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New Approaches for Bg-$T_{\frac{1}{2}}$ Spaces

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Abstract

The aim of this paper is to discuss and investigate some characterizations for Bg-$T_{\frac{1}{2}}$ spaces. Also, the implication of this notion among themselves and with the well known axioms such as Bg-T_2 are introduced. Further, we give the equivalence between Bg-$T_{\frac{1}{2}}$ space and some types of mappings. Furthermore we introduce and study the definition of Bg-T_D and Bg-symmetric spaces and some of thier properties are discussed.

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1 Introduction

In 1963, Levine [2] introduced the notion of a B-open set. So, many mathematicians turned their attention to the generalizations of various concepts of topology by considering B-open sets instead of open sets. In this way, M. E. Abd EI-Monsef etal, [1] defined the concept of B-generalized closed sets of a topological space taking help of the B-open sets. In the present paper, we continue to give some characterizations for Bg-$T_{\frac{1}{2}}$ spaces. Also, we introduce the equivalence between a Bg-$T_{\frac{1}{2}}$ spaces and some types of mappings. Further we discuss and investigate the definition of Bg-symmetric, Bg-T_D-spaces and some of their properties are introduced.

2 Preliminaries

Throughout the present paper, X and Y denote topological spaces. Let $A \subseteq X$, we denote the interior and the closure of A by $int(A)$ and $cl(A)$, respectively. Levine [2], 1963 defined $\tau(B) = \{O \cup (O' \cap B) : O,\ O' \in \tau\}$ and called it simple expansion of τ by B, where $B \notin \tau$. A subset of X belonging to $\tau(B)$ is denoted by *B-open set*, the complement of B-open set is denoted by *B-closed set*. The family of all B-open sets is denoted by $BO(X)$ and the family of all B-closed sets is denoted by $BC(X)$.

Now, we recall the following definitions which we shall require later.

Definition 2.1. A subset A of a topological space $(X,\ \tau)$ is said to be *B-generalized closed set* [1] (briefly Bg-closed) if $Bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in $(X,\ \tau)$, where $Bcl(A)$ is given by $Bcl(A) = \bigcap\{S \subseteq X : A \subseteq S$ and S is a closed set in $\tau(B)\}$.

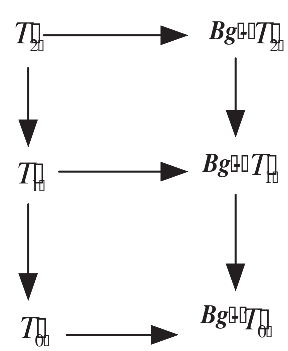
Definition 2.2. A space $(X,\ \tau)$ is said to be Bg-T_0 (resp. Bg-T_1) [3] space if for each $x,\ y \in X$ such that $x \neq y$, there exist Bg-open set containing x but not y or (resp. and) a Bg-open set containing y but not x.

3 B-generalized $T_{\frac{1}{2}}$ Space

We introduce the following definition

Definition 3.1. A space $(X,\ \tau)$ is said to be (1) a B-generalized $T_{\frac{1}{2}}$ (briefly, Bg-$T_{\frac{1}{2}}$) space if every Bg-closed set is B-closed,
(2) a Bg-T_2 space if for each $x,\ y \in X$ such that $x \neq y$, there exist Bg-open sets U_1 and U_2 such that $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 = \phi$.

The following implications hold:



Theorem 3.1. *A set V of a space X is Bg-closed if and only if $Bcl(V)\backslash V$ does not contain any non-empty B-closed set.*

Proof. Necessity: Obvious.

Sufficiency: Let $V \subseteq U$, where U is open in $(X,\ \tau)$. If $Bcl(V)$ is not contained in U, then $Bcl(V) \cap X\backslash U \neq \phi$. Now since $Bcl(V) \cap X\backslash U \subseteq Bcl(V)\backslash V$ and $Bcl(V) \cap X\backslash U$ is a non-empty B-closed set, then we obtain a contradiction and therefore V is Bg-closed. ■

Lemma 3.1. *If V is a Bg-closed set of a space X, then the following are equivalent:*
(1) V is B-closed,
(2) $Bcl(V)\backslash V$ is B-closed.

Proof. (1) $\Rightarrow$ (2) If V is a Bg-closed set which is also B-closed, then by Theorem 3.1., $Bcl(V)\backslash V = \phi$ which is B-closed.
(2) $\Rightarrow$ (1) Let $Bcl(V)\backslash V$ be a B-closed set and V be Bg-closed. Then by Theorem 3.1., $Bcl(V)\backslash V$ does not contain any non-empty B-closed subset, since $Bcl(V)\backslash V$ is B-closed and $Bcl(V)\backslash V = \phi$. This shows that V is B-closed. ■

Theorem 3.2. *For a space $(X,\ \tau)$, the following are equivalent:*
(1) $(X,\ \tau)$ is Bg-$T_{\frac{1}{2}}$-space,
(2) For each singleton set $\{x\}$ of X, $\{x\}$ is B-open or $\{x\}$ is B-closed.

Theorem 3.3. *If X is a space, then the following statements are equivalent*
(1) $(X,\ \tau)$ is Bg-$T_{\frac{1}{2}}$-space,
(2) Every subset of X is the intersection of all B-open sets and all B-closed sets containing it.

Proof. (1) $\Rightarrow$ (2) If X is a Bg-$T_{\frac{1}{2}}$-space with $B \subseteq X$, then $B = \cap\{X\backslash\{x\} : x \notin B\}$ is the intersection of B-open and B-closed sets containing it.

(2) $\Rightarrow$ (1) For each $x \in X$, then $X\backslash\{x\}$ is the intersection of all B-open and B-closed sets containing it, hence $X\backslash\{x\}$ is either B-open or B-closed. Therefore by Theorem 3.2., $(X,\ \tau)$ is Bg-$T_{\frac{1}{2}}$. ■

Lemma 3.2. *For a space $(X,\ \tau)$, the following equivalent:*
(1) every subset of X is Bg-closed,
(2) $BO(X,\ \tau) = BC(X,\ \tau)$.

Proof. (1) $\Rightarrow$ (2) Let $U \in BO(X,\ \tau)$. Then by hypothesis, U is Bg-closed which implies that $Bcl(U) \subseteq U$, so $Bcl(U) = U$, therefore $U \in BC(X,\ \tau)$. Also let $V \in BC(X,\ \tau)$. Then $X \backslash V \in BO(X,\ \tau)$, hence by hypothesis $X \backslash V$ is Bg-closed and then $X \backslash V \in BC(X,\ \tau)$, thus $V \in BO(X,\ \tau)$, according above we have, $BO(X,\ \tau) = BC(X,\ \tau)$.

(2) $\Rightarrow$ (1) If V is a subset of a space X such that $V \subseteq U$, where $U \in \tau$, since every B-open set is B-closed, and also, every open set is B-open which implies that $Bcl(V) \subseteq U$, which shows that V is Bg-closed. ■

Proposition 3.1. *The property of being a Bg-$T_{\frac{1}{2}}$-space is hereditary.*

Proof. If Y is subspace of a Bg-$T_{\frac{1}{2}}$ space X and $y \in Y \subseteq X$, then $\{y\}$ is B-open or B-closed in $(X,\ \tau)$ (by Theorem 3.2.). Therefore $\{y\}$ is either B-open or B-closed in Y. Hence Y is a Bg-$T_{\frac{1}{2}}$ space. ■

Theorem 3.4. *If $(X,\ \tau)$ is a topological space, then the following statements are hold:*
(i) every Bg-T_1 space is Bg-$T_{\frac{1}{2}}$,
(ii) every Bg-$T_{\frac{1}{2}}$-space is Bg-T_0.

Proof. Proof is obvious. ■

According the above Theorem, the following implication hold:
Bg-T_1-space $\rightarrow$ Bg-$T_{\frac{1}{2}}$-space $\rightarrow$ Bg-T_0-space.
but the converse is not true as shown in the following example.

Example 3.1 *If $X = \{a,\ b,\ c\}$ with $\tau = \{X,\ \phi,\ \{a\},\ \{b\},\ \{a,\ b\}\}$, a non open set $B = \{a,\ c\}$ and $\tau(B) = \{X,\ \phi,\ \{a\},\ \{b\},\ \{a,\ b\},\ \{a,\ c\}\}$ then $(X,\ \tau)$ is a Bg-$T_{\frac{1}{2}}$ but it is not Bg-T_1.*

Example 3.2 *Let $X = \{a,\ b,\ c\}$ with $\tau = \{X,\ \phi\}$, a non open set $B = \{a\}$ and $\tau(B) = \{X,\ \phi,\ \{a\}\}$ then $(X,\ \tau)$ is a Bg-T_0 space, but it is not Bg-$T_{\frac{1}{2}}$.*

4 Characterization of Bg-$T_{\frac{1}{2}}$

In this section, we give the definition of approximately B-irresolute and approximately B-closed mappings. Also, we introduce some characterizations of Bg-$T_{\frac{1}{2}}$-spaces on these mappings which are mentioned above.

Definition 4.1. A mapping $f : (X,\ \tau) \to (Y,\ \sigma)$ is called:
(1) approximately B-irresolute (briefly *ap*-B-irresolute), if $Bcl(V) \subseteq f^{-1}(H)$ whenever H is B-open subset of $(Y,\ \sigma)$, V is Bg-closed subset of $(X,\ \tau)$ and $V \subseteq f^{-1}(H)$,
(2) approximately B-closed (briefly *ap*-B-closed), if $f(A) \subseteq Bint(H)$, whenever H is a Bg-open subset of $(Y,\ \sigma)$, A is B-closed subset of $(X,\ \tau)$ and $f(A) \subseteq H$.

Example 4.1 *Let $X = \{a,\ b,\ c\}$, $Y = \{p,\ q,\ r\}$ with $\tau = \{X,\ \phi,\ \{a\},\ \{b\},\ \{a,\ b\}\}$, a non open set $B = \{b,\ c\}$ and $\sigma = \{Y,\ \phi,\ \{p\},\ \{r\},\ \{p,\ r\}\}$, a non open set $B = \{p,\ q\}$. A mapping $f : (X,\ \tau) \to (Y,\ \sigma)$ is defined by $f(a) = q$, $f(b) = r$, $f(c) = p$, then f is ap-B-irresolute and ap-B-closed.*

Theorem 4.1. *If $(X,\ \tau)$ is a Bg-$T_{\frac{1}{2}}$-space and $f : (X,\ \tau) \to (Y,\ \sigma)$ is surjective B-irresolute and M-B-open mapping, then $(Y,\ \sigma)$ is Bg-$T_{\frac{1}{2}}$.*

Proof. Let $A \subseteq Y$ be a Bg-closed set. Then we will prove that $f^{-1}(A)$ is Bg-closed in $(X,\ \tau)$. If $f^{-1}(A) \subseteq H$, where H is open in $(X,\ \tau)$.

Now,
$$\begin{aligned} f(Bcl(f^{-1}(A)) \cap X\backslash H)) &\subseteq f(Bcl(f^{-1}(A))) \cap f(X\backslash H) \\ &\subseteq f(Bcl(f^{-1}(A))) \cap X\backslash A \\ &\subseteq Bcl(f(f^{-1}(A)) \cap X\backslash A \\ &\subseteq Bcl(A) \cap X\backslash A \end{aligned}$$
$f(Bcl(f^{-1}(A)) \cap X\backslash A) = \phi$, hence $Bcl(f^{-1}(A)) \cap X\backslash H = \phi$. Then $Bcl(f^{-1}(A)) \subseteq H$. Therefore, $f^{-1}(A)$ is Bg-closed in $(X,\ \tau)$, where $(X,\ \tau)$ is Bg-$T_{\frac{1}{2}}$, then $f^{-1}(A)$ is B-closed in $(X,\ \tau)$. Hence $A = f(f^{-1}(A))$ is B-closed in $(Y,\ \sigma)$, then $(Y,\ \sigma)$ is Bg-$T_{\frac{1}{2}}$-space. ■

Theorem 4.2. *For a topological space $(X,\ \tau)$, the following are equivalent*
(1) $(X,\ \tau)$ is Bg-$T_{\frac{1}{2}}$-space,
(2) there exist a mapping $f : X \to Y$ such that f is ap-B-irresolute.

Proof. (1) $\Rightarrow$ (2) Let V be Bg-closed subset of $(X,\ \tau)$ and assume that there exists a mapping $f : X \to Y$ such that $V \subseteq f^{-1}(H)$, where H is B-open set of $(Y,\ \sigma)$. Since $(X,\ \tau)$ is a Bg-$T_{\frac{1}{2}}$-space, then V is B-closed, hence $V = Bcl(V)$. Therefore $Bcl(V) \subseteq f^{-1}(H)$. Then f is *ap*-B-irresolute.

(2) $\Rightarrow$ (1) Let A be a Bg-closed subset of $(X,\ \tau)$ and let Y be the set X with the topology $\sigma = \{\phi,\ A,\ Y\}$. Also, let $f : (X,\ \tau) \to (Y,\ \sigma)$ be the identity map. Since A is Bg-closed in $(X,\ \tau)$, B-open in $(X,\ \tau)$, and $A \subseteq f^{-1}(A)$. Since f is *ap*-B-irresolute, then $Bcl(A) \subseteq f^{-1}(A) = A$, therefore A is B-closed in $(X,\ \tau)$ and hence $(X,\ \tau)$ is a Bg-$T_{\frac{1}{2}}$-space. ■

Theorem 4.3. *For a topological space (Y, σ), the following are equivalent:*
(1) (Y, σ) is Bg-$T_{\frac{1}{2}}$-space,
(2) there exist a function $f : X \to Y$ such that f is ap-B-closed.

Proof. Analogus to Theorem 4.2., making the obvious change. ■

5 Bg-symmetric Space

In this section we introduce the definition of Bg-symmetric and a Bg-T_D spaces. Also, some of their properties are discussed.

Definition 5.1. A topological space (X, τ) is called a Bg-symmetric space, if for each x and y in X, $x \in Bcl(\{y\})$ implies that $y \in Bcl(\{x\})$.

Example 5.1 *Let $X = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a, b\}\}$, a non open set $B = \{c, d\}$, then (X, τ) is a Bg-symmetric space.*

Example 5.2 *Let $X = \{a, b, c\}$ with $\tau = \{X, \phi\}$, a non open set $B = \{a\}$ is not Bg symmetric, since $b \in Bcl(\{a\})$, does not imply that $a \in Bcl(\{b\})$.*

Theorem 5.1. *If (X, τ) is a topological space, then the following are equivalent*
(1) (X, τ) is a Bg-symmetric space,
(2) every singleton set is Bg-closed, for each $x \in X$.

Proof. (1) $\Rightarrow$ (2) Assume that $\{x\} \subseteq U \in \tau$, but $Bcl(\{x\}) \nsubseteq U$. Then $Bcl(\{x\}) \cap X \backslash U \neq \phi$. Now we take $y \in Bcl(\{x\}) \cap X \backslash U$, then by hypothesis $x \in Bcl(\{y\}) \subseteq X \backslash U$ and $x \notin U$ which is a contradiction. Therefore $\{x\}$ is Bg-closed, for each $x \in X$.

(2) $\Rightarrow$ (1) Assume that $x \in Bcl(\{y\})$, but $y \notin Bcl(\{x\})$. Then $\{y\} \subseteq X \backslash Bcl(\{x\})$ and hence $Bcl(\{y\}) \subseteq X \backslash Bcl(\{x\})$. Therefore $x \in X \backslash Bcl(\{x\})$ which is a contradiction and hence $y \in Bcl(\{x\})$. ■

Corollary 5.1. *If (X, τ) is a Bg-T_1-space, then (X, τ) is Bg-symmetric.*

Proof. Since (X, τ) is Bg-T_1-space, then every singleton set is B-closed, therefore it is Bg-closed. Then by Theorem 5.1., (X, τ) is B-symmetric. ■

Theorem 5.2. *If $(X,\ \tau)$ is a Bg-symmetric space, then the following are equivalent*
(1) $(X,\ \tau)$ is a Bg-T_0-space,
(2) $(X,\ \tau)$ is a Bg-$T_{\frac{1}{2}}$-space,
(3) $(X,\ \tau)$ is a Bg-T_1-space.

Proof. (1) $\Leftrightarrow$ (3) is obvious
(3) $\Rightarrow$ (2), (2) $\Rightarrow$ (1), directly from Theorem 3.2. ■

Definition 5.2. A topological space $(X,\ \tau)$ is called a Bg-T_D-space if every singleton set is either B-open (or) nowhere dense (or equivalently, the derived set $Bcl(\{x\})\backslash\{x\}$ is B-closed for each point $x \in X$).

Example 5.3 *Let $X = \{a,\ b,\ c\}$ with $\tau = \{X,\ \phi,\ \{a\},\ \{b\},\ \{a,\ b\}\}$, a non open set $B = \{a,\ c\}$, then $(X,\ \tau)$ is Bg-T_D-space.*

Example 5.4 *Let $X = \{a,\ b,\ c,\ d\}$ with $\tau = \{X,\ \phi,\ \{a,\ b\},\ \{c,\ d\}\}$, a non open set $B = \{b\}$, is not Bg-T_D. Since a subset $\{a\}$ is neither B-open nor nowhere dense.*

Remark 5.1. *For a topological space $(X,\ \tau)$, the following are equivalent:*
(1) $(X,\ \tau)$ is a Bg-T_D-space
(2) $(X,\ \tau)$ is a Bg-$T_{\frac{1}{2}}$-space.

Corollary 5.2. *If $(X,\ \tau)$ is a Bg-symmetric space, then the following are equivalent:*
(1) $(X,\ \tau)$ is a Bg-T_D-space,
(2) $(X,\ \tau)$ is a Bg-T_0-space,
(3) $(X,\ \tau)$ is a Bg-T_1-space.

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