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On Representation of Age-dependent Stretched Exponent in the Extended Weibull Model

E. S. Lakshminarayanan and M. Sumathi

Madurai Kamaraj University
mkueslnmath@yahoo.com
sumathimku@yahoo.co.in

Abstract

In this paper we provide a new representation for the age-dependent stretched exponent in the extended Weibull model which enable us to find the critical point and the zero of the mortality rate function in the absence of age-specific mortality data

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1 Introduction

The study of aging has traditionally been independently approached at two levels of biological organization : at the individual and sub-individual level by gerontologists interested in the physiology of human aging and at the population level by demographers primarily interested in patterns of survival and mortality in human populations [1]. The aging process have lately attracted the interest of researchers in a variety of disciplines, linking ideas and theories from such diverse fields as biochemistry to mathematics [7].

The fundamental law of population dynamics is the Gompertz law [2], in which the human mortality rate increases roughly exponentially with increasing age at senescence. The Gompertz model is most commonly employed to compare mortality rate between different populations [5]. For ages above 90 years, Thatcher et.al. [6] found that the classical Weibull model is inferior to a model described by Kannisto [3]. For this reason, Weon et.al. [8] modified the classical nature of the stretched exponent as a function of age. With this modification, they made an extended Weibull model adoptable at any shape of the empirical human survival curves [9].

In our previous work [4], in the absence of age specific mortality data, we gave a mathematical justification to predict the maximum human life span from the extended Weibull model, using the characteristic life. In this work, we provide a new representation for the age-dependent stretched exponent. Using this representation, we also found the critical point and the zero of the mortality rate function.

2 Extended Weibull model: Weibull model with age-dependent stretched exponent

Weon et al. put forward a general expression for human survival and mortality rate [8]. It is recently discovered that human survival and mortality curves are well described by a new mathematical model, derived from the Weibull survival function and it is simply described by two parameters, the age-dependent stretched exponent $\beta(t)$ and the characteristic life α,

$$S(t) = e^{(-(t/\alpha)^{\beta(t)})} \tag{1}$$

where $S(t)$ denotes the survival probability of surviving to age t. After graphically determining the α value, an adequate mathematical expression for $\beta(t)$ can be given by

$$\beta(t) = \frac{ln(-lnS(t))}{ln(t/\alpha)} \tag{2}$$

being a function of age t. On account of (1), we get

$$\mu(t) = \frac{d}{dt}((t/\alpha)^{\beta(t)})$$

or, equivalently

$$\mu(t) = (t/\alpha)^{\beta(t)} \left[\frac{\beta(t)}{t} + ln(t/\alpha)\frac{d\beta(t)}{dt}\right]. \tag{3}$$

In (3), α denotes the characteristic life ($t = \alpha$ when $S = e^{-1} \approx 36.79\%$). The trends and causes of increased characteristic life may be identical with those of average life (when $S = 50\%$). In empirical practice, an expression of $\beta(t)$ at $t > \alpha$ can be a quadratic (or a cubic formula) given by

$$\beta(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + ...$$

where the associated coefficients are determined by a regression analysis in the plot of $\beta(t)$ versus age. Particularly, the quadratic pattern

$$\beta(t) = \beta_0 + \beta_1 t + \beta_2 t^2$$

(the vertex is $\nu = \frac{-\beta_1}{2\beta_2}$) is mathematically valid before an age limit.

The survival probability $S(t)$ is mathematically a monotonic decay function of age ($\frac{dS(t)}{dt} < 0$) between $S(0) = 1$ and $S(\omega) = 0$. For this reason, the slope of the stretched exponent with age can be given by :

$$\frac{d\beta(t)}{dt} \begin{cases} < +\epsilon(t) & t < \alpha \\ > -\epsilon(t) & t > \alpha \quad ......(I) \end{cases}$$

where $\epsilon(t) = \left|\frac{-\beta(t)}{t\, ln(t/\alpha)}\right|$ is the mathematical constraint of $\beta(t)$. The maximum human lifespan t_m per each survival curve might be estimated at the mathematical limit of

$$\frac{d\beta(t)}{dt} = \frac{-\beta(t)}{t\, ln\frac{t}{\alpha}}. \tag{4}$$

E.S.Lakshminarayanan and M.Sumathi [4] gave an estimation for the age dependent stretched exponent $\beta(t)$ in the neighbourhood of the characteristic life

$$\beta(t) = \beta(\alpha) + \frac{\beta(\alpha)}{2\alpha}(t-\alpha) + \beta''(\alpha)\frac{(t-\alpha)^2}{2!}, \tag{5}$$

where

$$\beta'(t)/_{t=\alpha} = \frac{\beta(\alpha)}{2\alpha}, \tag{6}$$

$$\beta''(t)/_{t=\alpha} = -\left[\frac{\beta(t_m)}{t_m(t_m-\alpha)\, ln(t_m/\alpha)} + \frac{\beta(\alpha)}{2\alpha(t_m-\alpha)}\right]$$

and the mortality rate at the characteristic life α is given by

$$\mu(\alpha) = \frac{\beta(\alpha)}{\alpha}. \tag{7}$$

3 Properties of mortality function

(i) At the vertex $t = \nu$, $\beta'(t) = 0$ and hence the mortality function of the extended Weibull model given by (3) becomes

$$\mu(\nu) = \left(\frac{\nu}{\alpha}\right)^{\beta(\nu)}\left[\frac{\beta(\nu)}{\nu}\right] = \left(\frac{\nu}{\alpha}\right)^{\beta(\nu)-1}\left[\frac{\beta(\nu)}{\alpha}\right]. \tag{8}$$

As $\beta(\nu) > 1$ (8) gives

$$\mu(\nu) > \left(\frac{\nu}{\alpha}\right)^{\beta(\nu)-1} \left[\frac{1}{\alpha}\right]. \tag{9}$$

On the otherhand, $\nu > \alpha$ implies

$$\mu(\nu) > \frac{\beta(\nu)}{\alpha}. \tag{10}$$

From (9) and (10) we get

$$\mu(\nu) \geq max \left\{ \left(\frac{\nu}{\alpha}\right)^{\beta(\nu)-1} \left[\frac{1}{\alpha}\right], \frac{\beta(\nu)}{\alpha} \right\}.$$

When $\nu < \alpha$, obviously (8) satisfies

$$\mu(\nu) < \frac{\beta(\nu)}{\alpha}.$$

(ii) Now we prove that whenever $\nu > \alpha$, $\mu(\nu) > \mu(\alpha)$ implies $\beta(\nu) > \beta(\alpha)$.

Let $\mu(\nu) > \mu(\alpha)$. Then from (8) we have

$$\left(\frac{\nu}{\alpha}\right)^{\beta(\nu)-1} \left[\frac{\beta(\nu)}{\alpha}\right] > \frac{\beta(\alpha)}{\alpha}.$$

Simplifying further, we get

$$\frac{\beta(\nu)}{\beta(\alpha)} > \left(\frac{\alpha}{\nu}\right)^{\beta(\nu)-1} > \left(\frac{\alpha}{\nu}\right)^{\beta(\nu)}. \tag{11}$$

When $\nu > \alpha$

$$\left(\frac{\alpha}{\nu}\right)^{\beta(\nu)-1} = \left(\frac{\alpha}{\nu}\right)^{\beta(\nu)} \frac{\nu}{\alpha} < \frac{\nu}{\alpha}. \tag{12}$$

From (11) and (12) we get

$$\left(\frac{\alpha}{\nu}\right)^{\beta(\nu)} < \left(\frac{\alpha}{\nu}\right)^{\beta(\nu)-1} < \frac{\nu}{\alpha} = 1 + \frac{\nu-\alpha}{\alpha}. \tag{13}$$

Since

$$\frac{\beta(\nu)}{\beta(\alpha)} > \left(\frac{\alpha}{\nu}\right)^{\beta(\nu)}$$

and

$$\left(\frac{\alpha}{\nu}\right)^{\beta(\nu)} < 1 + \frac{\nu - \alpha}{\alpha},$$

it is possible to have $\frac{\beta(\nu)}{\beta(\alpha)} > 1$ which implies $\beta(\nu) > \beta(\alpha)$.

For instance, when
(i) $\alpha = 88.57223$, $\nu = 96.3447$, $\beta(\alpha) = 11.3941$, $\beta(\nu) = 11.6412$
(ii) $\alpha = 87.67320$, $\nu = 95.8000$, $\beta(\alpha) = 10.4413$, $\beta(\nu) = 10.5761$
(iii) $\alpha = 86.53680$, $\nu = 94.3000$, $\beta(\alpha) = 10.5005$, $\beta(\nu) = 10.6652$
(iv) $\alpha = 85.23810$, $\nu = 90.8000$, $\beta(\alpha) = 9.7210$, $\beta(\nu) = 9.7307$
(v) $\alpha = 84.26410$, $\nu = 96.0000$, $\beta(\alpha) = 9.5230$, $\beta(\nu) = 9.9152$

Interestingly, when $\nu < \alpha$, $\mu(\nu) > \mu(\alpha)$ also implies $\beta(\nu) > \beta(\alpha)$.

Let $\mu(\nu) > \mu(\alpha)$. Then

$$\frac{\beta(\nu)}{\beta(\alpha)} > \left(\frac{\alpha}{\nu}\right)^{\beta(\nu)-1} > 1$$

implies $\beta(\nu) > \beta(\alpha)$.

For instance, when
(i) $\alpha = 88.5278$, $\nu = 78.7$, $\beta(\alpha) = 12.1823$, $\beta(\nu) = 12.3262$
(ii) $\alpha = 88.4408$, $\nu = 87.0$, $\beta(\alpha) = 11.6768$, $\beta(\nu) = 11.6888$
(iii) $\alpha = 85.6169$, $\nu = 85.0$, $\beta(\alpha) = 9.7889$, $\beta(\nu) = 9.7946$

4 Determination of sign of the coefficients of the age-dependent stretched exponent

Weon et.al. [9] put forward that with the quadratic pattern of $\beta(t)$, the mortality pattern tends to decrease after a plateau and ultimately approach zero. The point where the mortality curve starts to decline is obtained by solving $\mu'(t) = 0$.

(i) To determine the sign of the coefficients of $\beta(t)$, first we determine the sign of β_2. Differentiating (3) with respect to t and equating it to zero gives

$$\left[\frac{\beta(t)}{t} + \beta'(t) ln\frac{t}{\alpha}\right]^2 + \frac{2}{t}\beta'(t) - \frac{\beta(t)}{t^2} + \beta''(t) ln\frac{t}{\alpha} = 0. \tag{14}$$

It follows from (14) that

$$\frac{2}{t}\beta'(t) - \frac{\beta(t)}{t^2} + \beta''(t) ln\frac{t}{\alpha} < 0, \quad t > \alpha \tag{15}$$

Since $\beta(t) > 0$, dividing each term of (15) by $\beta(t)$ we get

$$\frac{2\beta''(t)}{\beta(t)}\ ln\frac{t}{\alpha} + \frac{2\beta'(t)}{t\beta(t)} < \frac{1}{t^2} < \frac{1}{\alpha^2}. \tag{16}$$

Simplifying further and taking into account $\beta''(t) = \beta_2$ (16) satisfies the inequality

$$\frac{2\beta_2\alpha}{\beta(t)}\ ln\frac{t}{\alpha} - \frac{1}{\alpha} < \frac{-2\beta'(t)\alpha}{t\beta(t)} < \frac{-2\beta'(t)}{\beta(t)}, \quad t > \alpha$$

Since $\mu(t) > 0$, on account of (I) for $t > \alpha$ the above inequality takes the form

$$\frac{2\beta_2}{\beta(t)}\ ln\frac{t}{\alpha} < \frac{2}{\alpha t\ ln\frac{t}{\alpha}} + \frac{1}{\alpha^2}$$

or, equivalently

$$\frac{2\beta_2}{\beta(t)} < \frac{2}{\alpha t\ (ln\frac{t}{\alpha})^2} + \frac{1}{\alpha^2\ ln\frac{t}{\alpha}}. \tag{17}$$

When t is large, the terms on the RHS of (17) tends to zero and since $\beta(t) > 0$, it follows from (17) that

$$\beta_2 < 0. \tag{18}$$

(ii) Next we determine the sign of β_1. Since

$$\beta'(t) = \beta_1 + 2\beta_2 t < 0, \quad t > \alpha$$

we get

$$\alpha < t < \frac{-\beta_1}{2\beta_2}. \tag{19}$$

Since $\beta_2 < 0$, it follows from (19) that

$$\beta_1 > 0. \tag{20}$$

Setting $\beta_2 = -\beta_2^*,\ \beta_2^* > 0,\ \beta(t)$ takes the form

$$\beta(t) = \beta_0 + \beta_1 t - \beta_2^* t^2. \tag{21}$$

Remark: Note that for $t < \alpha$,

$$\beta'(t) = \beta_1 - 2\,\beta_2^* t > 0$$

implies $\beta_1 - \beta_2^* t > \beta_2^* t > 0$ and thus we have $t < \frac{\beta_1}{\beta_2^*}$.

(iii) Finally, we determine the sign of β_0.
Substitution of (21) into (6) gives

$$\beta_1 - 2\,\beta_2^* \alpha = \frac{\beta_0 + \beta_1 \alpha - \beta_2^* \alpha^2}{2\alpha}.$$

Simplifying further, the above equation reduces to

$$\beta_1 - 3\,\beta_2^* \alpha = \frac{\beta_0}{\alpha}.$$

Since $\beta_2^* > 0$, dividing each term of the above equaion by $2\,\beta_2^*$ gives

$$\frac{\beta_1}{2\,\beta_2^*} - \frac{3\alpha}{2} = \frac{\beta_0}{2\,\beta_2^* \alpha}. \tag{22}$$

To prevent potential statistical errors, Weon et.al. [9] selected the estimates with proper criteria that the vertex in the quadratic pattern is larger than age 92 years. Also from the observed data [9] the vertex cannot exceed 97.2 years . Hence (22) satisfies

$$97.2 > \frac{\beta_1}{2\,\beta_2^*} = \frac{\beta_0}{2\,\beta_2^* \alpha} + \frac{3\alpha}{2}.$$

Rewriting the above equation in the form

$$\frac{\beta_0}{2\,\beta_2^* \alpha} < 97.2 - \frac{3\alpha}{2} < 0$$

it is clear that $\beta_0 < 0$, since $\beta_2^* > 0$.

Setting $\beta_0 = -\beta_0^*,\ \beta_0^* > 0$, on account of (18) and (20) finally $\beta(t)$ takes the form, which is identical to the form recently found by Weon et al. [8].

$$\beta(t) = -\beta_0^* + \beta_1 t - \beta_2^* t^2. \tag{23}$$

5 Critical point of the mortality rate function

To find the critical point of the mortality rate function, we suggest an alternative expression for the quadratic pattern (23) given by Weon et.al. [9]. Consider [8]

$$\beta(t) = \beta(\nu) - c(\nu - t)^2 \tag{24}$$

where $\beta(\nu)$ and c are unknown constants., $\frac{\nu}{\alpha} > 1$ and $\frac{\nu}{\alpha}$ satisfies (35)
To determine c substitute (24) into (6) to get

$$2c(\nu-\alpha) = \frac{\beta(\nu) - c(\nu-\alpha)^2}{2\alpha}$$

and solving for c, we get

$$c = \frac{\beta(\nu)}{(\nu-\alpha)(\nu+3\alpha)}. \tag{25}$$

On account of (25), (24) takes the form

$$\beta(t) = \beta(\nu)\left\{1 - \frac{(\nu-t)^2}{(\nu-\alpha)(\nu+3\alpha)}\right\}. \tag{26}$$

The other unknown constant $\beta(\nu)$ can be found from (2) at $t = \nu$

$$\beta(\nu) = \frac{ln(-lnS(\nu))}{ln\frac{\nu}{\alpha}},$$

provided $S(\nu)$ is known from the given survival curve. Otherwise, $\beta(\nu)$ can be calculated numerically by graphical method.

For, let us find the critical point t^* of the mortality rate function $\mu(t)$. Here t^* is the age at which the mortality curve starts to decline. At $t = t^*$, it follows from (14) that

$$\left[\frac{\beta(t^*)}{t^*} + \beta'(t^*)ln\frac{t^*}{\alpha}\right]^2 + \frac{2}{t^*}\beta'(t^*) - \frac{\beta(t^*)}{t^{*2}} + \beta''(t)ln\frac{t}{\alpha} = 0, \quad t^* > \nu$$

since $\mu'(t^*) = 0$. Considering (26) at $t = t^*$ and susubstituting it into the last equation gives

$$\beta(\nu)\left\{\frac{1}{t^*}\left(1 - \frac{(\nu-t^*)^2}{(\nu-\alpha)(\nu+3\alpha)}\right) + \frac{2(\nu-t^*)^2}{(\nu-\alpha)(\nu+3\alpha)}ln\frac{t^*}{\alpha}\right\}^2$$
$$-\frac{1}{t^{*2}}\left(1 - \frac{(\nu-t^*)^2}{(\nu-\alpha)(\nu+3\alpha)}\right) - \frac{2\,ln\frac{t^*}{\alpha}}{(\nu-\alpha)(\nu+3\alpha)} + \frac{4(\nu-t^*)}{t^*(\nu-\alpha)(\nu+3\alpha)} = 0 \tag{27}$$

If $\beta(\nu)$ is known, then t^* in (27) can be determined by numerical methods. For instance, by Newton-Raphson method the computed values of t^* are listed in Table 1. In the absence of age-specific mortality data both $\beta(\nu)$ and t^* are unknown, we solved (27) graphically and the computed values are listed in Table 2.

Next, we prove that the alternative form of $\beta(t)$ given in (26) satisfies the properties of the quadratic pattern of $\beta(t)$ given by Weon et.al. [9].

From the computed values of $\beta(t)$, it is clear that $\beta(\nu) > 1$. Since

$$1 - \frac{(\nu - t)^2}{(\nu - \alpha)(\nu + 3\alpha)} < 1,$$

it follows from (26) that $\beta(t) < \beta(\nu)$.
At $t = \alpha$, (26) takes the form

$$\beta(\alpha) = \beta(\nu)\left\{1 - \frac{(\nu - t)^2}{(\nu - \alpha)(\nu + 3\alpha)}\right\}.$$

Simplifying further, the last equation reduces to

$$\beta(\alpha) = \beta(\nu)\left\{\frac{4\alpha}{\nu + 3\alpha}\right\} \tag{28}$$

or, equivalently

$$\frac{\beta(\nu)}{\beta(\alpha)} = \frac{\nu + 3\alpha}{4\alpha} > 1.$$

Differentiating (26) with respect to t we have

$$\beta'(t) = \frac{2\ \beta(\nu)(\nu - t)}{(\nu - \alpha)(\nu + 3\alpha)} \tag{29}$$

Clearly, $\beta(t)$ increases with age at $t < \alpha$ and decreases with age at $t > \alpha$. When $t = \nu$ (29) satsfies

$$\beta'(t)/_{t=\nu} = 0.$$

6 Zero of the mortality rate function

Finally, to satisfy (4), we find the point (maximum human lifespan) at which the mortality rate function $\mu(t)$ given by (3) is zero.

$$\frac{\beta(t)}{t} + \beta'(t)\ ln\frac{t}{\alpha} = 0. \tag{30}$$

Substitution of (26) into (30) gives

$$\frac{2\ \beta(\nu)(\nu - t)}{(\nu - \alpha)(\nu + 3\alpha)} = \frac{-\beta(\nu)\{1 - \frac{(\nu-t)^2}{(\nu-\alpha)(\nu+3\alpha)}\}}{t\ ln\frac{t}{\alpha}}.$$

After a little algebra, the last equation reduces to

$$t^2\ (1 + 2\ ln\frac{t}{\alpha}) - 2\ \nu\ t\ (1 + ln\frac{t}{\alpha}) - 2\ \alpha\ \nu + 3\ \alpha^2 = 0. \tag{31}$$

Notice that (31) does not involve $\beta(\nu)$. We now prove that equation (31) indeed has a solution. Divide (31) by t^2 and substitute $\frac{\alpha}{t} = x, x < 1$ to get

$$1 - \frac{2\ \nu x}{\alpha} - 2\ lnx\ (1 - \frac{\nu x}{\alpha}) + (3 - \frac{2\nu}{\alpha})x^2 = 0. \tag{32}$$

Since $x < 1$ and (32) involves logarithmic function, we seek the solution of the form $x = e^{-y}, y > 0$. Substitution of $x = e^{-y}$ into (32) gives

$$1 + 2\ y - \frac{2\nu}{\alpha}\ e^{-y}(1 + y) + (3 - \frac{2\nu}{\alpha})\ e^{-2y} = 0. \tag{33}$$

Theorem 1 : Equation (33) to admit a solution it is necessary that $e^{-y} > \frac{1}{2}$ or, equivalently $y < ln\ 2$.

Proof

We prove the theorem by contradiction.
Suppose that $e^{-y} < \frac{1}{2}$ or, equivalently $y > ln\ 2$. Notice that $e^{-y}\ (1+y)$ attains its maximum only at $y = 0$. Hence (33) can be written as

$$1 - \frac{\nu}{\alpha} + y\ (2 - \frac{\nu}{\alpha}) + (3 - \frac{2\nu}{\alpha})\ e^{-2y} < 0. \tag{34}$$

As $y > ln\ 2$, it follows from (34) that

$$e^{-2y}\ <\ \frac{(\frac{\nu}{\alpha} - 1)(1 + ln\ 2) - ln\ 2}{3 + \frac{2\nu}{\alpha}}\ <\ 0, \tag{35}$$

$for\ the\ tabulated\ values\ of\ \nu\ and\ \alpha.$ On the otherhand, $e^{-2y} > 0$, for any $y > 0$. Hence we conclude that (34) is false, implying that

$$1 - \frac{\nu}{\alpha} + y\ (2 - \frac{\nu}{\alpha}) + (3 - \frac{2\nu}{\alpha})\ e^{-2y} \geq 0 \text{ when } e^{-y} < \frac{1}{2}.$$

Further, when $e^{-y} = \frac{1}{2}$ (34) gives

$$1 - \frac{\nu}{\alpha} - ln\ 2\ (2 - \frac{\nu}{\alpha}) + (3 - \frac{2\nu}{\alpha})\ \frac{1}{4} > 0$$

$for\ the\ tabulated\ values\ of\ \nu\ and\ \alpha.$ Hence (33) admit a solution only for $e^{-y} > \frac{1}{2}$ or, equivalently $y < ln\ 2$. The proof is complete.

To find the interval of existence of solution, put $y = 0$ in (33). It is trivial to verify that (33) assumes negative value when $y = 0$. Hence the solution of the equation (33) exists only in the interval $0 < y < ln\ 2$.

Theorem 2 : Equation (33) has a unique solution in the interval $0 < y < ln\ 2$.

Proof

To prove the uniqueness, we show that the function

$$F(y) = 1 + 2\,y - \frac{2\nu}{\alpha}\,e^{-y}(1+y) + (3 - \frac{2\nu}{\alpha})\,e^{-2y}$$

is strictly increasing in the interval $0 < y < ln\ 2$. Upon differentiation, the last equation gives

$$F'(y) = 2 + \frac{2\nu}{\alpha}\,y\,e^{-y} - 6\,e^{-2y} + \frac{4\nu}{\alpha}\,e^{-2y}. \tag{36}$$

Suppose that $F'(y) \leq 0$. Then from (36) we have

$$2 + \frac{2\nu}{\alpha}\,y\,e^{-y} - 6\,e^{-2y} + \frac{4\nu}{\alpha}\,e^{-2y} \leq 0.$$

Since $e^{-y} > e^{-2y}$ and $2 > 2\,e^{-2y}$ for $y > 0$, the last equation reduces to

$$2\,e^{-2y}[1 - 3 + \frac{\nu}{\alpha}(y+2)] \leq 0. \tag{37}$$

After a little algebra we get

$$\frac{\nu}{\alpha}(y+2) \leq 2.$$

Since $\frac{\nu}{\alpha} > 1$ and $y + 2 > 2$ for any $y > 0$, we have $\frac{\nu}{\alpha}(y+2) > 2$. Thus we arrive at a contradiction. Hence $F'(y) > 0$ for any $y > 0$ implying that $F(y)$ is strictly increasing for any $y > 0$ and in particular, in the interval $0 < y < ln\ 2$. Hence the solution is unique.

By taking the midpoint ($\frac{ln\ 2}{2}$) of the interval $0 < y < ln\ 2$, we found that the LHS of (33) approximates to 0.0339408. Hence the approximate solution of equation (33) : $y = \frac{ln\ 2}{2}$. From this we can conclude that when $92 < \nu < 97.2$ and $\frac{\nu}{\alpha} > 1$, the maximum human lifespan t_m is approximately $\sqrt{2}$ times the characteristic life α. The numerical values of $t_m \approx \sqrt{2}\alpha$ are listed in Table 2.

Conclusion : The significance of the new expression of the age-dependent stretched exponent $\beta(t)$ is that the three unknown constants $\beta_0, \beta_1, \beta_2$ in the

Table 1: (reprinted from [8])

α(years)	ν(years)	$\beta(\nu) \geq$	$t^* > \nu$(years)
88.57223	96.3447	1.8	98.2
		5.2	105.562
		12	111.056
87.4567	96.9	1.6	97.14
		4	105.271
		10	112.034
86.6991	97.2	1.8	98.13
		4	106.25
		10	113.018

quadratic expression of $\beta(t)$ given by Weon et.al. [9] reduces to two unknown constants c and $\beta(\nu)$, if α and ν are measurable in a survival curve.

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Table 2 (reprinted from [8])

α(years)	ν(years)	$\beta(\nu) \geq$	$t^* > \nu$(years)	$t_{max} \approx \sqrt{2}\alpha$(years)
88.57223	96.3447	1.8	98	
		5.2	105	
		8	108	125.26
		12	111	
88.1960	95.9	1.9	97	
		4	103	
		10	109	124.728
		13	111	
87.4567	96.9	1.6	97	
		4	105	
		10	112	123.682
		13	113	
87.6732	95.8	1.9	97	
		4	103	
		10	110	123.989
		13	111	
86.7532	95.6	1.9	97	
		4	103	
		9	109	122.688
		12	111	
86.6991	97.2	1.8	98	
		4	106	
		8	111	122.611
		10	113	
85.0216	94	1.8	95	
		4	102	
		8	107	120.239
		13	110	
85.8874	95.4	1.7	96	
		3	101	
		10	110	121.463
		13	112	
84.2641	96	1.8	97	
		4	105	
		8.5	111	119.167
		13	114	
84.9134	94.7	1.7	95	
		3	100	
		10	110	120.086
		13	111	

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