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Graph Structure of Manifolds with Listing

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Abstract

An n-manifold M inherits a graph $G^{in}[M]$ structure consisting of its charts. Such structure enables one to characterize fundamental groups of manifolds, also combinatorial classifies those of locally compact manifolds with finite non-homotopic loops. Applying the Perelman's result, i.e., *every simply connected* 3*-manifold is homeomorphic to* S^3, we also conclude that every compact 3-manifold is a 3-dimensional graphs, particularly, every simply connected 3-manifold is a 3-dimensional tree in this paper.

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§1. Introduction

An n-manifold is a second countable Hausdorff space of locally Euclidean n-space without boundary. Terminologies and notions used in this paper are standard. We follow references [1], [10] for topology, [3], [4] for graphs or topological graphs and [5]-[8] for combinatorial manifolds. Here, we only mention some of them.

Denoted by $\mathbb{S}^n$ an n-sphere in $\mathbb{R}^{n+1}$ for an integer $n \geq 1$, i.e., the points $(x_1, x_2, \cdots, x_{n+1})$ with $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$ and $B^n = \{(x_1, x_2, \cdots, x_n) | x_1^2 + x_2^2 + \cdots + x_n^2 < 1\}$ in $\mathbb{R}^{n+1}$. It is well-known that for any point $p \in \mathbb{S}^n$, $\mathbb{S}^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$.

Let Λ be an index set. A *combinatorial Euclidean space* $\mathrm{E}_G(n_\nu; \nu \in \Lambda)$ underlying a connected graph G is a topological spaces consisting of Euclidean spaces $\mathbb{R}^{n_\nu}$, $\nu \in \Lambda$ such that

$V(G) = \{\mathbb{R}^{n_\nu} | \nu \in \Lambda\}$;

$$E(G) = \{ (\mathbb{R}^{n_\mu}, \mathbb{R}^{n_\nu}) | \ \mathbb{R}^{n_\mu} \cap \mathbb{R}^{n_\nu} \neq \emptyset, \mu, \nu \in \Lambda\}.$$

If $|\Lambda| = 1$, i.e., the dimension of Euclidean spaces in $\mathrm{E}_G(n_\nu; \nu \in \Lambda)$ are all in the same n, the notation $\mathrm{E}_G(n_\nu; \nu \in \Lambda)$ is abbreviated to $\mathrm{E}_G(n, \ \nu \in \Lambda)$ for simplicity.

It is well-known that a Euclidean space $\mathbb{R}^n$ is an n-dimensional vector space with a normal basis $\overline{\epsilon}_1 = (1, 0, \cdots, 0)$, $\overline{\epsilon}_2 = (0, 1, 0 \cdots, 0)$, $\cdots$, $\overline{\epsilon}_n = (0, \cdots, 0, 1)$, namely, it has n orthogonal orientations. We do not assume $\mathbb{R}^{n_\mu} \cap \mathbb{R}^{n_\nu} = \mathbb{R}^{\min\{n_\mu, n_\nu\}}$ in general. In fact, let $\mathcal{X}_{v_\mu}$ be the set of orthogonal orientations in $\mathbb{R}^{n_{v_\mu}}$, $\mu \in \Lambda$, respectively ([9]). Then $\mathbb{R}^{n_\mu} \cap \mathbb{R}^{n_\nu} = \mathcal{X}_{v_\mu} \cap \mathcal{X}_{v_\nu}$ in this paper.

A *combinatorial fan-space* $\widetilde{\mathbb{R}}(n_\nu; \nu \in \Lambda)$ is a combinatorial Euclidean space $\mathrm{E}_{K_{|\Lambda|}}(n_\nu; \nu \in \Lambda)$ of $\mathbb{R}^{n_\nu}$, $\nu \in \Lambda$, where $K_{|\Lambda|}$ is the complete graph of order $|\Lambda|$ such that for any integers $\mu, \nu \in \Lambda$, $\mu \neq \nu$, $\mathbb{R}^{n_\mu} \bigcap \mathbb{R}^{n_\nu} = \bigcap\limits_{\lambda \in \Lambda} \mathbb{R}^{n_\lambda}$. If $|\Lambda| = m < +\infty$, for $\forall p \in \widetilde{\mathbb{R}}(n_\nu; \nu \in \Lambda)$ we can present it by an $m \times n_m$ coordinate matrix $[\overline{x}]$ following with $x_{il} = \frac{x_l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$, where $\widehat{m}$ is the number of orthogonal orientations in $\mathcal{X}_{v_\mu} \bigcap \mathcal{X}_{v_\nu}$,

$$[\overline{x}] = \begin{bmatrix} x_{11} & \cdots & x_{1\widehat{m}} & x_{1(\widehat{m})+1)} & \cdots & x_{1n_1} & \cdots & 0 \\ x_{21} & \cdots & x_{2\widehat{m}} & x_{2(\widehat{m}+1)} & \cdots & x_{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ x_{m1} & \cdots & x_{m\widehat{m}} & x_{m(\widehat{m}+1)} & \cdots & \cdots & x_{mn_m-1} & x_{mn_m} \end{bmatrix},$$

which enables us to generalize the conception of manifold to combinatorial manifold, a locally combinatorial Euclidean space.

Definition 1.1 *A combinatorial manifold $\widetilde{M}$ is a Hausdorff space such that for any point $p \in \widetilde{M}$, there is a local chart (U_p, φ_p) of p, i.e., an open neighborhood U_p of p in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \to \widetilde{\mathbb{R}}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$ with*

$$\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_\nu, \ \nu \in \Lambda\},$$

$$\bigcup_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_\nu, \ \nu \in \Lambda\},$$

denoted by $\widetilde{M}(n_\nu, \ \nu \in \Lambda)$ or $\widetilde{M}$ on the context and

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \subsetneq \widetilde{M}(n_\nu, \ \nu \in \Lambda)\}$$

an atlas on $\widetilde{M}(n_\nu, \ \nu \in \Lambda)$.

A combinatorial manifold $\widetilde{M}(n_\nu, \ \nu \in \Lambda)$ is finite if it is just combined by finite manifolds without one manifold contained in the union of others.

If $|\Lambda| = 1$, then $\widetilde{M}(n_\nu, \ \nu \in \Lambda)$ is exactly a manifold by definition. Furthermore, if these manifolds $M_i, \ 1 \leq i \leq m$ in $\widetilde{M}(n_\nu, \ \nu \in \Lambda)$ are Euclidean spaces

$\mathbb{R}^{n_\nu}$, $\nu \in \Lambda$, then $\widetilde{M}(n_\nu,\ \nu \in \Lambda)$ is nothing but the combinatorial Euclidean space $\mathrm{E}_G(n_\nu; \nu \in \Lambda)$. For a combinatorial manifold $\widetilde{M}(n_\nu,\ \nu \in \Lambda)$ consisting of manifolds M_μ, $\mu \in \Lambda$, we construct its inherent underlying graph $G^{in}[\widetilde{M}]$ following.

Definition 1.2 *Let $\widetilde{M}$ be a finitely combinatorial manifold consisting of manifolds M_μ, $\mu \in \Lambda$. An inherent graph $G^{in}[\widetilde{M}]$ of $\widetilde{M}$ is such a graph with*

$$V(G^{in}[\widetilde{M}]) = \{M_\mu,\ \mu \in \Lambda\};$$

$$E(G^{in}[\widetilde{M}]) = \{(M_\mu, M_\nu)_i, 1 \leq i \leq \kappa_{\mu\nu} + 1 | M_\mu \cap M_\nu \neq \emptyset,\ \mu, \nu \in \Lambda\},$$

where $\kappa_{\mu\nu} + 1$ is the number of arcwise connected components in $M_\mu \cap M_\nu$ for $\mu, \nu \in \Lambda$. Denoted by $G[\widetilde{M}]$ the underlying graph of $\widetilde{M}$ defined by

$$V(G[\widetilde{M}]) = \{M_\mu,\ \mu \in \Lambda\};$$

$$E(G[\widetilde{M}]) = \{(M_\mu, M_\nu) | M_\mu \cap M_\nu \neq \emptyset,\ \mu, \nu \in \Lambda\}.$$

Particularly, if $\widetilde{M}$ is a manifold with atlas $\mathrm{A} = \{\ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\}$, then each U_λ is also an n-manifold. Whence, M is itself a combinatorial manifold by Definition 1.1 and we can also construct its inherent graph $G^{in}[M]$ and underlying graph $G[M]$ by definition 1.2. For example, a torus T is shown with its an inherent shown in Fig.1.1, where the inherent graph $G^{in}[T]$ is represented by dotted lines.

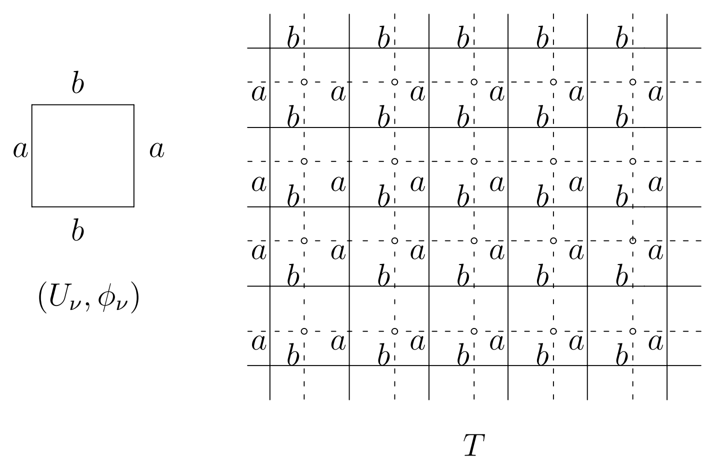

Fig.1.1

The objective of this paper is to characterize the combinatorial structure of locally compact n-manifolds by its graphs $G^{in}[M]$ and $G[M]$. For such manifolds, there is a well-known Poincaré conjecture ([2],[11],[15]) first proved by Perelman in 2003 following.

Theorem 1.3(Perelman,[12]-[14]) *Any closed simply connected* 3*-manifold is homeomorphic to* S^3*.*

The inherent graph $G^{in}[M]$ of an n-manifold M enables us to transfer the calculation problem of fundamental groups of manifolds to that of graphs and classify locally compact n-manifolds by that of labeled graphs $G^L[M]$. As a by-product, we also clarify the structure of compact 3-manifolds by showing that each of them is a 3-dimensional graph, particularly, each simply connected 3-manifold is a 3-dimensional tree by applying Theorems 1.3.

§2. Dimensional Graphs

The importance of combinatorial Euclidean spaces $\mathrm{E}_G(n)$ is shown in the next result.

Theorem 2.1 *Any locally compact n-manifold M with an alta* $\mathrm{A} = \{\ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\}$ *is a combinatorial manifold $\widetilde{M}$ homeomorphic to a Euclidean space* $\mathrm{E}_G(n, \lambda \in \Lambda)$ *with countable graphs* $G^{in}[M] \cong G$*.*

Proof By definition, each $U_\lambda,\ \lambda \in \Lambda$ is itself an n-manifold. According to Definitions 1.1 and 1.2, we get a combinatorial manifold $\widetilde{M}$ with its inherent graph $G^{in}[M]$. Define a combinatorial Euclidean space $\mathrm{E}_G(n, \lambda \in \Lambda)$ of spaces $\mathbb{R}^n$ by

$$V(G) = \{\varphi_\lambda(U_\lambda) | \lambda \in \Lambda\},$$

$$E(G) = \{\ (\varphi_\lambda(U_\lambda), \varphi_\iota(U_\iota))_i, 1 \le i \le \kappa'_{\lambda\iota} + 1 |\ \varphi_\lambda(U_\lambda) \bigcap \varphi_\iota(U_\iota) \neq \emptyset, \lambda, \iota \in \Lambda\},$$

where $\kappa'_{\lambda\iota}$ is the number of non-homotopic loops in formed between $\varphi_\lambda(U_\lambda)$ and $\varphi_\iota(U_\iota)$. Notice that $\varphi_\lambda(U_\lambda) \bigcap \varphi_\iota(U_\iota) \neq \emptyset$ if and only if $U_\lambda \bigcap U_\iota \neq \emptyset$ and $\kappa_{\lambda\iota} = \kappa'_{\lambda\iota}$ for $\lambda, \iota \in \Lambda$. We know that $G^{in}[M] \ \cong\ G$ by definition.

Now we prove that $\widetilde{M}$ is homeomorphic to $\mathrm{E}_G(n, \lambda \in \Lambda)$. By assumption, M is locally compact. Whence, there exists a partition of unity $c_\lambda : U_\lambda \to \mathbb{R}^n$, $\lambda \in \Lambda$ on the atlas $\mathrm{A}[M]$. Let $A_\lambda = \mathrm{supp}(\varphi_\lambda)$. Define functions $h_\lambda : M \to \mathbb{R}^n$ and $\mathbf{H} : M \to \mathrm{E}_G(n)$ by

$$h_\lambda(x) = \begin{cases} c_\lambda(x)\varphi_\lambda(x) & \text{if } x \in U_\lambda, \\ \mathbf{0} = (0, \cdots, 0) & \text{if } x \in U_\lambda - A_\lambda. \end{cases}$$

and

$$\mathbf{H} = \sum_{\lambda \in \Lambda} \varphi_\lambda c_\lambda, \quad \text{and} \quad \mathbf{J} = \sum_{\lambda \in \Lambda} c_\lambda^{-1} \varphi_\lambda^{-1}.$$

Then h_λ, $\mathbf{H}$ and $\mathbf{J}$ all are continuous by the continuity of φ_λ and c_λ for $\forall \lambda \in \Lambda$ on M. Notice that $c_\lambda^{-1}\varphi_\lambda^{-1}\varphi_\lambda c_\lambda$ =the unity function on M. We get that $\mathbf{J} = \mathbf{H}^{-1}$, i.e., $\mathbf{H}$ is a homeomorphism from M to $\mathrm{E}_{G'}(n, \lambda \in \Lambda)$. $\square$

Definition 2.2 *An n-dimensional graph $\widetilde{M}^n[G]$ is a combinatorial ball space $\widetilde{B}$ of B^n, $\mu \in \Lambda$ underlying a combinatorial structure G such that*

(1) *$V(G)$ is discrete consisting of B^n, i.e., $\forall v \in V(G)$ is an open ball B_v^n;*

(2) *$\widetilde{M}^n[G] \setminus V(\widetilde{M}^n[G])$ is a disjoint union of open subsets $e_1, e_2, \cdots, e_m$, each of which is homeomorphic to an open ball B^n;*

(3) *the boundary $\overline{e}_i - e_i$ of e_i consists of one or two B^n and each pair $(\overline{e}_i, e_i)$ is homeomorphic to the pair $(\overline{B}^n, B^n)$;*

(4) *a subset $A \subset \widetilde{M}^n[G]$ is open if and only if $A \cap \overline{e}_i$ is open for $1 \le i \le m$.*

A *topological graph* $\mathrm{T}[G]$ of a graph G is a 1-dimensional graph in a topological space P. We restate it in the following.

Definition 2.3 *A topological graph $\mathrm{T}[G]$ is a pair (X, X^0) of a Hausdorff space X with its a subset X^0 such that*

(1) *X^0 is discrete, closed subspaces of X;*

(2) *$X - X^0$ is a disjoint union of open subsets $e_1, e_2, \cdots, e_m$, each of which is homeomorphic to an open interval $(0, 1)$;*

(3) *the boundary $\overline{e}_i - e_i$ of e_i consists of one or two points. If $\overline{e}_i - e_i$ consists of two points, then $(\overline{e}_i, e_i)$ is homeomorphic to the pair $([0, 1], (0, 1))$; if $\overline{e}_i - e_i$ consists of one point, then $(\overline{e}_i, e_i)$ is homeomorphic to the pair $(S^1, S^1 - \{1\})$;*

(4) *a subset $A \subset \widetilde{T}[G]$ is open if and only if $A \cap \overline{e}_i$ is open for $1 \le i \le m$.*

Observation shows that there is a natural relation between a n-dimensional graph $\widetilde{M}^n[G]$ with that of its a topological graph $\mathrm{T}_0[G] = (X_0, X_0^0)$ constructed from $\widetilde{M}^n[G]_1$ by:

(1) $X_0^0 = \{\ O_v \mid O_v$ is the center of B_v^n for $v \in V(G)\}$;

(2) for $\forall uv \in E(G)$, choose uv being an arc $e_{uv} : [0, 1] \to B_u^n \cup B_v^n$ with $e_{uv}(0) = O_u$, $e_{uv}(1) = O_v$ and there is a deformation retract $E_{uv} : (B_u^n \cup B_v^n) \times [0, 1] \to (B_u^n \cup B_v^n)$ such that $E_{uv}(p, t) = p$ for $\forall p \in e_{uv},\ t \in [0, 1]$.

Then, the following result shows that an n-dimensional graph $\widetilde{M}^n[G]$ is in fact a blown up of a topological graph $\mathrm{T}[G]$ to dimensional n.

Theorem 2.4 *For any integer $n \ge 1$, $\mathrm{T}_0[G]$ is a deformation retract of $\widetilde{M}^n[G]$.*

Proof If $n = 1$, then $\widetilde{M}^n[G] = \mathrm{T}_0[G]$ is itself a topological graph. So we assume $n \ge 2$. By definition, we can choose compatible E_{uv} for $\forall uv \in E(G)$, i.e., $E_{uv} = E_{xy}$ on $(B_u^n \cup B_v^n) \cap (B_x^n \cup B_y^n)$ for $uv, xy \in E(G)$. Whence, applying the Gluing lemma, we can extend the deformation retract $E_{uv} : (B_u^n \cup B_v^n) \times [0, 1] \to (B_u^n \cup B_v^n)$ to that of $f : \widetilde{M}^n[G] \times I \to \widetilde{M}^n[G]$ with $f(p, t) = p$ for $\forall p \in \mathrm{T}_0[G],\ t \in [0, 1]$. Such a f is continuous by definition and for $\forall \overline{x} \in \widetilde{M}^n[G]$,

$$f(\overline{x},0)=\overline{x},\quad f(\overline{x},1)=p(\overline{x})$$

and $f(\overline{x},t)=\overline{x}$ for $\forall\overline{x}\in \mathrm{T}_0[G]$ and $t\in I$. Therefore, $\mathrm{T}_0[G]$ is a deformation retract of $\widetilde{M}^n[G]$ by definition. □

Notice that the inclusion mapping $i:\mathrm{T}_0[G]\to\widetilde{M}^n[G]$ is an isomorphism between groups $\pi(\mathrm{T}_0[G],v_0)$ and $\pi(\widetilde{M}^n[G]_1,v_0)$. We get a conclusion by Theorem 2.4 following.

Corollary 2.5 *Let $\widetilde{M}^n[G]$ be an n-dimensional graph. Then for $v_0\in\mathrm{T}_0[G]$,*

$$\pi(\widetilde{M}^n[G],v_0)\cong\pi(\mathrm{T}_0[G],v_0).$$

We have known the structure of fundamental group of a topological graph $\mathrm{T}[G]$ in [13]. Whence, we can characterize the fundamental group of a n-dimensional graph $\widetilde{M}^n[G]$ by applying Corollary 2.5 following.

Theorem 2.6 *Let T_{span} be a spanning tree in the topological graph $\mathrm{T}_0[G]$, $\{e_\lambda:\lambda\in\Lambda\}$ the set of edges of $\mathrm{T}_0[G]$ not in T_{span} and $\alpha_\lambda=A_\lambda e_\lambda B_\lambda\in\pi(\mathrm{T}_0[G],v_0)$ a loop associated with $e_\lambda=a_\lambda b_\lambda$ for $\forall\lambda\in\Lambda$, where $v_0\in\mathrm{T}_0[G]$ and A_λ, B_λ are unique paths from v_0 to a_λ or from b_λ to v_0 in T_{span}. Then*

$$\pi(\mathrm{T}_0[G],v_0)=\langle\alpha_\lambda|\lambda\in\Lambda\rangle.$$

An *n-dimensional tree* is an n-dimensional graph $\widetilde{M}^n[G]$ with a tree G, denoted by $\widetilde{M}^n[T]$. Applying Theorem 2.4, we know the next result.

Theorem 2.7 *An n-dimensional tree $\widetilde{M}^n[T]$ is contractible.*

Proof By Theorem 2.4, we know that there is a continuous mapping $f:\widetilde{M}^n[T]\times I\to\widetilde{M}^n[T]$ such that $\mathrm{T}_0[T]$ is a deformation retract of $\widetilde{M}^n[T]$, i.e., for $\forall\overline{x}\in\widetilde{M}^n[T]$,

$$f(\overline{x},0)=\overline{x},\quad f(\overline{x},1)=p(\overline{x}),$$

and $f(\overline{x},t)=\overline{x}$ for $\forall\overline{x}\in\mathrm{T}_0[T]$ and $t\in I$. Notice that $\mathrm{T}_0[T]$ is contractible ([13]), we have a continuous mapping $g:\mathrm{T}_0[T]\times I\to\mathrm{T}_0[T]$ such that $\{v_0\}$ is a deformation retract for $\forall v_0\in\mathrm{T}[T]$. Whence, the composition mapping $g\circ f:\widetilde{M}^n[T]\times I\to\widetilde{M}^n[T]$ is continuous such that for $\forall\overline{x}\in\widetilde{M}^n[T]$,

$$g\circ f(\overline{x},0)=\overline{x},\quad g\circ f(\overline{x},1)=p(\overline{x}),$$

and $g\circ f(v_0,t)=v_0$ for $\forall t\in I$, i.e., $\{v_0\}$ is a deformation retract of $\widetilde{M}^n[T]_1$. This completes the proof. □

Let $\mathrm{E}_G(n,\lambda\in\Lambda)$ be a Euclidean space of of Euclidean spaces $\mathbb{R}^{n_\nu}$, $\nu\in\Lambda$. Obviously, $\mathrm{A}=\{(\mathbb{R}^{n_\nu},\not\Vdash)|\nu\in\Lambda\}$ is its one alta. Denoted by $\mathrm{N}_C(\widetilde{\mathrm{E}})$ and

$\mathrm{N}_C(G^{in}[\mathrm{E}])$ the sets of non-equivalent homotopic loops in $\mathrm{E}_G(n, \lambda \in \Lambda)$ and cycles in its inherent graph $G^{in}[\mathrm{E}]$, respectively. Then we know the following interesting result.

Theorem 2.8 *There is a bijection* $\vartheta : \mathrm{N}_C(\mathrm{E}) \to \mathrm{N}_C(G^{in}[\mathrm{E}])$*, i.e.,*

$$\pi(\mathrm{E}_G(n, \lambda \in \Lambda)) \cong \pi(G^{in}[\mathrm{E}]).$$

Proof We only need to prove that a loop $L \in \mathrm{N}_C(\mathrm{E})$ if and only if there is a cycle $C_L \in \mathrm{N}_C(G^{in}[\mathrm{E}])$. The proof is divided into two cases following.

Case 1. L comes from the underlying graph $G[\mathrm{E}_G(n, \lambda \in \Lambda)]$.

According to Theorems 2.1 and 2.4, we know that the underlying graphs of $\mathrm{E}_G(n, \lambda \in \Lambda)$ and $G^{in}[\mathrm{E}]$ are isomorphic. Whence there exists a cycle $C_L \in \mathrm{T}_0[G]$, i.e., $C_L \in \mathrm{N}_C(G^{in}[\mathrm{E}])$ correspondent to L and verse via, for a cycle $C \in \mathrm{T}_0[G]$, there also exists a loop $L_C \in \mathrm{N}_C(\mathrm{E})$ for a cycle C. Whence, such a mapping $\vartheta : L \to C_L$ is a bijection.

Case 2. L comes from $U_\mu \cap U_\nu$ for two indexes $\mu, \nu \in \Lambda$.

Assume there are $\kappa_{\mu\nu}$ non-homotopic loos in $U \cap U_\nu$. Not loss of generality, let L be the ith loop. By Definition 1.1, we have a cycle C_L consisted of multiple edges $(U_\mu, U_\nu)_i$ with $(U_\mu, U_\nu)_{i+1}$ in the graph $G^{in}[M]$. Then $\vartheta : L \to C_L$ is a bijection by definition.

Combining the discussion of Cases 1 and 2, we get a bijection

$$\vartheta : \mathrm{N}_C(\mathrm{E}) \to \mathrm{N}_C(G^{in}[\mathrm{E}]).$$

Notice that $\pi(\mathrm{E}_G(n, \lambda \in \Lambda)) = \langle \mathrm{N}_C(\mathrm{E}) \rangle$ and $\pi(G^{in}[\mathrm{E}]) = \langle \mathrm{N}_C(G^{in}[\mathrm{E}]) \rangle$. The bijection $\vartheta : \mathrm{N}_C(\mathrm{E}) \to \mathrm{N}_C(G^{in}[\mathrm{E}])$ naturally induces an isomorphism $\vartheta* : \pi(\mathrm{E}_G(n, \lambda \in \Lambda)) \to \pi(G^{in}[\mathrm{E}])$ by defining $\vartheta * (L_1 + L_2) = \vartheta(L_1) + \vartheta(L_2)$ for $L_1,\ L_2 \in \mathrm{N}_C(\mathrm{E})$ in the field $\mathbb{Z}_2$. Hence, we conclude that

$$\pi(\mathrm{E}_G(n, \lambda \in \Lambda))\text{¸} \cong \pi(G^{in}[\mathrm{E}]). \qquad \square$$

§3 Graph Structures of Compact 3-Manifolds

There are many cycles in $G^{in}[M]$ are contractible. For example, these cycles on the plane $\mathbb{E}^2$ in Fig.1.1. We introduce the following conception for removing those dispensable.

Definition 3.1 *An atlas* $\mathrm{A}[M] = \{\ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\}$ *of an* n*-manifold* M *is minimal if there are no indexes* $\mu,\ \nu \in \Lambda$ *and a continuous mapping* $\varphi_{\mu\nu}$ *with* $\varphi_{\mu\nu} : U_\mu \cup U_\nu \to \mathbb{R}^n$ *such that*

$$\mathrm{A}' = \{\ (U_\lambda; \varphi_\lambda), (U_\mu \cup U_\nu, \varphi_{\mu\nu}) \mid \lambda \in \Lambda \setminus \{\mu, \nu\}\}$$

is also an atlas of M^n. An atlas $\mathrm{A}[M]$ *of an* n*-manifold* M *is minimum if it has minimum cardinality among all of its minimal atlases. Denoted such a minimal atlas by* $\mathrm{A}_{min}[M]$ *and its inherent graph by* $G^{in}_{min}[M]$.

Then we get the following result immediately by Definitions 1.1 and 3.1.

Theorem 3.2 *The number* $\varpi(G^{in}_{min}[M])$ *of cycles basis of* $G^{in}_{min}[M]$ *is*

$$\varpi(G^{in}_{min}[M]) = \varpi(G_{min}[M]) + \sum_{(U_\mu,U_\nu)\in E(G[M])} \kappa_{\mu\nu},$$

where $G_{min}[M]$ *denotes the underlying graph in the minimum atlas* $\mathrm{A}[M]$. *Therefore,* $\varpi(G^{in}_{min}[M]) = 0$ *if and only if* $\varpi(G_{min}[M]) = 0$ *and* $\kappa_{\mu\nu} = 0$ *for* $(U_\mu, U_\nu) \in E(G_{min}[M])$.

The following result characterizes the minimal atlas of n-manifolds.

Theorem 3.3 *Let* $\mathrm{A}[M] = \{\ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\}$ *be an atlas of a locally compact* n*-manifold* M. *Then*

(i) $\mathrm{A}[M]$ *is minimal if and only if there are no indexes* $\mu,\ \nu \in \Lambda$ *such that* $U_\mu \cap U_\nu$ *is arewise-connected, i.e.,* $\kappa_{\mu\nu} \geq 1$ *for* $\forall \mu, \nu \in \Lambda$ *with* $U_\mu \cap U_\nu \neq \emptyset$.

(ii) M *is with finite non-homotopic loops if and only if* $\mathrm{A}_{min}[M]$ *is finite.*

Proof (i) If the conclusion (i) is not true, then there exist indexes $\mu,\ \nu \in \Lambda$ such that $U_\mu \cap U_\nu$ is arewise-connected. Assume

$$\varphi_\mu(U_\mu \cap U_\nu) = S \subset \mathbb{R}^n \ \text{ and } \ \varphi_\nu(U_\mu \cap U_\nu) = T \subset \mathbb{R}^n.$$

Notice that S and T are homeomorphic to $\mathbb{R}^n$ by definition. We can always choose a continuous mapping $\tau :\ S \to T$, i.e., $\tau(S) = T$. Define

$$\varphi^\tau_\mu = \tau\varphi_\mu.$$

Then we get that $\varphi^\tau_\mu|_{U_\mu\cap U_\nu} = \varphi_\nu|_{U_\mu\cap U_\nu}$. Whence, there is a continuous mapping $\varphi_{\mu\nu} :\ U_\mu \cup U_\nu \to \mathbb{R}^n$. Therefore,

$$\mathrm{A}' = \{\ (U_\mu \cup U_\nu; \varphi_{\mu\nu}), (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda \setminus \{\mu, \nu\}\}$$

is also an atlas of M but with $|\mathrm{A}'| = |\Lambda| - 1$. This contradicts to the minimality of Λ. So the conclusion (i) holds.

(ii) If $\mathrm{A}_{min}[M]$ is finite, then M is obvious only with finite non-homotopic loops by the assumption of its locally compactness. Now let

$$\mathrm{A}_{min}[M^n] = \{\ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\}$$

be a minimum atlas of a locally compact manifold M with an infinite index set Λ. By (i), there are no indexes μ, ν in Λ such that $U_\mu \cap U_\nu$ is arcwise connected.

In other words, $U_\mu \cap U_\nu = \emptyset$ or with more than 2 arcwise components for $\forall \mu, \nu \in \Lambda$, i.e., $\kappa_{\mu\nu} \geq 1$ for $\forall (U_\mu, U_\nu) \in E(G^{in}_{min})[M]$. Applying Theorem 3.2, we know that the number

$$\varpi(G^L_{min}[M^n]) = \varpi(G_{min}[M^n]) + \sum_{e \in E(G_{min}[M^n])} (L(e) - 1)$$

of cycle basis of $G^{in}_{min}[M]$ is greater than any sufficient larger number $N > 0$, which contradicts the assumption that M is only with finite non-homotopic loops. □

Combining Theorems 2.1, 2.4 and 2.8, we get an important result on locally n-manifolds following.

Theorem 3.4 *For any locally compact n-manifold M, there always exists an inherent graph $G^{in}_{min}[M]$ such that*

$$\pi(M) \cong \pi(G^{in}_{min}[M]).$$

Proof According to Theorem 2.1, there is a combinatorial Euclidean space $\mathrm{E}_G(n, \lambda \in \Lambda)$ homeomorphic to M. Applying Theorems 2.4 and 2.8, we get a minimum atlas $\mathrm{A}[M]$ with an inherent graph $G^{in}_{min}[M]$ such that

$$\pi(\mathrm{E}_G(n, \lambda \in \Lambda)) \cong \pi(G^{in}_{min}[M]),$$

Whence, we know that

$$\pi(M) \cong \pi(\mathrm{E}_G(n, \lambda \in \Lambda)) \cong \pi(G^{in}_{min}[M]).$$ □

Corollary 3.5 *The number of non-homotopic loops in a locally compact n-manifold M is equal to the dimension of cycle space of its* $\mathrm{T}_0[G^{in}_{min}[M]]$.

This result alludes to that we can combinatorially determine the fundamental group of an n-manifold M with finite non-homotopic loops.

Corollary 3.6 *For an integer $n \geq 2$, a compact n-manifold M is simply connected if and only if $G^{in}_{min}[M]$ is a finite tree.*

Proof By definition, M is compact if and only if $|G^{in}_{min}[M]|$ is finite. Applying Theorems 3.2 and 3.4, we know that $\pi(M)$ is trivial if and only if $\pi(G^{in}_{min}[M])$ is trivial. Now by Corollary 2.5, there must be $\pi(\mathrm{T}_0[H_G], v_0) = \{v_0\}$ for $v_0 \in \mathrm{T}_0[H_G] \cap M$. But this can happens only if $\mathrm{T}_0[H_G]$ is a finite tree by Theorem 2.6. □

Applying Theorem 1.3, the Perelman's result, we can completely determine the structure of compact 3-manifolds following.

Lemma 3.7 *Let $X = U \cup V$ with U, V homeomorphic to $\mathbb{R}^3$ and let $C_1, C_2, \cdots, C_m$ be all arcwise-connected components in $U \cap V$, $m \geq 1$. Then each C_i, $1 \leq i \leq m$ is homeomorphic to S^3.*

Proof Choose points $x_{i-1} \in C_i$, $1 \leq i \leq m$ and arcs $a(x_0, x_i) \subset V$, $1 \leq i \leq m-1$. Define

$$U^E = U \bigcup \left(\bigcap_{i=1}^{m-1} a(x_0, x_i) \right).$$

Then we know that U^E is still open and $U^E \cap V$ an arcwise connected star $S_{1,m-1}$, such as those shown in Fig.3.2.

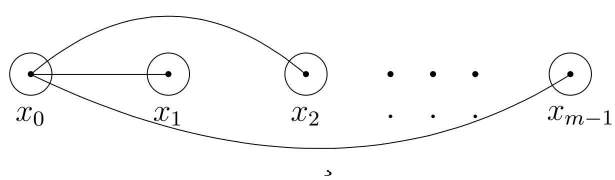

Fig.3.1

Applying the Seifert-Van Kampen theorem, we know that for $x_0 \in U \cap V$, the fundamental group of $U^E \cup V$

$$\begin{aligned} \pi_1(U^E \cup V, x_0) &\cong \frac{\pi_1(U^E, x_0)\pi_1(V, x_0)}{\left[i_1^{-1}(g) \cdot i_2(g) | \ g \in \pi_1(U^E \cap V, x_0) \right]} \\ &\cong \frac{1}{\left[i_1^{-1}(g) \cdot i_2(g) | \ g \in \prod_{i=1}^{m} \pi_1(C_i, x_{i-1}) \right]}, \end{aligned}$$

where $[A]$ denotes the minimal normal subgroup of a group G included $A \subset$ G. Whence, each fundamental group $\pi_1(C_i, x_{i-1})$, $1 \leq i \leq m$ is trivial. That is, a simply connected 3-manifold. Applying Theorem 1.3, we know that each $C_i, 1 \leq i \leq m$ is homeomorphic to S^3. □

Now we can determine the dimensional graph structure of compact 3-manifolds following.

Theorem 3.8 *Every compact 3-manifold M is a 3-dimensional graph underlying $G^{in}_{min}[M]$. Particularly, every simply connected 3-manifold is a 3-dimensional tree.*

Proof Let $\mathrm{A}[M] = \{ \ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda \}$ be an atlas of M. By definition, each $v \in V(G^{in}_{min}[M])$ is homeomorphic to $\mathbb{R}^3$ and consequently, S^3. Applying Lemma 3.7, we also know that each $e \in E(G^{in}_{min}[M])$ is homeomorphic to S^3. Whence, by the definition of n-dimensional graph we know that M is a 3-dimensional graph.

Applying Corollary 3.6, a 3-manifold M is simply connected if and only if $G^{in}_{min}[M]$ is a finite tree. Whence, M is a 3-dimensional tree. □

§4. Listing Compact Manifolds by Labeled Graphs

An inherent graph $G^{in}[M]$ of a compact manifold M can be simply transferred to an edge labeled graph following.

Definition 4.1 *Let* $\mathrm{A}[M] = \{ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\}$ *be an atlas of a compact* n*-manifold* M *and* $G^{in}[M]$ *the inherent graph of* M*. Define an edge labeled graph* $G^L[M]$ *following:*

$V(G^L[M]) = V(G^{in}[M])$;

$E(G^L[M]) = \{(U_\mu, U_\nu) \mid \mu, \nu \in \Lambda\}$,

and each edge $(U_\mu, U_\nu) \in E(G^L[M])$ *with a labeling* $L : (U_\mu, U_\nu) \to \kappa_{\mu\nu}$*, where* $\kappa_{\mu\nu} + 1$ *is the number of arcwise connected components in* $U_\mu \cap U_\nu$ *for* $\mu, \nu \in \Lambda$*.*

Then we know the labeled graph, or the equivalent inherent graph of an n-manifold is topological invariant following.

Theorem 4.2 *Let* $\mathrm{A}[M] = \{ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\}$ *be a atlas of a locally compact* n*-manifold* M*. Then the labeled graph* $G^L_{|\Lambda|}$ *of* M *is a topological invariant on* $|\Lambda|$*, i.e., if* $H^{L_1}_{|\Lambda|}$ *and* $G^{L_2}_{|\Lambda|}$ *are two labeled* n*-dimensional graphs of* M*, then there exists a self-homeomorphism* $h : M \to M$ *such that* $h : H^{L_1}_{|\Lambda|} \to G^{L_2}_{|\Lambda|}$ *naturally induces an isomorphism of graph.*

Proof Let $\mathrm{A}^1_{|\Lambda|}[M] = \{ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda_1\}$ and $\mathrm{A}^2_{|\Lambda|}[M] = \{ (V_\lambda; \phi_\lambda) \mid \lambda \in \Lambda_2\}$ be two minimum atlases of a locally compact n-manifold M with labeled graphs $H^{L_1}_{|\Lambda|}$ and $G^{L_2}_{|\Lambda|}$, respectively, where $\Lambda_1 = \Lambda_2 = \{1, 2, 3, \cdots, k, \cdots\}$ are countable index sets. Notice that $\varphi_\lambda : U_\lambda \to \mathbb{R}^n$ and $\phi_\lambda : V_\iota \to \mathbb{R}^n$ are homeomorphisms for $\forall\lambda \in \Lambda_1$ and $\iota \in \Lambda_2$. So there is a homeomorphism $\tau_\lambda : \varphi_\lambda(U_\lambda) \to \phi_\lambda(V_\lambda)$ for $\lambda \in \Lambda_1 = \Lambda_2$. Now define

$$h_\lambda = \phi_\lambda^{-1}\tau_\lambda\varphi_\lambda : x \to \phi_\lambda^{-1}(\tau_\lambda(\varphi_\lambda(x))) \ \text{ for } \ x \in U_\lambda.$$

Then h_λ with its inverse $h^{-1} = \varphi_\lambda^{-1}\tau_\lambda^{-1}\phi_\lambda$ is continuous on M. By the compactness of M^n there exists a partition of unity $c_\lambda : U_\lambda \to \mathbb{R}^n$, $\lambda \in \Lambda$ on the atlas $\mathrm{A}^1_{|\Lambda|}[M]$. Define a function $h : M \to M$ by

$$h = \sum_{\lambda\in\Lambda} \phi_\lambda^{-1}\tau_\lambda\varphi_\lambda c_\lambda,$$

where $A_\lambda = \mathrm{supp}(\varphi_\lambda)$ and

$$\phi_\lambda^{-1}\tau_\lambda\varphi_\lambda c_\lambda(x) = \begin{cases} c_\lambda(x)\phi_\lambda^{-1}\tau_\lambda\varphi_\lambda(x) & \text{if } x \in U_\lambda, \\ \mathbf{0} = (0, \cdots, 0) & \text{if } x \in U_\lambda - A_\lambda. \end{cases}$$

Then $h : M \to M$ is a homeomorphism with $h(U_\lambda) = V_\lambda$ for $\lambda \in \Lambda_1 = \Lambda_2$. Hence, $h : V(H^L_{|\Lambda|}) \to V(G^L_{|\Lambda|})$ is a bijection by definition.

Now if there are $\kappa_{\mu\nu}$ non-homotopic loops between U_μ and U_ν, then there are must be $\kappa_{\mu\nu}$ non-homotopic loops between V_μ and V_ν and vice via by the homeomorphic property. Therefore, $L_1 : (U_\mu, U_\nu) \to \kappa_{\mu\nu} + 1$ in $H^L_{|\Lambda|}$ if and only if $L_2 : (V_\mu, V_\nu) \to \kappa_{\mu\nu} + 1$ in $G^{L_2}_{|\Lambda|}$, i.e., $h(U_\mu, U_\nu) = (h(U_\mu), h(V_\nu))$ with $L_1(U_\mu, U_\nu) = L_2(h(U_\mu), h(U_\nu))$. By definition, two labeled graphs G^L_1 and G^L_2 with labeling mappings L_1 and L_2 are said to be isomorphic if there is an isomorphism $\varpi : G_1 \to G_2$ with $\varpi L_1 = L_2 \varpi$. Whence, $h : H^L_{|\Lambda|} \to G^L_{|\Lambda|}$ naturally induces an isomorphism between labeled graphs $H^{L_1}_{|\Lambda|}$ and $G^{L_2}_{|\Lambda|}$. $\square$

We get useful conclusions by Theorem 4.2 following.

Corollary 4.3 *The labeled graph $G^L_{|\Lambda|}$ of a locally compact n-manifold M^n is uniquely dependent on the minimum index set $|\Lambda|$.*

Corollary 4.6 *If the minimum labeled graph $G^L_{min}[M_1]$ of a locally compact n-manifold M_1 is not isomorphic to the minimum labeled graph $G^L_{min}[M_2]$ of M_2, then M_1 is not homeomorphic to M_2.*

Now by Theorem 4.2, let

$\mathrm{A}_{min}[M] = \{\ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda, |\Lambda| < +\infty\}$

be a minimum atlas of locally compact n-manifolds M. Then we can list n-manifolds, particularly, 3-manifolds following by Theorem 3.8 and Corollary 4.6 as follows.

(1) $|\Lambda| = 1$

In this case, $\mathrm{A}_{min}[M] = \{(U; \varphi)\}$, i.e., $M = \mathbb{R}^n$.

(2) $|\Lambda| = 2$

In this case,

$$\mathrm{A}_{min}[M] = \{(U_1; \varphi_1),\ (U_2; \varphi_2)\}$$

and M is double covered, which can be classified by labeled graphs $D^L_{0,1,0}$ shown in Fig.3.1,

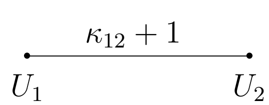

Fig.4.1

For example, if $n=2$ with $U_1=\mathbb{R}^2$, $\varphi_1=z$ and $U_2=(\mathbb{R}^2\setminus\{(0,0)\}\cup\{\infty\}$, $\varphi_2=1/z$, then $\kappa_{12}=0$ and the 2-manifold is just the Riemann sphere.

(3) $|\Lambda|=3$

In this case,

$\mathrm{A}_{min}[M]=\{(U_1;\varphi_1),\ (U_2;\varphi_2),\ (U_3;\varphi_3)\}$

and we can list such n-manifolds M by labeled graphs in Fig.3.2, where integers $k,l,s\geq 2$.

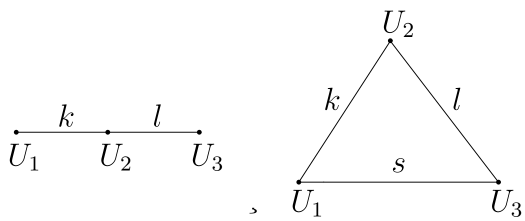

Fig.4.2

(4) $|\Lambda|=4$

In this case,

$\mathrm{A}_{min}[M]=\{(U_1;\varphi_1),\ (U_2;\varphi_2),\ (U_3;\varphi_3),\ (U_4;\varphi_4)\}$

and we can list such n-manifolds M by labeled graphs in Fig.3.3, where integers $k,l,r,q,s,t\geq 2$.

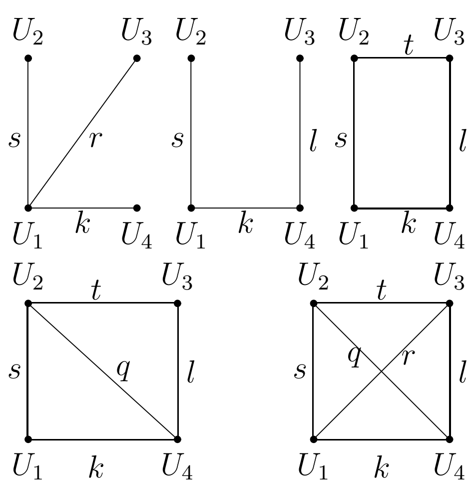

Fig.4.3

. . . .

(p) $|\Lambda| = p$

In this case,

$$\mathrm{A}_{min}[M] = \{(U_i; \varphi_i), 1 \leq i \leq p\}.$$

and there are $G_1, G_2, \cdots, G_{k(p)}$ such non-isomorphic graphs known in graph theory, where

$$k(p) \sim \frac{2^{\frac{p(p-1)}{2}}}{n!}$$

is the number of non-isomorphic graphs of order p with

p	1	2	3	4	5	6	7	8	9
$k(p)$	1	1	2	6	21	112	853	11117	261080

for $p \leq 9$. Then we can list such n-manifolds M by labeled graphs following:

$$G_1^L, G_2^L, \cdots, G_{k(p)}^L, \quad L_i(e_{ij}) \geq 2 \text{ for } \forall e_i \in E(G_i), 1 \leq i \leq k(p).$$

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