & \\ (X \times I, \Im_1, \Im_2) & \longrightarrow & (Y \times I, \mathcal{L}_1, \mathcal{L}_2) \\ p \downarrow & & \downarrow q \\ (TX, \Im_1', \Im_2') & \longrightarrow & (TY, \mathcal{L}_1', \mathcal{L}_2') \\ & T_f & \end{array}$$

Now, we have to show that T_f is pairwise H-Singular i.e.

Case I : T_f is $(L'_1, \Im'_2)$ H-Singular

Case II : T_f is $(L'_2, \Im'_1)$ H-Singular

Case I : Suppose that T_f is not $(L_1', \Im_2')$ H-Singular. Then there exists an open set $U \in L_1'$ of T_Y satisfying that $\Im_2'ClT_f^{-1}(U)$ is compact. Since p is a $\Im_2'$ compact mapping $p^{-1}(\Im_2'Cl\ T_f^{-1}(U))$ is $\Im_2$-compact. From continuity of p and commutativity of the above diagram, it follows that

$p^{-1}(\Im_2'ClT_f^{-1}(U)) \supseteq \Im_2Cl\ (p^{-1}(T_f^{-1}(U))) = \Im_2Cl\ [(f \times I)^{-1}(q^{-1}(U))]$ Thus $\Im_2Cl\ (f \times I)^{-1}q^{-1}(U)$ is compact, a contradiction to the hypothesis that f is a pairwise H-Singular map. This implies that T_f is a $(L_1', \Im_2')$ H-Singular. The proof of case (II) is similar. So T_f is a pairwise H-Singular map.

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