Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 4, 169 - 175

On Semi Pseudo Symmetric Manifolds Admitting a Type of Quarter Symmetric Metric Connection

M. Tarafdar

Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Kolkata: 700019, India
drmanjushamajumdar@gmail.com

J. Sengupta

Department of Mathematics
University of North Bengal
P.O. North Bengal University
District Darjeeling, 734013, India
joydeep1972@yahoo.com

S. Chakraborty

Directorate of Distance Education
University of North Bengal
P.O. North Bengal University
District Darjeeling, 734013, India
sarani121179@yahoo.co.in

Abstract

The present paper deals with Semi Pseudo Symmetric manifolds $(\mathrm{SPS})_{\mathrm{n}}$ $(n > 3)$ admitting a quarter-symmetric metric connection $\overline{\nabla}$ whose torsion tensor $\overline{T}$ is given by

$$\overline{T}(X, Y) = A(Y)LX - A(X)LY$$

and whose curvature tensor $\overline{R}$ satisfy the condition

$$\overline{R}(X, Y)Z = 0.$$

It is shown that a $(\mathrm{SPS})_{\mathrm{n}}$ $(n>3)$ admitting such a quarter symmetric metric connection is a $(\mathrm{PRS})_{\mathrm{n}}$ with zero scalar curvature provided the vector field ρ as defined above is a parallel vector field.

Further it is shown that in such a $(\mathrm{SPS})_{\mathrm{n}}$ the Ricci tensor is a Codazzi tensor.

It is further shown that such a $(\mathrm{SPS})_{\mathrm{n}}$ is a Ricci recurrent manifold, whose 1-form of recurrence is four times the associated 1-form of the manifold, whose scalar curvature is zero. Finally, it is shown that, in such a $(\mathrm{SPS})_{\mathrm{n}}$, the conformal curvature tensor is conservative.

Mathematics Subject Classification: Primary 53C25

Keywords: Semi Pseudo Symmetric Manifold, Pseudo Ricci Symmetric Manifold, Quarter Symmetric Connection, Parallel Vector Field.

1 Introduction

In a paper [6], Tarafdar and Jawarneh introduced a type of non-flat Riemannian manifold (M^n, g) $(n > 3)$ whose curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)W &= 2A(X)R(Y,Z)W + A(Y)R(X,Z)W \\ &\quad + A(Z)R(Y,X)W + A(W)R(Y,Z)X\end{aligned} \tag{1}$$

where A is a non-zero 1-form

$$g(X,\rho) = A(X) \tag{2}$$

for every smooth vector field X and ∇ denotes the operator of covariant differentiation with respect to the metric g. Such a manifold was called a semi-Pseudo Symmetric manifold, A was called its associated 1-form and an n-dimensional manifold of this kind was denoted by $(\mathrm{SPS})_{\mathrm{n}}$. In a paper [1], Chaki introduced a type of non-flat Riemannian manifold $(M^n, g)(n > 3)$ whose Ricci tensor S of type (0, 2) satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X) \tag{3}$$

where A and ∇ have the meanings already stated. Such a manifold was called a Pseudo Ricci Symmetric manifold and an n-dimensional manifold of this kind was denoted by $(PRS)_n$.

The present paper deals with Semi Pseudo Symmetric manifolds $(\mathrm{SPS})_{\mathrm{n}}$ $(n > 3)$ admitting a quarter-symmetric metric connection $\overline{\nabla}$ whose torsion tensor $\overline{T}$ is given by

$$\overline{T}(X,Y) = A(Y)LX - A(X)LY \tag{4}$$

and whose curvature tensor $\overline{R}$ satisfy the condition

$$\overline{R}(X,Y)Z = 0. \tag{5}$$

It is shown that a $(\mathrm{SPS})_{\mathrm{n}}$ $(n > 3)$ admitting such a quarter symmetric metric connection is a $(\mathrm{PRS})_{\mathrm{n}}$ with zero scalar curvature provided the vector field ρ defined by (2) is a parallel vector field.

Further it is shown that in such a $(\mathrm{SPS})_{\mathrm{n}}$ the Ricci tensor is a Codazzi tensor.

Using a result in [1], it is further shown that such a $(\mathrm{SPS})_{\mathrm{n}}$ is a Ricci recurrent manifold, whose 1-form of recurrence is four times the associated 1-form of the manifold, whose scalar curvature is zero.

Finally, it is shown that, in such a $(\mathrm{SPS})_{\mathrm{n}}$, the conformal curvature tensor is conservative.

2 Preliminaries

Let r denote the scalar curvature and L denote the symmetric endomorphism of the tangent space at each point of $(\mathrm{SPS})_{\mathrm{n}}$ corresponding to the Ricci tensor S i.e.

$$g(LX,Y) = S(X,Y) \tag{1.1}$$

for every vector field X, Y.

From (1) we get [6]

$$\begin{aligned}(\nabla_X S)(Y,Z) &= 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X) \\ &\quad + A(R(X,Y)Z).\end{aligned} \tag{1.2}$$

Contracting (1.2) we get

$$dr(X) = 2A(X)r + 3A(LX). \tag{1.3}$$

Again from (1.2),

$$\begin{aligned}(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) &= A(X)S(Y,Z) - A(Y)S(X,Z) \\ &\quad +2\,A(R(X,Y)Z),\end{aligned} \tag{1.4}$$

$$\begin{aligned}(\nabla_X L)Y - (\nabla_Y L)X &= A(X)LY - A(Y)LX \\ &\quad +2\,R(X,Y)\rho.\end{aligned} \tag{1.5}$$

Contracting (1.4) we get

$$dr(X) = 2A(X)r + 2A(LX). \tag{1.6}$$

Hence from (1.3) and (1.6) it follows that

$$A(LX) = g(LX,\rho) = S(X,\rho) = 0. \tag{1.7}$$

We shall use these formulas later.

2 $(\mathbf{SPS})_{\mathbf{n}}$ $(n > 3)$ admitting a special type of quarter symmetric metric connection

A linear connection $\overline{\nabla}$ is called a metric connection if for all vector fields X, Y, Z,

$$(\overline{\nabla}_X g)(Y,Z) = 0.$$

If ∇ is the Levi-Civita connection of the manifold with respect to the metric g, then the quarter symmetric metric connection $\overline{\nabla}$ [4] is given by

$$\overline{\nabla}_X Y = \nabla_X Y + A(Y)LX - S(X,Y)\rho, \tag{2.1}$$

where ρ is given by (2).

In virtue of (2.1), from a known result [4] we get,

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z - M(Y,Z)LX + M(X,Z)LY \\ &\quad -S(Y,Z)QX + S(X,Z)QY \\ &\quad +A(Z)\{(\nabla_X L)(Y) - (\nabla Y L)(X)\} \\ &\quad -\{(\nabla_X S)(Y,Z) - (\nabla Y S)(X,Z)\}\,\rho\end{aligned} \tag{2.2}$$

where M is a tensor field of type (0,2) defined by

$$\begin{aligned}M(X,Y) &= g(QX,Y) \\ &= (\nabla_X A)Y - A(LX)A(Y) + \frac{1}{2}A(\rho)S(X,Y),\end{aligned} \tag{2.3}$$

and Q is a tensor field of type (1,1) defined by

$$QX = \nabla_X\rho - A(LX)\rho + \frac{1}{2}\,A(\rho)LX. \tag{2.4}$$

In this paper, on using (1.7) we get

$$M(X,Y) = (\nabla_X A)(Y) + \frac{1}{2}\,A(\rho)\,S(X,Y) \quad \text{and} \tag{2.5}$$

$$QX = \nabla_X\rho + \frac{1}{2}\,A(\rho)LX. \tag{2.6}$$

Using (1.4), (1.5) (2.5) and (2.6), we get from (2.2)

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z \\ &+ \{(\nabla_X A)(Z) + A(X)A(Z) + \frac{1}{2}\,S(X,Z)A(\rho)\}LY \\ &- \{(\nabla_Y A)(Z) + A(Y)A(Z) + \frac{1}{2}\,S(Y,Z)A(\rho)\}LX \\ &+ S(X,Z)\{\nabla_Y\rho + A(Y)\rho + \frac{1}{2}\,A(\rho)LY\} \\ &- S(Y,Z)\{\nabla_X\rho + A(X)\rho + \frac{1}{2}\,A(\rho)LX\} \\ &- 2\,A(Z)R(X,Y)\rho - 2A(R(X,Y)Z)\rho.\end{aligned} \tag{2.7}$$

On using (5) we get

$$\begin{aligned} R(X,Y)Z &= \{(\nabla_Y A)(Z) + A(Y)A(Z) + \frac{1}{2}\, S(Y,Z)A(\rho)\}LX \\ &- \{(\nabla_X A)(Z) + A(X)A(Z) + \frac{1}{2}\, S(X,Z)A(\rho)\}LY \\ &- S(X,Z)\{\nabla_Y \rho + A(Y)\rho + \frac{1}{2}\, A(\rho)LY\} \\ &+ S(Y,Z)\{\nabla_X \rho + A(X)\rho + \frac{1}{2}\, A(\rho)LX\} \\ &+ 2\, A(Z)R(X,Y)\rho + 2A(R(X,Y)Z)\rho. \end{aligned}$$

Again using (1.7) we get from above

$$\begin{aligned} g(R(X,Y)Z,\rho) &= S(Y,Z)g(\nabla_X \rho + A(X)\rho, \rho) \\ &- S(X,Z)g(\nabla_Y \rho + A(Y)\rho, \rho) \\ &+ 2\, A(Z)g(R(X,Y)\rho, \rho) + 2A(R(X,Y)Z)g(\rho,\rho). \end{aligned} \tag{2.8}$$

If in particular, ρ is a parallel vector field, then

$$\nabla_X \rho = 0 \tag{2.9}$$

and hence

$$R(X,Y)\rho = \nabla_X \nabla_Y \rho - \nabla_Y \nabla_X \rho - \nabla_{[X,Y]}\rho = 0 \tag{2.10}$$

and

$$A(R(X,Y)Z) = g(R(X,Y)Z,\rho) = -g(R(X,Y)\rho, Z) = 0. \tag{2.11}$$

Using (2.11) and (2.10) we find from (2.8)

$$A(X)S(Y,Z)A(\rho) - A(Y)S(X,Z)A(\rho) = 0,$$

and hence

$$A(X)S(Y,Z) - A(Y)S(X,Z) = 0, \tag{2.12}$$

as $A(\rho) \neq 0$.

Also, from (1.2), we find on using (2.11)

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X).$$

This shows in virtue of (3), that the $(\mathrm{SPS})_{\mathrm{n}}$ under consideration is a $(\mathrm{PRS})_{\mathrm{n}}$.

Next, contracting (2.12) we obtain

$$A(X)r = 0,$$

as $g(LX, \rho) = 0$ by (1.7).

Thus

$$r = 0, \tag{2.13}$$

as $A(X) \neq 0$.

Summing up, we state

Theorem 2.1. *If a (SPS)$_n$ ($n > 3$) admits a quarter symmetric metric connection, whose curvature tensor vanishes, then it reduces to $(PRS)_n$ with zero scalar curvature.*

Using (2.12), we see from (1.40) that

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = 0.$$

Thus the Ricci tensor is a Codazzi tensor. We know [1] that if the Ricci tensor of a (PRS)$_n$ is a Codazzi tensor, then the manifold is a Ricci recurrent manifold whose 1-form of recurrence is four times the associated 1-form of the manifold. Hence we state

Theorem 2.2. *If a (SPS)$_n$ ($n > 3$) admits a quarter symmetric metric connection whose curvature tensor vanishes, then it is a Ricci recurrent manifold, where ρ, defined by (2)is a parallel vector field.*

It is known [2] that in a Riemannian manifold (M^n, g) $(n > 3)$,

$$\begin{aligned}(\text{div}C)(X, Y)Z \;&=\; \frac{n-3}{n-2}\,\{(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X)\} \\ &+\; \frac{1}{2(n-1)}\{g(X, Y)dr(Z) - g(Y, Z)dr(X)\}.\end{aligned}$$

Using (2.13) and the fact that the Ricci tensor is a Codazzi tensor, we now claim that

Theorem 2.3. *If a (SPS)$_n$ ($n > 3$) admits a quarter symmetric metric connection whose curvature tensor vanishes, then its conformal curvature is conservative [3], provided ρ defined by (2) is a parallel vector field.*

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Received: July, 2010