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# Invariant Approximation Results for Pointwise $R$-Subweakly Commuting Maps

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**Abstract**

A common fixed point result for pointwise $R$-subweakly commuting maps in strongly $M$-starshaped metric spaces is obtained. As application, invariant approximation theorems are derived. Our results unify, and extend various known results existing in the literature.

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**Keywords:** Common fixed point, pointwise $R$-subweakly commuting maps, strongly $M$-starshaped metric space, invariant approximation

## 1. Introduction and preliminaries

We first review needed definitions. Let $X$ be a metric space with metric d, $M \subset X$ and J=[0,1]. The space $X$ is called;
(1) M-starshaped [22] if there exists a continuous mapping $W : X \times M \times J \to X$ satisfying

$$d(x, W(y, q, \lambda)) \leq \lambda d(x, y) + (1 - \lambda)d(x, q)$$

for all $x, y \in X$, $q \in M$ and all $\lambda \in J$; (2) strongly $M$-starshaped [1, 16] if it is $M$-starshaped and satisfies the property $(I)$, that is,

$$d(W(x, q, \lambda), W(y, q, \lambda)) \leq \lambda d(x, y)$$

for all $x, y \in X$, $q \in M$ and all $\lambda \in J$; (3) (strongly) convex if it is (strongly)

$X$-starshaped; (4) starshaped if it is $\{q\}$-starshaped for some $q \in X$. Strongly convex metric space is also said to be a metric space of hyperbolic type(see Ciric [4]). Obviously, every normed space X is a strongly convex metric space with W defined by $W(x,q,\lambda) = \lambda x + (1-\lambda)q$ for all $x, q \in X$ and all $\lambda \in J$. More generally, if $X$ is a linear space with a translation invariant metric satisfying $d(\lambda x+(1-\lambda)y, 0) \leq \lambda d(x,0)+(1-\lambda)d(y,0)$, then $X$ is a strongly convex metric space. A subset $D$ of a $M$-starshaped metric space $X$ is called $q$-starshaped if there exists $q \in D \cap M$ such that $W(x,q,\lambda) \in D$ for all $x \in D$ and all $\lambda \in J$. For details, we refer the reader to Al-Thagafi [1], Guay et al.[6] and Takahashi [22]).

Let $I, T : X \to X$ be two mappings and $D \subset X$. Then $T$ is called; (5) $I$-nonexpansive on $D$ if $d(Tx,Ty) \leq d(Ix,Iy)$, for all $x, y \in D$; (6) $I$-contraction on $D$ if there exists $k \in [0,1)$ such that $d(Tx,Ty) \leq kd(Ix,Iy)$, for all $x, y \in D$. A point $x \in D$ is a coincidence point (common fixed point) of $I$ and $T$ if $Ix = Tx$ ($x = Ix = Tx$). The set of coincidence points of $I$ and $T$ is denoted by $C(I,T)$. The mappings $I$ and $T$ are called (7) commuting on $D$ if $ITx = TIx$ for all $x \in D$; (8) pointwise $R$-weakly commuting on $D$ if for given $x$ in $D$, there exists a real number $R > 0$ such that $d(TIx, ITx) \leq Rd(Tx, Ix)$; (9) weakly compatible if they commute at their coincidence points, i.e.,if $ITx = TIx$ whenever $Ix = Tx$. Suppose that $D$ is $q$-starshaped with $q \in F(S) \cap M$ and is both $T$- and $I$-invariant. Then $T$ and $I$ are said to be;
(10) $R$-subcommuting on $D$ if there exists a real number $R > 0$ such that $d(TIx, ITx) \leq \frac{R}{\lambda} d(W(Tx,q,\lambda), Ix)$ for all $x \in D$ and all $\lambda \in (0,1]$; (11) $R$-subweakly commuting on $D$ if for all $x \in M$, there exists a real number $R > 0$ such that $d(ITx, TIx) \leq$R$dist(Ix, seg[q,Tx])$; (12) pointwise $R$-subcommuting on $D$ if for given $x \in D$, there exists a real number $R > 0$ such that $d(TIx, ITx) \leq \frac{R}{\lambda} d(W(Tx,q,\lambda), Ix)$ for all $k \in (0,1]$; (13) pointwise $R$-subweakly commuting [18] on $D$ if for given $x \in D$, there exists a real number $R > 0$ such that $d(ITx, TIx) \leq Rdist(Ix, seg[q,Tx])$. Clearly, pointwise $R$-subweakly commuting maps are weakly compatible but not conversely in general and $R$-subweakly commuting maps are pointwise $R$-subweakly commuting but the converse does not hold in general. The mapping $I$ is called affine on $D$ if $I(W(x,q,\lambda)) = W(Ix, Iq, \lambda)$ for all $x \in D$ and all $\lambda \in J$. Let $S \subset X$ and $\widehat{x} \in X$. Then $P_S(\widehat{x}) = \{x \in S : d(x,\widehat{x}) = d(\widehat{x}, S)\}$ is called the set of best $S$-approximants to $\widehat{x}$, where $d(\widehat{x}, S) = inf\{d(\widehat{x}, y) : y \in S\}$ and $C_S^I(\widehat{x}) = \{x \in S : Ix \in P_S(\widehat{x})\}$.

In 1963, Meinardus [17] employed the Schauder fixed point theorem to prove a result regarding invariant approximation. In 1979, Singh [20] proved the following extension of the result of Meinardus.

**Theorem 1.1.** Let $T$be a nonexpansive operator on a normed space $X$ , $M$ a nonempty subset of $X$, $T(M) \subset M$ and $u \in F(T)$. If $P_M(u)$ is nonempty compact and starshaped, then $P_M(u) \cap F(T) \neq \emptyset$.

Hicks and Humphries [8] found that Singh's results remain true if $T(M) \subset M$ is replaced by $T(\partial M) \subset M$. In 1988, Sahab, Khan and Sessa [19] established the following result which contains the result of Hicks and Humphries and Theorem 1.1.

**Theorem 1.2.** Let $I$ and $T$ be selfmaps of a normed space $X$ with $u \in F(I) \cap F(T)$, $M \subset X$ with $T(\partial\ M) \subset M$, and $q \in F(I)$. If $D = P_M\ (u)$ is compact and $q$-starshaped, $I(D) = D$, $I$ is continuous and linear on $D$, $I$ and $T$ are commuting on $D$ and $T$ is $I$-nonexpansive on $D \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(I) \neq \emptyset$.

Invariant approximation results for commuting maps due to Al-Thagafi [2] extended and generalized Theorems 1.1-1.2 and the works of [7, 8, 21]. Al-Thagafi results have been further extended by [10, 11, 12, 18] to $R$-subweakly commuting and pointwise $R$-subweakly commuting.

The aim of this paper is to establish a common fixed point theorem for pointwise $R$-subweakly commuting maps in the setup of strongly $M$-starshaped metric spaces. As application, invariant approximation results for pointwise $R$-subweakly commuting and $R$-subcommuting maps are derived. Our results extend and unify the work of Al-Thagafi [2], Dotson [5], Habiniak [7], Hicks and Humphries [8], Hussain and Berinde [9], Hussain, O'Regan and Agarwal [11], Naz [16], Latif [15], Sahab, Khan and Sessa [19] and Singh [20, 21].

The following result will be needed.

**Lemma 1.4[1].** Let $D$ be a subset of a $M$-starshaped metric space $(X, d)$ and $\widehat{x} \in X$. Then $P_D(\widehat{x}) \subset \partial D \cap D$.

## 2. **Main Results**

The following result will be needed.

**Theorem 2.1 [18].** Let $M$ be a closed subset of a metric space $(X,d)$, and let $I$ and $T$ be pointwise $R$-weakly commuting self-mappings of $M$. If $cl(T(M)) \subset I(M)$, $cl(T(M))$ is complete, $T$ is $I$-continuous and $I$ and $T$ satisfy for all $x,y \in M$ and $0 \le h < 1$,

$$d(Tx,Ty) \le h\ max\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), d(Ix,Ty), d(Iy,Tx)\}, \tag{2.1}$$

then $M \cap F(I) \cap F(T)$ is a singleton.

**Theorem 2.2.** Let $S$ be a $q$-starshaped subset of a strongly $M$-starshaped metric space $X$ and $I,T : S \to S$ pointwise $R$-subweakly commuting mappings such that $clT(S) \subset I(S)$. Suppose that $q \in F(I) \cup M$ and $I$ is affine, $clT(S)$ is compact, $T$ is $I$-continuous, $I$ is continuous and $I$ and $T$ satisfy

$$d(Tx,Ty) \le max \left\{ \begin{array}{c} d(Ix,Iy), dist(Ix, seg[q,Tx]), dist(Iy, seg[q,Ty]), \\ dist(Ix, seg[q,Ty]), dist(Iy, seg[q,Tx]) \end{array} \right\} \tag{2.2}$$

for all $x,y \in M$. Then $S \cap F(I) \cap F(T) \neq \emptyset$.

**Proof.** Choose a sequence $\{k_n\} \subset (0,1)$ such that $k_n \to 1$ as $n \to \infty$. Define for each n, a map $T_n x = W(Tx,q,k_n)$ for each x in S. Since $clT(S) \subset I(S)$ and $I$ is affine with $Iq = q$, then $clT_n(S) \subset I(S)$. As $I$ and $T$ are pointwise $R$-subweakly commuting and $I$ is affine with $Iq = q$, so for each $x \in C_q(I,T)$

$$\begin{aligned} d(T_nIx, IT_nx) &= d(W(TIx,q,k_n), I(W(Tx,q,k_n))) \\ &= d(W(TIx,q,k_n), W(ITx,q,k_n)) \\ &= k_n d(TIx, ITx) \\ &= k_n R\ dist(Ix, seg[q,Tx]). \end{aligned}$$

Thus $I$ and $T_n$ are pointwise $k_nR$-weakly commuting for all $n$. Also by (2.2),

$$\begin{aligned} d(T_nx, T_ny) &= d(W(Tx,q,k_n), W(Ty,q,k_n)) \\ &\le k_n d(Tx,Ty) \\ &\le k_n max\{d(Ix,Iy), dist(Ix, seg[q,Tx]), dist(Iy, seg[q,Ty]), \\ &\quad dist(Ix, seg[q,Ty]), dist(Iy, seg[q,Tx])\} \\ &\le k_n max\{d(Ix,Iy), d(Ix,T_nx), d(Iy,T_ny), \\ &\quad d(Ix,T_ny), d(Iy,T_nx)\}, \end{aligned}$$

for each $x, y \in S$ and $0 < k_n < 1$. Since cl(T(S)) is compact, each $cl(T_n(S))$ is compact. By Theorem 2.1, for each $n \geq 1$, there exists $x_n \in S$ such that $x_n = Ix_n = T_n x_n$. The compactness of $cl(T(S))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ and $y \in clT(S)$ such that $Tx_m \to y$ as $m \to \infty$. Since $k_m \to 1$, $x_m = \{W(Tx_m, q, k_m)\}$ converges to $y$. Since $T$ is continuous, $Tx_m \to Ty$ as $m \to \infty$. Thus $y = Ty$. Since $I$ is also continuous, we have $Iy = y$. Thus $S \cap F(I) \cap F(T) \neq \emptyset$.

**Theorem 2.3.** Let $S$ be a $q$-starshaped subset of a strongly $M$-starshaped metric space $X$ and $I, T : S \to S$ pointwise $R$-subweakly commuting mappings such that $clT(S) \subset I(S)$. Suppose that $q \in F(I) \cup M$ and $I$ is affine, $clT(S)$ is compact and $T$ is continuous and $I$-nonexpansive. Then $S \cap F(I) \cap F(T) \neq \emptyset$.
**Proof.** We follow the proof of Theorem 2.2 up to the equation $y = Ty$. Since $T(S) \subset I(S)$, we can choose $z \in S$ such that $y = Ty = Iz$. Also,

$$d(Tx_m, Tz) \leq d(Ix_m, Iz) = d(x_m, Iz) = d(x_m, y).$$

Taking limit when $m \to \infty$, we obtain $Ty = Tz$. Thus $y = Ty = Tz = Iz$. Hence $C(I, T)$ is nonempty. Since $I$ and $T$ are also weakly compatible, we have $Iy = ITz = TIz = Ty = y$. Thus $S \cap F(I) \cap F(T) \neq \emptyset$.

As $R$-subcommuting mappings are pointwise $R$-subweakly commuting, so we obtain the following extension of the recent result of Naz ([18], Theorem 4).

**Corollary 2.4.** Let $S$ be a $q$-starshaped subset of a strongly $M$-starshaped metric space $X$ and $I, T : S \to S$ $R$-subcommuting mappings such that $clT(S) \subset I(S)$. Suppose that $q \in F(I) \cap M$, I is affine, $clT(S)$ is compact and $T$ is continuous and $I$-nonexpansive. Then $S \cap F(I) \cap F(T) \neq \emptyset$.

**Remark 2.5.** Theorems 2.2-2.3 extend and improve Theorem 2.2 of Al-Thagafi [2], Theorem 1 of Dotson [5], Theorem 4 of Habiniak [7], Theorem 2.2 of Hussain and Berinde [9], Theorem 6 of Jungck and Sessa [13] and the corresponding results of Hussain and Jungck [10].

**Theorem 2.6.** Let $X$ be a strongly $M$-starshaped metric space, $I, T : X \to X$ two mappings, $S$ be a subset of $X$ such that $T(\partial S \cap S) \subset S$ and $\widehat{x} \in F(T) \cap F(I)$. Suppose that $P_S(\widehat{x})$ is nonempty closed and $q$-starshaped with $q \in F(I) \cap M$, $I$ is affine on $P_S(\widehat{x})$, $cl(T(P_S(\widehat{x})))$ is compact, and $I(P_S(\widehat{x})) = P_S(\widehat{x})$.

If $T$ is $I$-continuous, the pair $\{T, I\}$ is pointwise $R$-subweakly commuting and continuous on $P_S(\widehat{x})$ and satisfy for all $x \in P_S(\widehat{x}) \cup \{\widehat{x}\}$,

$$d(Tx, Ty) \le \begin{cases} d(Ix, Iu) & \text{if } y = u, \\ max\{d(Ix, Iy), dist(Ix, [q, Tx]), dist(Iy, [q, Ty]), & \\ \quad dist(Ix, [q, Ty]), dist(Iy, [q, Tx])\} & \text{if } y \in P_S(\widehat{x}), \end{cases} \tag{2.3}$$

then $P_S(\widehat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

**Proof.** Let $x \in P_S(\widehat{x})$. Then by Lemma 1.4, $x \in \partial S \cap S$ and so $Tx \in S$ since $T(\partial S \cap S) \subset S$. As $T$ satisfies (2.3) on $P_S(\widehat{x}) \cup \{\widehat{x}\}$ and $I(P_S(\widehat{x})) = P_S(\widehat{x})$, we have

$$d(Tx, \widehat{x}) = d(Tx, T\widehat{x}) \le d(Ix, I\widehat{x}) = d(Ix, \widehat{x}) = d(\widehat{x}, S).$$

This implies that $Tx \in P_S(\widehat{x})$. Thus $\mathrm{T}(\mathrm{P}_S(\widehat{x})) \subset \mathrm{P}_S(\widehat{x}) = \mathrm{I}(\mathrm{P}_S(\widehat{x}))$. Now Theorem 2.2 implies that $P_S(\widehat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

**Theorem 2.7.** Let $X$ be a strongly $M$-starshaped metric space, $I, T : X \to X$ two mappings, $S$ be a subset of $X$ such that $T(\partial S \cap S) \subset S$ and $\widehat{x} \in F(T) \cap F(I)$. Suppose that $P_S(\widehat{x})$ is nonempty closed and $q$-starshaped with $q \in F(I) \cap M$, $I$ is affine on $P_S(\widehat{x})$, $cl(T(P_S(\widehat{x})))$ is compact, and $I(P_S(\widehat{x})) = P_S(\widehat{x})$. If $T$ and $I$ are pointwise $R$-subweakly commuting on $P_S(\widehat{x})$, T is continuous and $I$-nonexpansive on $P_S(\widehat{x}) \cup \{\widehat{x}\}$, then $P_S(\widehat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

**Remark 2.8.** Theorems 2.6-2.7 extend Theorems 1.1-1.2, Theorem 6 of Naz [16], main results of Singh [20, 21] and many others.

The following result extends Theorem 3.3 [2] and corresponding results in [10] for the case $D = C_S^I(\widehat{x})$.

**Theorem 2.9.** Let $X$ be a strongly $M$-starshaped metric space, $I, T : X \to X$ two mappings, $S$ be a subset of $X$ such that $T(\partial S \cap S) \subset I(S) \cap S$ and $\widehat{x} \in F(T) \cap F(I)$. Suppose that $P_S(\widehat{x})$ is nonempty closed and $q$-starshaped with $q \in F(I) \cap M$, $I$ is affine on $C_S^I(\widehat{x})$, $cl(T(C_S^I(\widehat{x})))$ is compact, and $I(C_S^I(\widehat{x})) = C_S^I(\widehat{x})$. If $T$ and $I$ are pointwise $R$-subweakly commuting on $C_S^I(\widehat{x})$, T is continuous, $I$-nonexpansive on $C_S^I(\widehat{x}) \cup \{\widehat{x}\}$, then $P_S(\widehat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

**Proof.** Let $x \in C_S^I(\widehat{x})$. Since $I(C_S^I(\widehat{x})) = C_S^I(\widehat{x})$, then $C_S^I(\widehat{x}) \subset P_S(\widehat{x})$. Thus by Lemma 1.2, $x \in \partial S \cap S$. Since $T(\partial S \cap S) \subset I(S) \cap S$,it follows that $Tx \in I(S)$. So there exists $z \in S$ such that $Tx = Iz$. Thus $z \in C_S^I(\widehat{x})$

and hence $cl(T(C_S^I(\widehat{x})) \subset C_S^I(\widehat{x}) = I(C_S^I(\widehat{x}))$. Now Theorem 2.2 implies that $P_S(\widehat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

The following result contains Theorem 3.2 [2].

**Theorem 2.10.** Let $X$ be a strongly $M$-starshaped metric space, $I, T : X \to X$ two mappings, $S$ be a subset of $X$ such that $T(\partial S \cap S) \subset S$ and $\widehat{x} \in F(T) \cap F(I)$. Suppose that $D = P_S(\widehat{x}) \cap C_S^I(\widehat{x})$ is nonempty closed and $q$-starshaped with $q \in F(T) \cap M$, $I$ is affine on $D$, $clT(D)$ is compact, and $I(D) = D$. If $T$ and $I$ are pointwise $R$-subweakly commuting on $D$, T is $I$-nonexpansive on $D \cup \{\widehat{x}\}$ and $I$ is nonexpansive on $P_S(\widehat{x}) \cup \{\widehat{x}\}$, then $P_S(\widehat{x}) \cap F(T) \cap F(I) \neq \emptyset$.
**Proof.** Let $x \in D$. Then $x \in P_S(\widehat{x})$ and $\|x - \widehat{x}\| = dist(\widehat{x}, S)$. Proceeding as in the proof of Theorem 2.6, we obtain $Tx \in P_S(\widehat{x})$. As $I$ is nonexpansive on $P_S(\widehat{x}) \cup \{\widehat{x}\}$, we have

$$d(ITx, \widehat{x}) = d(ITx, I\widehat{x}) \leq d(Tx, T\widehat{x}) = d(Ix, I\widehat{x}) = d(Ix, \widehat{x}) = d(\widehat{x}, S).$$

Thus $ITx \in P_S(\widehat{x})$. This implies that $Tx \in C_S^I(\widehat{x})$, and hence $Tx \in D$. Thus $T$ maps $D$ into itself. Theorem 2.2 further guarantees that $P_S(\widehat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

**Remarks 2.11.** A subset $S$ of a strongly $M$-starshaped metric space $X$ is said to have property $(N)$ w.r.t. $T$ [9, 11] if,
(i) $T : S \to S$,
(ii) $W(Tx, q, k_n) \in S$, for some $q \in S \cap M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1 and for each $x \in S$.
A mapping $I$ is said to have property $(C)$ on a set $S$ with property $(N)$ if $I(W(Tx, q, k_n)) = W(ITx, Iq, k_n)$ for each $x \in S$ and $n \geq 0$.
All results of the paper (Theorem 2.2-Theorem 2.10) remain valid provided $I$ is assumed to be surjective and affineness of $I$ and $q$-starshapedness of the set $S$ is replaced by the property $(C)$ and property $(N)$ respectively. Consequently, recent results due to Hussain and Berinde [9] and Hussain, O'Regan and Agarwal [11] are improved and extended.

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