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Integro Differential Equation and Fundamental Problems of an Infinite Plate with a Curvilinear Hole Having Strong Pole

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Abstract

In the present paper, Cauchy singular method has been applied to derive exact and closed expressions for Goursat functions for the first and second fundamental problems of an infinite plate weakened by a hole having strong pole and arbitrary shape. The hole is conformally mapped onto domain outside a unit circle by means of rational mapping function. Some applications are investigated. Many interesting cases when the shape of the hole takes different shapes are included as special cases.

Keywords: Complex variables method, rational mapping, first and second fundamental problems, an infinite plate

1. Introduction

Problems dealing with isotropic homogeneous perforated infinite plate have been investigated by several authors [1-8]. Some of them [1, 2, 3] used Laurent's theorem to express each potentials as a power series, others [4-8] used complex variables method of Cauchy type integrals, to obtain the solution in the form of Goursat functions.

Consider a thin infinite plate of thickness h with a curvilinear hole C, where the origin lies inside the hole, conformally mapped on the domain outside a unit circle γ by means of a rational mapping function $z = c\omega(\zeta)$,
$c > 0$, subject to the condition $\omega'(\zeta)$ dose not vanish or become infinite outside the unit circle γ, $\xi = e^{i\psi}, 0 \leq \psi \leq 2\pi$.

It is known that [4], the first and second fundamental problems in the plane theory of elasticity are equivalent to finding two analytic functions, $\phi_1(z)$ and $\psi_1(z)$ of one complex argument $z = x + iy$. These functions must satisfy the boundary conditions

$$k\phi_1(t) - t\overline{\phi_1'(t)} - \overline{\psi(t)} = f(t), \qquad (1.1)$$

where for the first fundamental problem $k = -1$, $f(t)$ is a given function of stresses, while for the second fundamental problem $k = \chi = \frac{\lambda+3\mu}{\lambda+\mu} > 1$,
$f(t)$ is a given function of the displacement, χ is called Muskhelishvili's constant; λ, μ are called the Lame's constants and t denotes the affix of appoint on the boundary.

A very powerful method for solving the elastic problems makes use of conformal mapping to reduce the problem, for any given region whose boundary C satisfies certain regularity conditions, to corresponding problem for a region having a unit circle. In terms of the rational mapping function $z - c\omega(\zeta), c > 0,\ \omega'(\zeta)$ dose not vanish or become infinite for $|\zeta| > 1$, then the infinite region outside a closed contour may be conformally mapped outside the unit circle γ. This method is known as Cauchy method; where the fundamental problems of the first and second kind are directly take the form of an integro differential equation with Cauchy kernel.

In the absence of body forces, it is known that [4], the stress components, in the theory of elasticity, take the form

$$\sigma_{xx} + \sigma_{yy} = a\,Re\{\phi'(z)\}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2[\bar{z}\phi''(z) + \psi'(z)] \qquad (1.2)$$

where the two complex functions of potential $\phi_1(t)$ and $\psi_1(t)$ respectively, take the form (see [3]).

$$\phi_1(\zeta) = -\frac{X + iY}{2\pi(1+\chi)}\ln\zeta + \phi(\zeta),$$

$$\psi_1(\zeta) = -\frac{\chi(X - iY)}{2\pi(1+\chi)}\ln\zeta + c\Gamma^*\zeta + \psi(\zeta), \qquad (1.3)$$

Here, X, Y are the components of resultant vector of all external forces; acting on the boundary; Γ, Γ^* represented the stresses at infinity. Generally, the two complex functions $\phi(\zeta), \psi(\zeta)$ are single valued analytic within the region outside the unit circle γ and $\phi(\infty) = \psi(\infty) = 0$, which means that $\phi(z)$ and $\psi(z)$ are homomorphic functions at infinity. It will be assumed that $\Gamma = \bar{\Gamma}$ and $X = Y = 0$ for the first fundamental problem.

The rational function mapping $z = c\omega(\zeta)$ has a singularity is discussed by several authors (see [4-8]), but for strong singularity, in this domain of fundamental problems in the theory of elasticity, there are no papers, for this time, explained it. For this, our attention is obtaining the two analytic functions $\phi(z)$ and$\psi(z)$, when the rational mapping $z = c\omega(\zeta)$ has a strong singularity.

In this work, the two analytic functions, Goursat functions, are obtained, when the rational mapping has a pole of strong singularity. Many special cases are derived from the work. Also, holes corresponding to certain combinations of the parameters of the rational mapping are sketched (see Figs 1-4). Some applications of the first and second fundamental problems, in this case, are investigated.

2. Method of solution

$$, \left(c > 0, |n| < 1, \left| \frac{1-m}{m+3} \right| < 1 \right)$$

Using the conformal mapping

$$z = c\,\omega(\zeta) = c\frac{\zeta + m\zeta^{-1}}{(1 - n\zeta^{-1})^2} \qquad (2.1)$$

where m and n are real parameters restricted such that $\omega'(\zeta)$ dose not vanish or become infinite outside γ ,see figs (1-4). The expression $\omega(\zeta^{-1})/\omega'(\zeta)$ can be written in the from

$$\frac{\omega(\zeta^{-1})}{\omega'(\zeta)} = (\zeta^{-1}) + \beta(\zeta), \qquad (2.2)$$

where $(\zeta) = \frac{h}{(\zeta - n)^2}$,

$$h = (1-n^2)^2 J^{-2}(n)[3J(n)(m - 6n^2 - 3n^3 - 4mn^2) \\ - nJ'(n)(1-n^2)(n^2+m)],$$

$$J(\zeta) = 1 - 3n\zeta - m\zeta^2 - nm\zeta^3 \tag{2.3}$$

and $\beta(\zeta)$ is a regular function for $|\zeta| > 1$.
Using (1.2), (2.1), (2.2), in (1.1), we have

$$k\phi(\zeta) - \alpha(\sigma)\overline{\phi'(\alpha)} - \psi(\sigma) = f^*(\sigma), \tag{2.4}$$

where

$$\psi_*(\zeta) = \psi(\zeta) + \beta(\zeta)\phi'(\zeta)$$
$$f^*(\zeta) = F(\zeta) - ck\Gamma\zeta + c\overline{\Gamma^*}\zeta^{-1} + N(\zeta)\big(\alpha(\zeta) + \overline{\beta(\zeta)}\big),$$
$$N(\zeta) = c\Gamma - \frac{(X - iY)}{2\pi(1+\chi)}\zeta, \tag{2.5}$$

and

$$F(\zeta) = f(t). \tag{2.6}$$

Assume that the derivatives of $F(\sigma)$ must satisfy the Hölder condition.

$$|F(\sigma_1) - F(\sigma_2)| = |\sigma_1 - \sigma_2|^{e} \qquad (0 < e < 1) \tag{2.7}$$

Multiplying both sides of (2.4) by $\frac{d\sigma}{2\pi i(\sigma-\zeta)}$ and integrating with respect to σ on γ, one has

$$k\phi(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\alpha(\sigma)\overline{\phi'(\zeta)}}{\sigma - \zeta}\, d\sigma =$$
$$-cF^*\zeta^{-1} - \frac{h(X - iY)}{2\pi(1+\chi)(n-\zeta)} + \frac{hN(n)}{(n-\zeta)^2} - A(\zeta), \tag{2.8}$$

where

$$A(\zeta) = -\frac{1}{2\pi i}\sum_{\eta=0}^{\infty} \zeta^{-\eta-1}\int \sigma^n F(\sigma)d\sigma, \qquad |\zeta| > 1 \tag{2.9}$$

Using (2.2), we can write

$$\frac{1}{2\pi i}\int_\gamma \frac{\alpha(\sigma)\overline{\phi'(\zeta)}}{\sigma - \zeta}\, d\sigma = h\left[\frac{b_1}{n-\zeta} + \frac{b_2}{(n-\zeta)^2}\right], \tag{2.10}$$

where b_1, b_2 are complex constants to be determined.

For determining b_1, b_2 we introduce (2.10) in (2.8) then, we differentiate the result to obtain $\phi'(\zeta)$ and substitute it again in (2.10), to get

$$b_1 = \frac{(k+2hv^3)Re\,L - 2kc_2 Re\,M}{(c_1 h - k)(k+2hv^3) - 2khc_2 v^2} + i\,\frac{(k-2hv^3)Im\,L - 2c_2 h\,Im\,M}{(c_1 h + k)(k-2hv^3) - 2c_2 v^2 h^2},$$

$$b_2 = \frac{hv^2 Re\,L + (k - c_1 h)Re\,M}{(c_1 h - k)(k+2hv^3) + 2khc_2 v^2} \tag{2.11}$$

where

$$L = 2cn\Gamma^* + c_1 h\left(\frac{X - iY}{2\pi(1+\chi)}\right) + 2c_2 h\,\overline{N(n)} + \overline{A''(n)},$$

$$M = \overline{A'(n)} + 2cn^2\Gamma^* + v^2 h\left(\frac{X - iY}{2\pi(1+\chi)}\right) + 2hv^3\overline{N(n)}$$

$$c_1 = \frac{2v^2}{n}(1+v), \quad c_2 = \frac{3v}{2}c_1, \quad v = \frac{n}{1-n^2}, \quad (c > 0)\,. \tag{2.12}$$

Hence, the first Goursat function $\phi(\zeta)$ takes the form

$$k\phi(\zeta) = -cF^*\zeta^{-1} + \frac{h}{\zeta - n}\left[b_1 + \frac{X - iY}{2\pi(1+\chi)}\right] + \frac{h}{(\zeta - n)^2}[N(n) - b_2] - A(\zeta) \tag{2.13}$$

Also, following the same previous way, using the boundary conditions (2.4), the second Goursat function $\psi(\zeta)$, for the first and second fundamental problems, takes the form

$$\psi(\zeta) = ck\bar{\Gamma}\zeta^{-1} - \frac{\omega(\zeta^{-1})}{\omega'(\zeta)}\phi^*(\zeta) + \frac{h\zeta^2}{(1-n\zeta)^2}\phi^*(n^{-1}) + B(\zeta) - B\,, \tag{2.14}$$

where

$$\phi^*(\zeta) = \phi'(\zeta) + \overline{N(\zeta)}\,, \qquad B(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\overline{F(\sigma)}}{\sigma - \zeta}\,d\sigma$$

$$B = \frac{1}{2\pi i}\int_\gamma \overline{F(\sigma)}\frac{d\sigma}{\sigma} \tag{2.15}$$

3. Special cases

(i) For $n = 0$, $0 \le m \le 1$, we get the function $z = c(\zeta + m\zeta^{-1})$ and (3.12), (3.13) agree with (82.4), (82.5), (83.10) and (83.11) of Muskelishvilis, result obtained for the elliptic hole [4].

(ii) For $m = -1$ the boundary *C* degenerate into a circular cut having strong pole. For values of m near -1 the edge of the hole resembles the shape of a crescent of strong pole.

(iii) For $m = -n^2$, we have a rational mapping function $\zeta(\zeta + n)/(\zeta - n)$, which has a pole at $\zeta = n$.

4. Applications

(i) For $k = -1$, $\Gamma = \frac{1}{4}P$, $\Gamma^* = -\frac{1}{2}Pe^{-2i\theta}$ and $X - Y - f - 0$, we have an infinite plate weakened by the curvilinear hole *C* which having singularity and is free from stresses. The plate stretched at infinity by the application of a uniform tensile of intensity p, making an angle θ with the x-axis.

The two complex functions can be represented by the formulae

$$\phi(\zeta) = \frac{-cp}{2}e^{2i\theta}\zeta^{-1} + \frac{ch}{n-\zeta}\left(b_1 + \frac{p}{4}\right) + \frac{ch}{(n-\zeta)^2}\left(b_2 + \frac{p}{4}\right), \tag{4.1}$$

$$\psi(\zeta) = \frac{-cp}{\zeta} - \frac{\omega(\zeta^{-1})}{\omega'(\zeta)}\phi^*(\zeta) + \frac{h\zeta^2}{(1-n\zeta^2)}\phi^*(n^{-1}), \tag{4.2}$$

where b_1, b_2 are given, respectively, in the form

$$b_1 = p\frac{(1-2hv^2)\left(n\cos 2\theta + \frac{1}{2}c_2 k\right) - c_2(2n^2\cos 2\theta + hv^3)}{(1+hc_1)(2hv^3-1) + 2hc_2v^3}$$
$$+inp\sin 2\theta\frac{(1+2hv^3-2nhc_2)}{(1+2hv^3)(1-hc_1) - 2c_2v^2h^2},$$

$$b_2 = phv^2\left(\frac{c_2}{2}h - n\cos 2\theta\right) + (1+hc_1)\left(n^2\cos 2\theta + \frac{hv^2}{2}\right)$$
$$+inp\sin 2\theta\frac{n(c_1h-1) - hv^2}{(1+2hv^3)(1-hc_1) + 2v^2c_2h^2} \tag{4.3}$$

and

$$\phi^*(\zeta) = \phi'(\zeta) + \frac{cp}{4} \tag{4.4}$$

(ii) For $k = -1$, $X = Y = \Gamma = \Gamma^* = 0$ and $f(t) = Pt$, we have an infinite plate which is weakened by a curvilinear hole C, when there is no external forces and the edge of the hole is subject to a uniform pressure P. The two analytic functions, in this case, are given by the following formulas

$$\phi(\zeta) = \frac{-hb_3}{(\zeta - n)} + \frac{hb_4}{(\zeta - n)^2} + P\frac{m\zeta + \zeta^3}{(\zeta - n)^2} \tag{4.5}$$

$$\psi(\zeta) = -\frac{\omega(\zeta^{-1})}{\omega'(\zeta)}\left\{\frac{hb_3}{(\zeta - n)^2} - \frac{2hb_4}{(\zeta - n)^3} - 2P\frac{m\zeta + \zeta^3}{(\zeta - n)^3} + 2P\frac{m + 3\zeta^2}{(\zeta - n)^2}\right.$$
$$\left. + \frac{h\zeta^2\nu^2}{(1 - n\zeta)^2}\left[h(b_3 + 2b_2\nu) + 2P\nu\left(m + \frac{1}{n^2}\right) - 2P\left(m + \frac{3}{n^2}\right)\right] - P\zeta^{-1}\right\} \tag{4.6}$$

where

$$b_3 = \frac{(2h\nu^3 - 1)A''(n) + 2c_2A'(n)}{(1 + c_1h)(2h\nu^3 - 1) + 2hc_2\nu^2}\ , \quad b_4 = \frac{h\nu^2\overline{A_1''(n)} - (1 + c_1h)A_1'(n)}{(2h\nu^3 - 1)(1 + c_1h) - 2hc_2\nu^2}$$

$$A_1'(n) = P\nu^2[m + 3n^2 - 2n\nu(m + n^2)]\ , \quad \nu = \frac{n}{(1 - n^2)}$$

$$\overline{A_1''(n)} = 2\nu^3P[3n\nu(m + n^2) - (m + 3n^2)] \tag{4.7}$$

(iii) For $k = -1$, $X = Y = \Gamma = \Gamma^* = 0$ and $f(t) - iTt$, we have the case of the first fundamental problem, when the edge of the hole is subjected to uniform tangential stress T and the result of Goursat functions is obtained directly from (4.5) and (4.6) by putting $-iT$ instead of P.

(iv) For $k = x$, $\Gamma = \Gamma^* = \bar{\Gamma} = f(t) = 0$, the Goursat functions become

$$\phi(\zeta) = \frac{hb_5}{\kappa(n - \zeta)} + \frac{hb_6}{\kappa(n - \zeta)^2}, \tag{4.8}$$

$$\psi(\zeta) = \frac{h\omega(\zeta^{-1})}{\kappa\omega'(\zeta)}\left[\frac{b_5}{(\zeta - n)} - \frac{b_6}{(\zeta - n)^2}\right] + \frac{\nu^2h^2\zeta^2}{\kappa(1 - n\zeta)^2}\left(\overline{b_5} + 2\nu\overline{b_6}\right)$$
$$- \frac{\omega(\zeta^{-1})}{\omega'(\zeta)}\frac{X - iY}{2\pi(1 + \chi)}\zeta \tag{4.9}$$

where

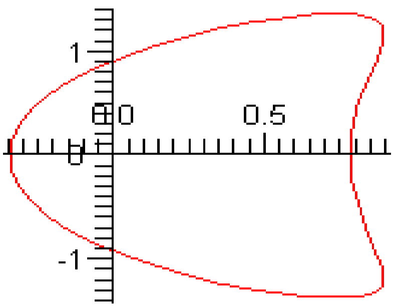
$$b_5 = \frac{X}{2\pi(1 + \kappa)}\left[\frac{(\kappa + 2h\nu^3)(c_1h - 2nc_2) - 2\kappa c_2\nu^2h(1 - 2n\nu)}{(c_1h + \kappa)(\kappa + 2h\nu^3) + 2\kappa hc_2\nu^2}\right]$$

$$+i\frac{Y}{2\pi(1+\kappa)}\left[\frac{(\kappa-2h\nu^3)(2nc_2-c_1h)-2c_2h^2\nu^2(2n\nu-1)}{(\kappa-2h\nu^3)(c_1h+\kappa)-2c_2h^2\nu^2}\right]$$

$$b_6=\frac{X}{2\pi(1+\kappa)}\left[\frac{h\nu^2(c_1h-2nc_2)+(\kappa-c_1h)(1-2n\nu)\nu^2h}{(\kappa+2h\nu^3)(c_1h-\kappa)+2h\kappa c_2\nu^2}\right]$$

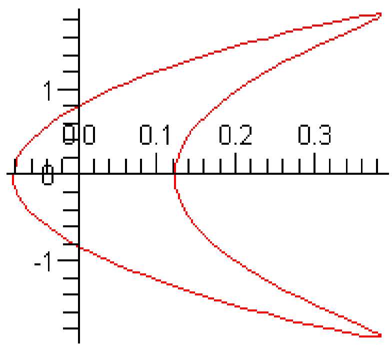
$$+i\frac{Y}{2\pi(1+\kappa)}\left[\frac{\nu^2h(2n\nu-1)(c_1h+\kappa)-\nu^2h(2nc_2-c_1h)}{(\kappa-2h\nu^3)(c_1h+\kappa)+2c_2h^2\nu^2}\right] \qquad (4.10)$$

Therefore, we have the solution of the second fundamental problem in the case when a force (X,Y) acts on the centre of the curvilinear kernel.



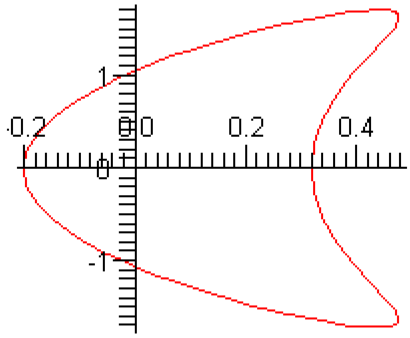
$$z=c\frac{\zeta-0.5\zeta^{-1}}{(1-0.2\zeta^{-1})^2}$$

Fig (1)



$$z=c\frac{\zeta-0.9\zeta^{-1}}{(1-0.095\zeta^{-1})^2}$$

Fig (2)



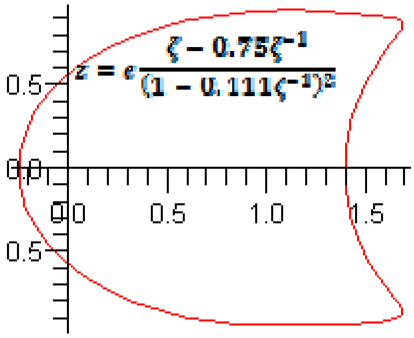


Fig (3)

$$z = c\frac{\zeta - 0.5\zeta^{-1}}{(1 - 0.4\zeta^{-1})^2}$$

Fig (4)

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