Invariant Approximation Results for Pointwise $R$-Subweakly Commuting Maps

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Abstract
A common fixed point result for pointwise $R$-subweakly commuting maps in strongly $M$-starshaped metric spaces is obtained. As application, invariant approximation theorems are derived. Our results unify, and extend various known results existing in the literature.

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1. Introduction and preliminaries

We first review needed definitions. Let $X$ be a metric space with metric $d$, $M \subset X$ and $J=[0,1]$. The space $X$ is called;

(1) M-starshaped [22] if there exists a continuous mapping $W : X \times M \times J \rightarrow X$ satisfying

$$d(x, W(y, q, \lambda)) \leq \lambda d(x, y) + (1 - \lambda) d(x, q)$$

for all $x, y \in X$, $q \in M$ and all $\lambda \in J$;

(2) strongly $M$-starshaped [1, 16] if it is $M$-starshaped and satisfies the property (I), that is,

$$d(W(x, q, \lambda), W(y, q, \lambda)) \leq \lambda d(x, y)$$

for all $x, y \in X$, $q \in M$ and all $\lambda \in J$; (3) (strongly) convex if it is (strongly)
X-starshaped; (4) starshaped if it is \( \{q\}\)-starshaped for some \( q \in X \). Strongly convex metric space is also said to be a metric space of hyperbolic type (see Ciric [4]). Obviously, every normed space \( X \) is a strongly convex metric space with \( W \) defined by

\[
W(x, q, \lambda) = \lambda x + (1 - \lambda)q
\]

for all \( x, q \in X \) and all \( \lambda \in J \). More generally, if \( X \) is a linear space with a translation invariant metric satisfying

\[
d(\lambda x + (1 - \lambda)y, 0) \leq \lambda d(x, 0) + (1 - \lambda)d(y, 0),
\]

then \( X \) is a strongly convex metric space. A subset \( D \) of a \( M \)-starshaped metric space \( X \) is called \( q \)-starshaped if there exists \( q \in D \cap M \) such that \( W(x, q, \lambda) \in D \) for all \( x \in D \) and all \( \lambda \in J \). For details, we refer the reader to Al-Thagafi [1], Guay et al.[6] and Takahashi [22]).

Let \( I, T : X \to X \) be two mappings and \( D \subset X \). Then \( T \) is called;
(5) \( I \)-nonexpansive on \( D \) if \( d(Tx, Ty) \leq d(Ix, Iy) \), for all \( x, y \in D \); (6) \( I \)-contraction on \( D \) if there exists \( k \in [0, 1) \) such that \( d(Tx, Ty) \leq kd(Ix, Iy) \), for all \( x, y \in D \). A point \( x \in D \) is a coincidence point (common fixed point) of \( I \) and \( T \) if \(Ix = Tx \) \( (x = Ix = Tx) \). The set of coincidence points of \( I \) and \( T \) is denoted by \( C(I,T) \). The mappings \( I \) and \( T \) are called (7) commuting on \( D \) if \( ITx = TIx \) for all \( x \in D \); (8) pointwise \( R \)-weakly commuting on \( D \) if for given \( x \) in \( D \), there exists a real number \( R > 0 \) such that \( d(TIx, ITx) \leq Rd(Tx, Ix) \); (9) weakly compatible if they commute at their coincidence points, i.e., if \( ITx = TIx \) whenever \( Ix = Tx \). Suppose that \( D \) is \( q \)-starshaped with \( q \in F(S) \cap M \) and is both \( T \)- and \( I \)-invariant. Then \( T \) and \( I \) are said to be;
(10) \( R \)-subcommuting on \( D \) if there exists a real number \( R > 0 \) such that \( d(TIx, ITx) \leq \frac{R}{\lambda}d(W(Tx, q, \lambda), Ix) \) for all \( x \in D \) and all \( \lambda \in (0, 1] \); (11) \( R \)-subweakly commuting on \( D \) if for all \( x \in M \), there exists a real number \( R > 0 \) such that \( d(ITx, TIx) \leq R\text{dist}(Ix, \text{seg}[q, Tx]) \); (12) pointwise \( R \)-subcommuting on \( D \) if for given \( x \in D \), there exists a real number \( R > 0 \) such that \( d(TIx, ITx) \leq \frac{R}{\lambda}d(W(Tx, q, \lambda), Ix) \) for all \( k \in (0, 1] \); (13) pointwise \( R \)-subweakly commuting \([18] \) on \( D \) if for given \( x \in D \), there exists a real number \( R > 0 \) such that \( d(ITx, TIx) \leq R\text{dist}(Ix, \text{seg}[q, Tx]) \). Clearly, pointwise \( R \)-subweakly commuting maps are weakly compatible but not conversely in general and \( R \)-subweakly commuting maps are pointwise \( R \)-subweakly commuting but the converse does not hold in general. The mapping \( I \) is called affine on \( D \) if \( I(W(x, q, \lambda)) = W(Ix, Iq, \lambda) \) for all \( x \in D \) and all \( \lambda \in J \). Let \( S \subset X \) and \( \hat{x} \in X \). Then \( P_S(\hat{x}) = \{x \in S : d(x, \hat{x}) = d(\hat{x}, S)\} \) is called the set of best \( S \)-approximants to \( \hat{x} \), where \( d(\hat{x}, S) = \inf\{d(\hat{x}, y) : y \in S\} \) and \( C_S^I(\hat{x}) = \{x \in S : Ix \in P_S(\hat{x})\} \).
In 1963, Meinardus [17] employed the Schauder fixed point theorem to prove a result regarding invariant approximation. In 1979, Singh [20] proved the following extension of the result of Meinardus.

**Theorem 1.1.** Let \( T \) be a nonexpansive operator on a normed space \( X \), \( M \) a nonempty subset of \( X \), \( T(M) \subset M \) and \( u \in F(T) \). If \( P_M(u) \) is nonempty compact and starshaped, then \( P_M(u) \cap F(T) \neq \emptyset \).

Hicks and Humphries [8] found that Singh’s results remain true if \( T(M) \subset M \) is replaced by \( T(\partial M) \subset M \). In 1988, Sahab, Khan and Sessa [19] established the following result which contains the result of Hicks and Humphries and Theorem 1.1.

**Theorem 1.2.** Let \( I \) and \( T \) be selfmaps of a normed space \( X \) with \( u \in F(I) \cap F(T) \), \( M \subset X \) with \( T(\partial M) \subset M \), and \( q \in F(I) \). If \( D = P_M(u) \) is compact and \( q \)-starshaped, \( I(D) = D \), \( I \) is continuous and linear on \( D \), \( I \) and \( T \) are commuting on \( D \) and \( T \) is \( I \)-nonexpansive on \( D \cup \{u\} \), then \( P_M(u) \cap F(T) \cap F(I) \neq \emptyset \).

Invariant approximation results for commuting maps due to Al-Thagafi [2] extended and generalized Theorems 1.1-1.2 and the works of [7, 8, 21]. Al-Thagafi results have been further extended by [10, 11, 12, 18] to \( R \)-subweakly commuting and pointwise \( R \)-subweakly commuting.

The aim of this paper is to establish a common fixed point theorem for pointwise \( R \)-subweakly commuting maps in the setup of strongly \( M \)-starshaped metric spaces. As application, invariant approximation results for pointwise \( R \)-subweakly commuting and \( R \)-subcommuting maps are derived. Our results extend and unify the work of Al-Thagafi [2], Dotson [5], Habiniak [7], Hicks and Humphries [8], Hussain and Berinde [9], Hussain, O’Regan and Agarwal [11], Naz [16], Latif [15], Sahab, Khan and Sessa [19] and Singh [20, 21].

The following result will be needed.

**Lemma 1.4**[1]. Let \( D \) be a subset of a \( M \)-starshaped metric space \( (X, d) \) and \( \hat{x} \in X \). Then \( P_D(\hat{x}) \subset \partial D \cap D \).

2. **Main Results**
The following result will be needed.

**Theorem 2.1 [18].** Let $M$ be a closed subset of a metric space $(X,d)$, and let $I$ and $T$ be pointwise $R$-weakly commuting self-mappings of $M$. If $\text{cl}(T(M)) \subset I(M)$, $\text{cl}(T(M))$ is complete, $T$ is $I$-continuous and $I$ and $T$ satisfy for all $x, y \in M$ and $0 \leq h < 1$,

$$d(Tx, Ty) \leq h \max \{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx)\},$$

(2.1) then $M \cap F(I) \cap F(T)$ is a singleton.

**Theorem 2.2.** Let $S$ be a $q$-starshaped subset of a strongly $M$-starshaped metric space $X$ and $I, T : S \to S$ pointwise $R$-subweakly commuting mappings such that $\text{cl}T(S) \subset I(S)$. Suppose that $q \in F(I) \cup M$ and $I$ is affine, $\text{cl}T(S)$ is compact, $T$ is $I$-continuous, $I$ is continuous and $I$ and $T$ satisfy

$$d(Tx, Ty) \leq \max \left\{ \frac{d(Ix, Iy), \text{dist}(Ix, \text{seg}[q, Tx]), \text{dist}(Iy, \text{seg}[q, Ty])}{d(Ix, \text{seg}[q, Ty]), \text{dist}(Iy, \text{seg}[q, Tx])} \right\}$$

(2.2) for all $x, y \in M$. Then $S \cap F(I) \cap F(T) \neq \emptyset$.

**Proof.** Choose a sequence $\{k_n\} \subset (0, 1)$ such that $k_n \to 1$ as $n \to \infty$. Define for each $n$, a map $T_nx = W(Tx, q, k_n)$ for each $x$ in $S$. Since $\text{cl}T(S) \subset I(S)$ and $I$ is affine with $Iq = q$, then $\text{cl}T_n(S) \subset I(S)$. As $I$ and $T$ are pointwise $R$-subweakly commuting and $I$ is affine with $Iq = q$, so for each $x \in C_q(I,T)$

$$d(T_nIx, IT_nx) = d(W(TIx, q, k_n), I(W(Tx, q, k_n))) = d(W(TIx, q, k_n), W(ITx, q, k_n)) = k_n d(TIx, ITx) = k_n R \text{ dist}(Ix, \text{seg}[q, Tx]).$$

Thus $I$ and $T_n$ are pointwise $k_n R$-weakly commuting for all $n$. Also by (2.2),

$$d(T_nx, T_ny) = d(W(Tx, q, k_n), W(Ty, q, k_n)) \leq k_n d(Tx, Ty) \leq k_n \max \{d(Ix, Iy), \text{dist}(Ix, \text{seg}[q, Tx]), \text{dist}(Iy, \text{seg}[q, Ty]), \text{dist}(Ix, \text{seg}[q, Ty]), \text{dist}(Iy, \text{seg}[q, Tx])\} \leq k_n \max \{d(Ix, Iy), d(Ix, T_nx), d(Iy, T_ny), d(Ix, T_ny), d(Iy, T_nx)\}.$$
for each $x, y \in S$ and $0 < k_n < 1$. Since $\text{cl}(T(S))$ is compact, each $\text{cl}(T_n(S))$ is compact. By Theorem 2.1, for each $n \geq 1$, there exists $x_n \in S$ such that $x_n = Ix_n = T_n x_n$. The compactness of $\text{cl}(T(S))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ and $y \in \text{cl}T(S)$ such that $Tx_m \to y$ as $m \to \infty$. Since $k_m \to 1$, $x_m = \{W(Tx_m, q, k_m)\}$ converges to $y$. Since $T$ is continuous, $Tx_m \to Ty$ as $m \to \infty$. Thus $y = Ty$. Since $I$ is also continuous, we have $Iy = y$. Thus $S \cap F(I) \cap F(T) \neq \emptyset$.

**Theorem 2.3.** Let $S$ be a $q$-starshaped subset of a strongly $M$-starshaped metric space $X$ and $I, T : S \to S$ pointwise $R$-subweakly commuting mappings such that $\text{cl}T(S) \subset I(S)$. Suppose that $q \in F(I) \cup M$ and $I$ is affine, $\text{cl}T(S)$ is compact and $T$ is continuous and $I$-nonexpansive. Then $S \cap F(I) \cap F(T) \neq \emptyset$.

**Proof.** We follow the proof of Theorem 2.2 up to the equation $y = Ty$. Since $T(S) \subset I(S)$, we can choose $z \in S$ such that $y = Ty = I z$. Also,

$$d(Tx_m, Tz) \leq d(Ix_m, I z) = d(x_m, I z) = d(x_m, y).$$

Taking limit when $m \to \infty$, we obtain $Ty = Tz$. Thus $y = Ty = Tz = I z$. Hence $C(I, T)$ is nonempty. Since $I$ and $T$ are also weakly compatible, we have $I y = IT z = T I z = Ty = y$. Thus $S \cap F(I) \cap F(T) \neq \emptyset$.

As $R$-subcommuting mappings are pointwise $R$-subweakly commuting, so we obtain the following extension of the recent result of Naz ([18], Theorem 4).

**Corollary 2.4.** Let $S$ be a $q$-starshaped subset of a strongly $M$-starshaped metric space $X$ and $I, T : S \to S$ $R$-subcommuting mappings such that $\text{cl}T(S) \subset I(S)$. Suppose that $q \in F(I) \cup M$, $I$ is affine, $\text{cl}T(S)$ is compact and $T$ is continuous and $I$-nonexpansive. Then $S \cap F(I) \cap F(T) \neq \emptyset$.

**Remark 2.5.** Theorems 2.2-2.3 extend and improve Theorem 2.2 of Al-Thagafi [2], Theorem 1 of Dotson [5], Theorem 4 of Habiniak [7], Theorem 2.2 of Hussain and Berinde [9], Theorem 6 of Jungck and Sessa [13] and the corresponding results of Hussain and Jungck [10].

**Theorem 2.6.** Let $X$ be a strongly $M$-starshaped metric space, $I, T : X \to X$ two mappings, $S$ be a subset of $X$ such that $\partial S \cap S \subset S$ and $\hat{x} \in F(T) \cap F(I)$. Suppose that $P_S(\hat{x})$ is nonempty closed and $q$-starshaped with $q \in F(I) \cap M$, $I$ is affine on $P_S(\hat{x})$, $\text{cl}(T(P_S(\hat{x})))$ is compact, and $I(P_S(\hat{x})) = P_S(\hat{x})$. 
If $T$ is $I$-continuous, the pair $\{T, I\}$ is pointwise $R$-subweakly commuting and continuous on $P_S(\hat{x})$ and satisfy for all $x \in P_S(\hat{x}) \cup \{\hat{x}\}$,

$$d(Tx, Ty) \leq \begin{cases} 
    d(Ix, Iu) & \text{if } y = u, \\
    \max\{d(Ix, Iy), dist(Ix, [q, Tx]), dist(Iy, [q, Ty]), \\
    dist(Ix, [q, Ty]), dist(Iy, [q, Tx])\} & \text{if } y \in P_S(\hat{x}),
\end{cases} \quad (2.3)$$

then $P_S(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

**Proof.** Let $x \in P_S(\hat{x})$. Then by Lemma 1.4, $x \in \partial S \cap S$ and so $Tx \in S$ since $T(\partial S \cap S) \subset S$. As $T$ satisfies (2.3) on $P_S(\hat{x}) \cup \{\hat{x}\}$ and $I(P_S(\hat{x})) = P_S(\hat{x})$, we have

$$d(Tx, \hat{x}) = d(Tx, T\hat{x}) \leq d(Ix, I\hat{x}) = d(Ix, \hat{x}) = d(\hat{x}, S).$$

This implies that $Tx \in P_S(\hat{x})$. Thus $T(P_S(\hat{x})) \subset P_S(\hat{x}) = I(P_S(\hat{x}))$. Now Theorem 2.2 implies that $P_S(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

**Theorem 2.7.** Let $X$ be a strongly $M$-starshaped metric space, $I, T : X \rightarrow X$ two mappings, $S$ be a subset of $X$ such that $T(\partial S \cap S) \subset S$ and $\hat{x} \in F(T) \cap F(I)$. Suppose that $P_S(\hat{x})$ is nonempty closed and $q$-starshaped with $q \in F(I) \cap M$, $I$ is affine on $P_S(\hat{x})$, $cl(T(P_S(\hat{x})))$ is compact, and $I(P_S(\hat{x})) = P_S(\hat{x})$. If $T$ and $I$ are pointwise $R$-subweakly commuting on $P_S(\hat{x})$, $T$ is continuous and $I$-nonexpansive on $P_S(\hat{x}) \cup \{\hat{x}\}$, then $P_S(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

**Remark 2.8.** Theorems 2.6-2.7 extend Theorems 1.1-1.2, Theorem 6 of Naz [16], main results of Singh [20, 21] and many others.

The following result extends Theorem 3.3 [2] and corresponding results in [10] for the case $D = C_S^I(\hat{x})$.

**Theorem 2.9.** Let $X$ be a strongly $M$-starshaped metric space, $I, T : X \rightarrow X$ two mappings, $S$ be a subset of $X$ such that $T(\partial S \cap S) \subset I(S) \cap S$ and $\hat{x} \in F(T) \cap F(I)$. Suppose that $P_S(\hat{x})$ is nonempty closed and $q$-starshaped with $q \in F(I) \cap M$, $I$ is affine on $C_S^I(\hat{x})$, $cl(T(C_S^I(\hat{x})))$ is compact, and $I(C_S^I(\hat{x})) = C_S^I(\hat{x})$. If $T$ and $I$ are pointwise $R$-subweakly commuting on $C_S^I(\hat{x})$, $T$ is continuous, $I$-nonexpansive on $C_S^I(\hat{x}) \cup \{\hat{x}\}$, then $P_S(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

**Proof.** Let $x \in C_S^I(\hat{x})$. Since $I(C_S^I(\hat{x})) = C_S^I(\hat{x})$, then $C_S^I(\hat{x}) \subset P_S(\hat{x})$. Thus by Lemma 1.2, $x \in \partial S \cap S$. Since $T(\partial S \cap S) \subset I(S) \cap S$, it follows that $Tx \in I(S)$. So there exists $z \in S$ such that $Tx = Ix$. Thus $z \in C_S^I(\hat{x})$
and hence $cl(T(C^I_S(\hat{x}))) \subset C^I_S(\hat{x})$. Now Theorem 2.2 implies that $P_S(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

The following result contains Theorem 3.2 [2].

**Theorem 2.10.** Let $X$ be a strongly $M$-starshaped metric space, $I, T : X \to X$ two mappings, $S$ be a subset of $X$ such that $T(\partial S \cap S) \subset S$ and $\hat{x} \in F(T) \cap F(I)$. Suppose that $D = P_S(\hat{x}) \cap C^I_S(\hat{x})$ is nonempty closed and $q$-starshaped with $q \in F(T) \cap M$, $I$ is affine on $D$, $clT(D)$ is compact, and $I(D) = D$. If $T$ and $I$ are pointwise $R$-subweakly commuting on $D$, $T$ is $I$-nonexpansive on $D \cup \{\hat{x}\}$ and $I$ is nonexpansive on $P_S(\hat{x}) \cup \{\hat{x}\}$, then $P_S(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

**Proof.** Let $x \in D$. Then $x \in P_S(\hat{x})$ and $\|x - \hat{x}\| = dist(\hat{x}, S)$. Proceeding as in the proof of Theorem 2.6, we obtain $Tx \in P_S(\hat{x})$. As $I$ is nonexpansive on $P_S(\hat{x}) \cup \{\hat{x}\}$, we have

$$d(ITx, \hat{x}) = d(ITx, I\hat{x}) \leq d(Tx, T\hat{x}) = d(Ix, I\hat{x}) = d(Ix, \hat{x}) = d(\hat{x}, S).$$

Thus $ITx \in P_S(\hat{x})$. This implies that $Tx \in C^I_S(\hat{x})$, and hence $Tx \in D$. Thus $T$ maps $D$ into itself. Theorem 2.2 further guarantees that $P_S(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

**Remarks 2.11.** A subset $S$ of a strongly $M$-starshaped metric space $X$ is said to have property ($N$) w.r.t. $T$ [9, 11] if,

(i) $T : S \to S$,

(ii) $W(Tx, q, k_n) \in S$, for some $q \in S \cap M$ and a fixed sequence of real numbers $k_n(0 < k_n < 1)$ converging to 1 and for each $x \in S$.

A mapping $I$ is said to have property ($C$) on a set $S$ with property ($N$) if $I(W(Tx, q, k_n)) = W(ITx, Iq, k_n)$ for each $x \in S$ and $n \geq 0$.

All results of the paper (Theorem 2.2-Theorem 2.10) remain valid provided $I$ is assumed to be surjective and affineness of $I$ and $q$-starshapedness of the set $S$ is replaced by the property ($C$) and property ($N$) respectively. Consequently, recent results due to Hussain and Berinde [9] and Hussain, O’Regan and Agarwal [11] are improved and extended.

**References**


Invariant approximation results


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