New Approaches for $B$-Generalized Closed Functions and Associated Properties

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Abstract

In this paper we offer a new class of functions called $B$-closed, $Bg$-closed, regular $Bg$-closed and almost $Bg$-closed functions. Moreover, we investigate not only some of their basic properties but also their relationships with other types of already well-known functions.

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1 Introduction and preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, separation axioms etc., by utilizing generalized open sets. One of the most well-known notions and also an inspiration source in the notion of $B$-open sets [4] introduced by Levine in 1963.

Throughout this paper, $X$ and $Y$ refer always to topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of $X$, $cl(A)$ and $int(A)$ denote the closure of $A$ and the interior of $A$ in $X$, respectively. The family of all $Bg$-closed (resp. $B$-regular, regular open, regular closed) sets of $X$ is denoted by $BGC(X)$ (resp. $BR(X)$, $RO(X)$, $RC(X)$).
Definition 1.1. A subset $A$ of $X$ is said to be regular closed [3] if $A = \text{clint}(A)$, if its complement $A^c$ is regular open subset of $X$.

Definition 1.2. Levine [2], 1963 defined $\tau(B) = \{O \cup (O' \cap B) : O, O' \in \tau\}$ and called it simple expansion of $\tau$ by $B$, where $B \notin \tau$.

Definition 1.3. A subset $A$ of a topological space $(X, \tau)$ is said to be $B$-generalized closed set [1] (briefly $Bg$-closed) if $B\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$, where $B\text{cl}(A)$ is given by $B\text{cl}(A) = \bigcap \{S \subseteq X : A \subseteq S$ and $S$ is a closed set in $\tau(B)\}$. A subset of $X$ belonging to $\tau(B)$ is denoted by $B$-open set, the complement of $B$-open set is denoted by $B$-closed set. The family of all $B$-open sets is denoted by $BO(X)$ and the family of all $B$-closed sets is denoted by $BC(X)$.

Definition 1.4. A subset $A$ of a space $X$ is $B$-regular [4] if $A$ is both $B$-open and $B$-closed.

Definition 1.5. A map $f : X \to Y$ is called $M$-$B$-open[4] (resp. $M$-$B$-closed) if $f(V)$ is $B$-open (resp. $B$-closed) set in $Y$ for every $B$-open (resp. $B$-closed) set $V$ of $X$.

2 $Bg$-closed function

We introduce the following definitions

Definition 2.1. A function $f : X \to Y$ is said to be $B$-closed (resp. $B$-open) if for each closed (resp. open) set $F$ in $X$, $f(F)$ is $B$-closed (resp. $B$-open) set in $Y$.

Definition 2.2. A function $f : X \to Y$ is said to be $B$-generalized closed (briefly $Bg$-closed) (resp. regular $Bg$-closed, almost $Bg$-closed) if for each closed set $F$ in $X$ ($F \in BR(X)$, $F \in RC(X)$), $f(F)$ is $Bg$-closed set in $Y$.

From the above definitions, we obtain the following diagram:

\[ \text{B-closed} \quad \longrightarrow \quad \text{regular Bg-closed} \]
\[ \text{Bg-closed} \quad \longmapsto \quad \text{almost Bg-closed} \]
Remark 2.1. None of all implications in the above diagram is reversible as the following three examples show.

Example 2.1 If \( X = \{a, b, c\} \) with topology \( \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( Y = \{a, b, c, d\} \) with topology \( \sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\} \), a non open set \( B = \{a, c\} \). Define a function \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = d, f(b) = c, f(c) = a \). Then \( f \) is almost \( Bg \)-closed, but it is not \( Bg \)-closed as \( \{c\} \) is closed in \((X, \tau)\) but \( f(c) = \{a\} \) is not \( Bg \)-closed in \((Y, \sigma)\).

Example 2.2 If \( X = \{a, b, c\} \) with topology \( \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( Y = \{a, b, c, d\} \) with topology \( \sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\} \), a non open set \( B = \{a, c\} \). Define a function \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = c, f(b) = d, f(c) = a \) which is regular \( Bg \)-closed, but not \( B \)-closed as \( \{a\} \) is closed in \((X, \tau)\) but \( f(a) = \{c\} \) is not \( B \)-closed in \((Y, \sigma)\).

Example 2.3 If \( X = \{a, b, c\} \) with topology \( \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( Y = \{a, b, c, d\} \) with topology \( \sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\} \), a non open set \( B = \{a, c\} \). Define a function \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = b, f(b) = c, f(c) = d \) which is almost \( Bg \)-closed, \( Bg \)-closed but not regular \( Bg \)-closed, \( B \)-closed.

The proof of the following Lemma follows using a standard technique, and thus omitted.

Lemma 2.1. A function \( f : X \to Y \) is \( B \)-closed (resp. regular \( Bg \)-closed) if and only if for each subset \( B \) of \( Y \) and each \( U \in \tau \) (resp. \( U \in BR(X) \)) containing \( f^{-1}(B) \), there exists a \( B \)-open set \( V \) of \( Y \) such that \( B \subseteq V \) and \( f^{-1}(V) \subseteq U \).

Corollary 2.1. If \( f : X \to Y \) is \( B \)-closed (resp. regular \( Bg \)-closed) then for each closed set \( K \) of \( Y \) and each \( U \in \tau \) (resp. \( U \in BR(X) \)) containing \( f^{-1}(K) \), there exists \( V \in BO(Y) \) containing \( K \) such that \( f^{-1}(V) \subseteq U \).

Proof. Suppose that \( f : X \to Y \) is \( B \)-closed (resp. regular \( Bg \)-closed). Let \( K \) be any closed set of \( Y \) and \( U \in \tau \) (resp. \( U \in BR(X) \)) containing \( f^{-1}(K) \). By Lemma 2.1. there exists a \( B \)-open set \( G \) of \( Y \) such that \( K \subseteq G \) and \( f^{-1}(G) \subseteq U \). Since \( K \) is closed, \( K \subseteq Bint(G) \). Put \( V = Bint(G) \). Then \( K \subseteq V \in BO(Y) \) and \( f^{-1}(V) \subseteq U \).

Theorem 2.1. If \( f : X \to Y \) is continuous \( B \)-closed then \( f(H) \) is \( B \)-closed in \( Y \) for each closed set \( H \) of \( X \).
Proof. Let $H$ be any closed set of $X$ and $V$ an open set of $Y$ containing $f(H)$. Since $f^{-1}(V)$ is an open set of $X$ containing $H$, $cl(H) \subseteq f^{-1}(V)$. Since $Bcl(H) \subseteq cl(H)$. This implies $Bcl(H) \subseteq f^{-1}(V)$ and hence $f(Bcl(H)) \subseteq V$. Since $f$ is $B$-closed and $Bcl(H) \subseteq Bc(X)$. We have $Bcl(f(H)) \subseteq Bcl(f(Bcl(H))) \subseteq V$. Therefore $f(H)$ is $B$-closed in $Y$. ■

Definition 2.3. [1] A function $f : X \to Y$ is said to be $gB$-continuous if $f^{-1}(K)$ is $Bg$-closed for every $K \in C(Y)$.

It is obvious that a function $f : X \to Y$ is $gB$-continuous if and only if $f^{-1}(V)$ is $Bg$-open in $X$ for every $V \in O(Y)$.

Theorem 2.2. If $f : X \to Y$ is closed $gB$-continuous then $f^{-1}(K)$ is $Bg$-closed in $X$ for each $Bg$-closed set $K$ of $Y$.

Proof. Let $K$ be a $Bg$-closed set of $Y$ and $U$ an open set of $X$ containing $f^{-1}(K)$. Put $V = Y/f(X/U)$, then $V$ is open in $Y$, $K \subseteq V$ and $f^{-1}(V) \subseteq U$. Therefore, we have $Bcl(K) \subseteq V$ and hence $f^{-1}(K) \subseteq f^{-1}(Bcl(K)) \subseteq f^{-1}(V) \subseteq U$. Since $f$ is $gB$-continuous, $f^{-1}(Bcl(K))$ is $Bg$-closed in $X$ and hence $Bcl(f^{-1}(K)) \subseteq Bcl(f^{-1}(Bcl(K))) \subseteq U$. This shows that $f^{-1}(K)$ is $Bg$-closed in $X$. ■

Recall that a function $f : X \to Y$ is said to be $B$-irresolute [4], if $f^{-1}(V) \in BO(X)$ for every $V \in BO(Y)$.

Corollary 2.2. If $f : X \to Y$ is closed $B$-irresolute then $f^{-1}(K)$ is $Bg$-closed in $X$ for each $Bg$-closed set $K$ of $Y$.

Corollary 2.3. Let $f : X \to Y$ be a closed open continuous function. If $K$ is a $Bg$-closed set of $Y$ then $f^{-1}(K)$ is $Bg$-closed in $X$.

Proof. Follows from the fact that a continuous open function is $B$-irresolute. ■

For the composition of $B$-closed functions, we have the following Theorems.

Theorem 2.3. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition $g \circ f : X \to Z$ is $B$-closed if $f$ is $B$-closed and $g$ is continuous $B$-closed.

Proof. The proof follows immediately from Theorem 2.2. ■

Theorem 2.4. Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition $g \circ f : X \to Z$ be $B$-closed. Then the following hold
(i) If $f$ is a closed continuous surjection, then $g$ is $B$-closed.
(ii) If $g$ is a closed $B$-continuous injection, then $f$ is $B$-closed.
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**Proof.** (i) Let $K$ be a closed set in $Y$. Since $f$ is closed continuous and surjective, $f^{-1}(K) \in BC(X)$ and $(g \circ f)(f^{-1}(K)) = g(K)$. Therefore, $g(K)$ is $B$-closed in $Z$ and hence $g$ is $B$-closed.

(ii) Let $H$ be a closed set in $X$. Then $(g \circ f)(H)$ is $B$-closed in $Z$ and $g^{-1}(g \circ f)(H)) = f(H)$. By Theorem 2.2., $f(H)$ is $B$-closed in $Y$ and hence $f$ is $B$-closed.

The following Lemma is analogous to Lemma 2.1. the straightforward proof is omitted.

**Lemma 2.2.** A function $f : X \rightarrow Y$ is almost $Bg$-closed if and only if for each subset $B$ of $Y$ and each $U \in RO(X)$ containing $f^{-1}(B)$, there exists a $Bg$-open set $V$ of $Y$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

**Corollary 2.4.** If $f : X \rightarrow Y$ is almost $Bg$-closed then for each closed set $K$ of $Y$ and each $U \in RO(X)$ containing $f^{-1}(K)$, there exists $V \in BO(Y)$ such that $K \subseteq V$ and $f^{-1}(V) \subseteq U$.

**Proof.** The proof is similar to that of Corollary 2.1.

**Definition 2.4.** A topological space $(X, \tau)$ is said to be $B$ normal (briefly, $B$-normal) if for any pair of disjoint closed sets $A$ and $B$, there exist disjoint $B$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Theorem 2.5.** Let $f : X \rightarrow Y$ be a continuous almost $Bg$-closed surjection. If $X$ is normal, then $Y$ is $B$-normal.

**Proof.** Let $K_1$ and $K_2$ be any disjoint closed sets of $Y$. Since $f$ is continuous, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are disjoint closed sets of $X$. By the normality of $X$, there exist disjoint open sets $U_1$ and $U_2$ such that $f^{-1}(K_i) \subseteq U_i$, where $i = 1, 2$. Now, put $G_i = \text{int}(\text{cl}(U_i))$ for $i = 1, 2$, then $G_i \in RO(X)$, $f^{-1}(K_i) \subseteq U_i \subseteq G_i$ and $G_1 \cap G_2 = \phi$. By Corollary 2.4., there exists $V_i \in BO(Y)$ such that $K_i \subseteq V_i$ and $f^{-1}(V_i) \subseteq G_i$, $i = 1, 2$. Since $G_1 \cap G_2 = \phi$, $f$ is surjective we have $V_1 \cap V_2 = \phi$. This shows that $Y$ is $B$-normal.

The following two Corollaries are immediate consequence of Theorem 2.5.

**Corollary 2.5.** If $f : X \rightarrow Y$ is a continuous $B$-closed surjection and $X$ is normal, then $Y$ is $B$-normal.

**Corollary 2.6.** If $f : X \rightarrow Y$ is a continuous and closed surjection and $X$ is normal, then $Y$ is $B$-normal.
Remark 2.2. It is clear that every M-B-closed is B-closed. The converse is not true for the following example.

Example 2.4  $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{X, \phi, \{a\}, \{b, c\}\}$, a non open set $B = \{a, c\}$. Define a function $f : (X, \tau) \rightarrow (X, \sigma)$ by $f(a) = a$, $f(b) = c$, $f(c) = b$ which is B-closed but not M-B-closed as $\{b\}$ is B-closed in $(X, \tau)$ but $f(b) = \{c\}$ which is not M-B-closed in $(X, \sigma)$.

Theorem 2.6. Let $A$ be a subset of a topological space $X$. Then
(i) $A \in BO(X)$ iff $Bcl(A) \in BR(X)$. (ii) $A \in BC(X)$ iff $Bint(A) \in BR(X)$.

Theorem 2.7. Let $f : X \rightarrow Y$ be a continuous regular $B_g$-closed surjection. If $X$ is B-normal, then $Y$ is B-normal.

Proof. Although the proof is similar to that Theorem 2.5. we will state it for the convenience of the reader. Let $K_1$ and $K_2$ be any disjoint closed sets of $Y$. Since $f$ is continuous, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are disjoint closed sets of $X$. By the B-normality of $X$, there exist disjoint sets $U_1, U_2 \in BO(X)$ such that $f^{-1}(K_i) \subseteq U_i$, for $i = 1, 2$. Now, put $G_i = Bcl(U_i)$ for $i = 1, 2$, then by Theorem 2.6. $G_i \in BR(X)$, $f^{-1}(K_i) \subseteq U_i \subseteq G_i$ and $G_1 \cap G_2 = \phi$. By Corollary 2.1., there exists $V_i \in BO(Y)$ such that $K_i \subseteq V_i$ and $f^{-1}(V_i) \subseteq G_i$, where $i = 1, 2$. Since $f$ is surjective and $G_1 \cap G_2 = \phi$, we obtain $V_1 \cap V_2 = \phi$. This shows that $Y$ is B-normal.

Corollary 2.7. Let $f : X \rightarrow Y$ be a continuous $B$-closed surjection. If $X$ is $B$-normal, then $Y$ is $B$-normal.

Corollary 2.8. If $f : X \rightarrow Y$ be continuous $M$-$B$-closed surjection and $X$ is $B$-normal, then $Y$ is $B$-normal.

Theorem 2.8. Let $f : X \rightarrow Y$ be a closed $B$-irresolute injection. If $Y$ is $B$-normal, then $X$ is $B$-normal.

Proof. Let $H_1$ and $H_2$ be disjoint closed sets of $X$. Since $f$ is closed injection, $f(H_1)$ and $f(H_2)$ are disjoint closed sets of $Y$. By the B-normality of $Y$, there exist disjoint $B$-open sets $V_1, V_2$ such that $f(H_i) \subseteq V_i$ for $i = 1, 2$. Since $f$ is $B$-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint $B$-open sets of $X$ and $H_i \subseteq f^{-1}(V_i)$ for $i = 1, 2$. Now put $U_i = Bint(f^{-1}(V_i))$ for $i = 1, 2$. Then $U_i \in BO(X)$, $H_i \subseteq U_i$ and $U_1 \cap U_2 = \phi$. This shows that $X$ is $B$-normal.
Theorem 2.9. For a topological space $X$, the following properties are equivalent:

(i) $X$ is $B$-regular,

(ii) For each $U$ open in $X$ and each $x \in U$, there exists $V \in BO(X)$ such that $x \in V \subseteq Bcl(V) \subseteq U$,

(iii) For each $U$ open in $X$ and each $x \in U$, there exists $V \in BR(X)$ such that $x \in V \subseteq U$.

Theorem 2.10. Let $f : X \to Y$ be a continuous $B$-open almost $Bg$-closed surjection. If $X$ is regular, then $Y$ is $B$-regular.

Proof. Let $y \in Y$ and $V$ be an open neighbourhood of $y$. Take a point $x = f^{-1}(y)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is $B$-open in $X$. By regularity of $X$, there exists an open set $U$ of $X$ such that $x \in U \subseteq cl(U) \subseteq f^{-1}(V)$. Then $y \in f(U) \subseteq f(cl(U)) \subseteq V$, $f(U) \in BO(Y)$ and $f(cl(U))$ is $B$-closed in $Y$. Therefore, we obtain $y \in f(U) \subseteq Bcl(f(U)) \subseteq Bcl(f(cl(U))) \subseteq V$. It follows from Theorem 2.9. that $Y$ is $B$-regular.

Corollary 2.9. If $f : X \to Y$ is a continuous $B$-open, $B$-closed surjection and $X$ is regular, then $Y$ is $B$-regular.

Theorem 2.11. Let $f : X \to Y$ be a continuous $M$-$B$-open regular $Bg$-closed surjection. If $X$ is $B$-regular, then $Y$ is $B$-regular.

Proof. Let $F$ be any closed set of $Y$ and $y \in Y \setminus F$. Then $f^{-1}(F)$ is closed in $X$ and $f^{-1}(F) \cap f^{-1}(y) = \phi$. Take a point $x = f^{-1}(y)$. Since $X$ is $B$-regular, there exists disjoint sets $U_1, U_2 \in BO(X)$ such that $x \in U_1$ and $f^{-1}(F) \subseteq U_2$. Therefore we have $f^{-1}(F) \subseteq U_2 \subseteq Bcl(U_2)$, $Bcl(U_2) \subseteq BR(X)$ and $U_1 \cap Bcl(U_2) = \phi$. Since $f$ is regular $B$-closed, By Corollary 2.1. there exists $V \in BO(Y)$ such that $F \subseteq V$ and $f^{-1}(V) \subseteq Bcl(U_2)$. Since $f$ is $M$-$B$-open we have $f(U_1) \in BO(Y)$. Moreover $U_1 \cap f^{-1}(V) = \phi$ and hence $f(U_1) \cap V = \phi$. Consequently, we obtain $y \in f(U_1) \in BO(Y)$, $F \subseteq V \subseteq BO(Y)$ and $f(U_1) \cap V = \phi$. This shows that $Y$ is $B$-regular.

Corollary 2.10. If $f : X \to Y$ is a continuous $M$-$B$-open, $M$-$B$-closed surjection and $X$ is $B$-regular, then $Y$ is $B$-regular.
References


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