Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 1, 1 - 8

New Approaches for B-Generalized Closed Functions and Associated Properties

R. Vijayalakshmi and D. Krishnaswamy

Department of Mathematics, Annamalai University
Annamalainagar, Tamil Nadu-608 002, India
viji_lakshmi80@rediffmail.com
krishna_swamy2004@yahoo.com

Abstract

In this paper we offer a new class of functions called B-closed, Bg-closed, regular Bg-closed and almost Bg-closed functions. Moreover, we investigate not only some of their basic properties but also their relationships with other types of already well-known functions.

Mathematics Subject Classification: 54C10, 54C08, 54C05

Keywords: B-closed, Bg-closed, regular Bg-closed, almost Bg-closed, B-normal

1 Introduction and preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, separation axioms etc., by utiliaing generalized open sets. One of the most well-known notions and also an inspiration source in the notion of B-open sets [4] inroduced by Levine in 1963.

Throuhout this paper, X and Y refer always to topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A in X, respectively. The family of all Bg-closed (resp. B-regular, regular open, regular closed) sets of X is denoted by $BGC(X)$ (resp. $BR(X)$, $RO(X)$, $RC(X)$).

Definition 1.1. A subset A of X is said to be regular closed [3] if $A = clint(A)$, if its complement A^c is regular open subset of X.

Definition 1.2. Levine [2], 1963 defined $\tau(B) = \{O \cup (O' \cap B) : O,\ O' \in \tau\}$ and called it simple expansion of τ by B, where $B \notin \tau$.

Definition 1.3. A subset A of a topological space $(X,\ \tau)$ is said to be B-*generalized closed set* [1] (briefly Bg-closed) if $Bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in $(X,\ \tau)$, where $Bcl(A)$ is given by $Bcl(A) = \bigcap\{S \subseteq X : A \subseteq S$ and S is a closed set in $\tau(B)\}$. A subset of X belonging to $\tau(B)$ is denoted by B-*open set*, the complement of B-open set is denoted by B-*closed set*. The family of all B-open sets is denoted by $BO(X)$ and the family of all B-closed sets is denoted by $BC(X)$.

Definition 1.4. A subset A of a space X is B-regular [4] if A is both B-open and B-closed.

Definition 1.5. A map $f : X \to Y$ is called M-B-*open*[4] (resp. M-B-closed) if $f(V)$ is B-open (resp. B-closed) set in Y for every B-open (resp. B-closed) set V of X.

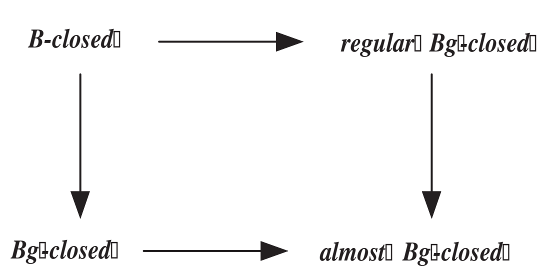
2 Bg-closed function

We introduce the following definitions

Definition 2.1. A function $f : X \to Y$ is said to be B-closed (resp. B-open) if for each closed (resp. open) set F in X, $f(F)$ is B-closed (resp. B-open) set in Y.

Definition 2.2. A function $f : X \to Y$ is said to be B-generalized closed (briefly,Bg-closed) (resp. regular Bg-closed, almost Bg-closed) if for each closed set F in X ($F \in BR(X)$, $F \in RC(X)$), $f(F)$ is Bg-closed set in Y.

From the above definitions, we obtain the following diagram:



Remark 2.1. *None of all implications in the above diagram is reversible as the following three examples show.*

Example 2.1 If $X = \{a,\ b,\ c\}$ with topology $\tau = \{X,\ \phi,\ \{a\},\ \{b\},\ \{a,\ b\}\}$ and $Y = \{a,\ b,\ c,\ d\}$ with topology $\sigma = \{Y,\ \phi,\ \{a\},\ \{b\},\ \{a,\ b\}\}$, a non open set $B = \{a,\ c\}$. Define a function $f : (X,\ \tau) \to (Y,\ \sigma)$ by $f(a) = d$, $f(b) = c$, $f(c) = a$. Then f is almost Bg-closed, but it is not Bg-closed as $\{c\}$ is closed in $(X,\ \tau)$ but $f(c) = \{a\}$ is not Bg-closed in $(Y,\ \sigma)$.

Example 2.2 If $X = \{a,\ b,\ c\}$ with topology $\tau = \{X,\ \phi,\ \{a\},\ \{b,\ c\}\}$ and $Y = \{a,\ b,\ c,\ d\}$ with topology $\sigma = \{Y,\ \phi,\ \{a,\ b\}\}$ a non open set $B = \{a,\ c\}$. Define a function $f : (X,\ \tau) \to (Y,\ \sigma)$ by $f(a) = c$, $f(b) = d$, $f(c) = a$ which is regular Bg-closed, but not B-closed as $\{a\}$ is closed in $(X,\ \tau)$ but $f(a) = \{c\}$ is not B-closed in $(Y,\ \sigma)$.

Example 2.3 If $X = \{a,\ b,\ c\}$ with topology $\tau = \{X,\ \phi,\ \{a\},\ \{b\},\ \{a,\ b\}\}$ and $Y = \{a,\ b,\ c,\ d\}$ with topology $\sigma = \{Y,\ \phi,\ \{a\},\ \{b\},\ \{a,\ b\}\}$ a non open set $B = \{a,\ c\}$. Define a function $f : (X,\ \tau) \to (Y,\ \sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = d$ which is almost Bg-closed, Bg-closed but not regular Bg-closed, B-closed.

The proof of the following Lemma follows using a standard technique, and thus omitted.

Lemma 2.1. *A function $f : X \to Y$ is B-closed (resp. regular Bg-closed) if and only if for each subset B of Y and each $U \in \tau$ (resp. $U \in BR(X)$) containing $f^{-1}(B)$, there exists a B-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Corollary 2.1. *If $f : X \to Y$ is B-closed (resp. regular Bg-closed) then for each closed set K of Y and each $U \in \tau$ (resp. $U \in BR(X)$) containing $f^{-1}(K)$, there exists $V \in BO(Y)$ containing K such that $f^{-1}(V) \subseteq U$*

Proof. Suppose that $f : X \to Y$ is B-closed (resp. regular Bg-closed). Let K be any closed set of Y and $U \in \tau$ (resp. $U \in BR(X)$) containing $f^{-1}(K)$. By Lemma 2.1. there exists a B-open set G of Y such that $K \subseteq G$ and $f^{-1}(G) \subseteq U$. Since K is closed, $K \subseteq Bint(G)$. Put $V = Bint(G)$. Then $K \subseteq V \in BO(Y)$ and $f^{-1}(V) \subseteq U$. ■

Theorem 2.1. *If $f : X \to Y$ is continuous B-closed then $f(H)$ is B-closed in Y for each closed set H of X.*

Proof. Let H be any closed set of X and V an open set of Y containing $f(H)$. Since $f^{-1}(V)$ is an open set of X containing H, $cl(H) \subseteq f^{-1}(V)$. Since $Bcl(H) \subseteq cl(H)$. This implies $Bcl(H) \subseteq f^{-1}(V)$ and hence $f(Bcl(H)) \subseteq V$. Since f is B-closed and $Bcl(H) \subseteq Bcl(X)$. We have $Bcl(f(H)) \subseteq Bcl(f(Bcl(H))) \subseteq V$. Therefore $f(H)$ is B-closed in Y. ■

Definition 2.3. *[1]* A function $f : X \to Y$ is said to be gB-continuous if $f^{-1}(K)$ is Bg-closed for every $K \in C(Y)$.

It is obvious that a function $f : X \to Y$ is gB-continuous if and only if $f^{-1}(V)$ is Bg-open in X for every $V \in O(Y)$.

Theorem 2.2. *If $f : X \to Y$ is closed gB-continuous then $f^{-1}(K)$ is Bg-closed in X for each Bg-closed set K of Y.*

Proof. *Let K be a Bg-closed set of Y and U an open set of X containing $f^{-1}(K)$. Put $V = Y/f(X/U)$, then V is open in Y, $K \subseteq V$ and $f^{-1}(V) \subseteq U$. Therefore, we have $Bcl(K) \subseteq V$ and hence $f^{-1}(K) \subseteq f^{-1}(Bcl(K)) \subseteq f^{-1}(V) \subseteq U$. Since f is gB-continuous, $f^{-1}(Bcl(K))$ is Bg-closed in X and hence $Bcl(f^{-1}(K)) \subseteq Bcl(f^{-1}(Bcl(K))) \subseteq U$. This shows that $f^{-1}(K)$ is Bg-closed in X.* ■

Recall that a function $f : X \to Y$ is said to be B-irresolute [4], if $f^{-1}(V) \in BO(X)$ for every $V \in BO(Y)$.

Corollary 2.2. *If $f : X \to Y$ is closed B-irresolute then $f^{-1}(K)$ is Bg-closed in X for each Bg-closed set K of Y.*

Corollary 2.3. *Let $f : X \to Y$ be a closed open continuous function. If K is a Bg-closed set of Y then $f^{-1}(K)$ is Bg-closed in X.*

Proof. *Follows from the fact that a continuous open function is B-irresolute.* ■

For the composition of B-closed functions, we have the following Theorems.

Theorem 2.3. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition $g \circ f : X \to Z$ is B-closed if f is B-closed and g is continuous B-closed.*

Proof. *The proof follows immediately from Theorem 2.2.* ■

Theorem 2.4. *Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition $g \circ f : X \to Z$ be B-closed. Then the following hold*
(i) If f is a closed continuous surjection, then g is B-closed.
(ii) If g is a closed B-continuous injection, then f is B-closed.

Proof. *(i) Let K be a closed set in Y. Since f is closed continuous and surjective, $f^{-1}(K) \in BC(X)$ and $(g \circ f)(f^{-1}(K)) = g(K)$. Therefore, $g(K)$ is B-closed in Z and hence g is B-closed.*

(ii) Let H be a closed set in X. Then $(g \circ f)(H)$ is B-closed in Z and $g^{-1}(g \circ f)(H)) = f(H)$. By Theorem 2.2., $f(H)$ is B-closed in Y and hence f is B-closed. ■

The following Lemma is analogous to Lemma 2.1. the straightforward proof is omitted.

Lemma 2.2. *A functon $f : X \to Y$ is almost Bg-closed if and only if for each subset B of Y and each $U \in RO(X)$ containing $f^{-1}(B)$, there exists a Bg-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Corollary 2.4. *If $f : X \to Y$ is almost Bg-closed then for each closed set K of Y and each $U \in RO(X)$ containing $f^{-1}(K)$, there exists $V \in BO(Y)$ such that $K \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. *The proof is smilar to that of Corollary 2.1.* ■

Definition 2.4. A topological space $(X,\ \tau)$ is said to be B *normal (briefly, B-normal)* if for any pair of disjoint closed sets A and B, there exist disjoint B-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 2.5. *Let $f : X \to Y$ be a continuous almost Bg-closed surjection. If X is normal, then Y is B-normal.*

Proof. *Let K_1 and K_2 be any disjoint closed sets of Y. Since f is continuous, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are disjoint closed sets of X. By the normality of X, there exist disjoint open sets U_1 and U_2 such that $f^{-1}(K_i) \subseteq U_i$, where $i = 1, 2$. Now, put $G_i = int(cl(U_i))$ for $i = 1, 2$, then $G_i \in RO(X)$, $f^{-1}(K_i) \subseteq U_i \subseteq G_i$ and $G_1 \cap G_2 = \phi$. By Corollary 2.4., there exists $V_i \in BO(Y)$ such that $K_i \subseteq V_i$ and $f^{-1}(V_i) \subseteq G_i$, $i = 1, 2$. Since $G_1 \cap G_2 = \phi$, f is surjective we have $V_1 \cap V_2 = \phi$. This shows that Y is B-normal.* ■

The following two Corollaries are immediate consequence of Theorem 2.5.

Corollary 2.5. *If $f : X \to Y$ is a continuous B-closed surjection and X is normal, then Y is B-normal.*

Corollary 2.6. *If $f : X \to Y$ is a continuous and closed surjection and X is normal, then Y is B-normal.*

Remark 2.2. *It is clear that every M-B-closed is B-closed. The converse is not true for the following example.*

Example 2.4 $X = \{a,\ b,\ c\}$ with topology $\tau = \{X,\ \phi,\ \{a\},\ \{b\},\ \{a,\ b\}\}$ and $\sigma = \{X,\ \phi,\ \{a\},\ \{b,\ c\}\}$, a non open set $B = \{a,\ c\}$. Define a function $f : (X,\ \tau) \to (X,\ \sigma)$ by $f(a) = a,\ f(b) = c,\ f(c) = b$ which is B-closed but not M-B-closed as $\{b\}$ is B-closed in $(X,\ \tau)$ but $f(b) = \{c\}$ which is not M-B-closed in $(X,\ \sigma)$.

Theorem 2.6. *Let A be a subset of a topological space X. Then*
(i) $A \in BO(X)$ iff $Bcl(A) \in BR(X)$. (ii) $A \in BC(X)$ iff $Bint(A) \in BR(X)$.

Theorem 2.7. *Let $f : X \to Y$ be a continuous regular Bg-closed surjection. If X is B-normal, then Y is B-normal.*

Proof. *Although the proof is similar to that Theorem 2.5. we will state it for the convenience of the reader. Let K_1 and K_2 be any disjoint closed sets of Y. Since f is continuous, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are disjoint closed sets of X. By the B-normality of X, there exist disjoint sets $U_1,\ U_2 \in BO(X)$ such that $f^{-1}(K_i) \subseteq U_i$, for $i = 1,\ 2$. Now, put $G_i = Bcl(U_i)$ for $i = 1, 2$, then by Theorem 2.6. $G_i \in BR(X)$, $f^{-1}(K_i) \subseteq U_i \subseteq G_i$ and $G_1 \cap G_2 = \phi$. By Corollary 2.1., there exists $V_i \in BO(Y)$ such that $K_i \subseteq V_i$ and $f^{-1}(V_i) \subseteq G_i$, where $i = 1,\ 2$. Since f is surjective and $G_1 \cap G_2 = \phi$, we obtain $V_1 \cap V_2 = \phi$. This shows that Y is B-normal.* ■

Corollary 2.7. *Let $f : X \to Y$ be a continuous B-closed surjection. If X is B-normal, then Y is B-normal.*

Corollary 2.8. *If $f : X \to Y$ be continuous M-B-closed surjection and X is B-normal, then Y is B-normal.*

Theorem 2.8. *Let $f : X \to Y$ be a closed B-irresolute injection. If Y is B-normal, then X is B-normal.*

Proof. *Let H_1 and H_2 be disjoint closed sets of X. Since f is closed injection, $f(H_1)$ and $f(H_2)$ are disjoint closed sets of Y. By the B-normality of Y, there exist disjoint B-open sets $V_1,\ V_2$ such that $f(H_i) \subseteq V_i$ for $i = 1,\ 2$. Since f is B-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint B-open sets of X and $H_i \subseteq f^{-1}(V_i)$ for $i = 1, 2$. Now put $U_i = Bint(f^{-1}(V_i))$ for $i = 1,\ 2$. Then $U_i \in BO(X)$, $H_i \subseteq U_i$ and $U_1 \cap U_2 = \phi$. This shows that X is B-normal.* ■

Theorem 2.9. *For a topological space X, the following properties are equivalent:*
(i) X is B-regular,
(ii) For each U open in X and each $x \in U$, there exists $V \in BO(X)$ such that $x \in V \subseteq Bcl(V) \subseteq U$,
(iii) For each U open in X and each $x \in U$, there exists $V \in BR(X)$ such that $x \in V \subseteq U$.

Theorem 2.10. *Let $f : X \to Y$ be a continuous B-open almost Bg-closed surjection. If X is regular, then Y is B-regular.*

Proof. *Let $y \in Y$ and V be an open neighbourhood of y. Take a point $x = f^{-1}(y)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is B-open in X. By regularity of X, there exists an open set U of X such that $x \in U \subseteq cl(U) \subseteq f^{-1}(V)$. Then $y \in f(U) \subseteq f(cl(U)) \subseteq V$, $f(U) \in BO(Y)$ and $f(cl(U))$ is B-closed in Y. Therefore, we obtain $y \in f(U) \subseteq Bcl(f(U)) \subseteq Bcl(f(cl(U))) \subseteq V$. It follows from Theorem 2.9. that Y is B-regular.* ■

Corollary 2.9. *If $f : X \to Y$ is a continuous B-open, B-closed surjection and X is regular, then Y is B-regular.*

Theorem 2.11. *Let $f : X \to Y$ be a continuous M-B-open regular Bg-closed surjection. If X is B-regular, then Y is B-regular.*

Proof. *Let F be any closed set of Y and $y \in Y \backslash F$. Then $f^{-1}(F)$ is closed in X and $f^{-1}(F) \cap f^{-1}(y) = \phi$. Take a point $x = f^{-1}(y)$. Since X is B-regular, there exists disjoint sets U_1, $U_2 \in BO(X)$ such that $x \in U_1$ and $f^{-1}(F) \subseteq U_2$. Therefore we have $f^{-1}(F) \subseteq U_2 \subseteq Bcl(U_2)$, $Bcl(U_2) \in BR(X)$ and $U_1 \cap Bcl(U_2) = \phi$. Since f is regular B-closed, By Corollary 2.1. there exists $V \in BO(Y)$ such that $F \subseteq V$ and $f^{-1}(V) \subseteq Bcl(U_2)$. Since f is M-B-open we have $f(U_1) \in BO(Y)$. Moreover $U_1 \cap f^{-1}(V) = \phi$ and hence $f(U_1) \cap V = \phi$. Consequently, we obtain $y \in f(U_1) \in BO(Y)$, $F \subseteq V \in BO(Y)$ and $f(U_1) \cap V = \phi$. This shows that Y is B-regular.* ■

Corollary 2.10. *If $f : X \to Y$ is a continuous M-B-open, M-B-closed surjection and X is B-regular, then Y is B-regular.*

References

[1] M. E. Abd El. Monsef, A. M. Kozae and R. A. Abu-Gdairi, *New Approches for Generalized continuous Functions*, Int. Journal of Math. Analysis, Vol.4, 2010, no.27, 1329-1339.

[2] N. Levine, *Simple Extention of Topologies*, Amer. Math. Japan. Monthly, 71 (1964), 22-105.

[3] M. Stone, *Application of the Theory of Boolean Rings to General Topology*, Trans. Amer. Math. Soc., 41(1937), 374-481.

[4] A. Vadivel, R. Vijayalakshmi and D. Krishnaswamy, *B-generalized Regular and B-generalized Normal Spaces* (in press).

Received: July, 2010