Refinement of Generalized Jacobi (RGJ) Method for Solving System of Linear Equations

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Abstract

In this paper Refinement of Generalized Jacobi (RGJ) method for solving systems of linear algebraic equations is proposed and its convergence is discussed. Few numerical examples are considered to show the efficiency of the Refinement of Generalized Jacobi method over generalized Jacobi method.

Mathematics Subject Classification: 65F10, 65F50

Keywords: Generalized Jacobi method (GJ), M-matrix, Row strictly diagonally matrix, Convergence

1 Introduction

In many applications, one faces with the problem of solving large sparse linear systems of the form

$$Au = b$$ (1.1)

where A is a given non-singular real matrix of order N and b is a given N-dimensional real vector. As it is discussed by F.Naeimi Dafchahi [4], Ibrahim B.Kalambi [5] and Davod K. Salkuyeh [2], that Jacobi method is easier method to use for determination of the N-dimensional solution vector u of (1.1) but slow to converge. Davod K. Salkuyeh [2], introduced generalized method of Jacobi, which is more efficient than conventional Jacobi method. On the other hand F.Naeimi Dafchahi [4] developed the method called Refinement of Jacobi method and mentioned that the refinement Jacobi is as fast as successive over relaxation (SOR) method.
Consider linear system of equations (1.1) and the splitting made by Davod K. Salkuyeh [2] as

\[ A = T_m - E_m - F_m \]  \hspace{1cm} (1.2)

where \( T_m = (a_{ij}) \) be a banded matrix with band length \( 2m+1 \) is defined as

\[
t_{ij} = \begin{cases} 
  a_{ij}, & |j - i| \leq m \\
  0, & \text{otherwise}
\end{cases}
\]

where \(-E_m\) and \(-F_m\) are strictly lower and strictly upper triangular parts of \( A - T_m \) respectively and they are defined as follows:

\[
T_m = \begin{pmatrix}
  a_{1,1} & \cdots & a_{1,m+1} \\
  \vdots & \ddots & \vdots \\
  a_{m+1} & \cdots & a_{n-m,n} \\
  \vdots & \ddots & \vdots \\
  a_{n,m-n} & \cdots & a_{n,n}
\end{pmatrix}
\]

\[
E_m = \begin{pmatrix}
  -a_{m+2,1} \\
  \vdots \\
  -a_{n,1} \cdots -a_{n-m-1,n}
\end{pmatrix}
\]

\[
F_m = \begin{pmatrix}
  -a_{1,m+2} \cdots -a_{1,n} \\
  \vdots \\
  -a_{n-m-1,n}
\end{pmatrix}
\]

Then the generalized Jacobi method for solving equation (1.1) is defined as

\[
u^{n+1} = T_m^{-1}(E_m + F_m)u^n + T_m^{-1}b, \hspace{0.5cm} n = 0, 1, 2, \ldots \]  \hspace{1cm} (1.3)

where \( B_{G,I}^m = T_m^{-1}(E_m + F_m) \) is generalized Jacobi iteration matrix and \( T_m^{-1}b \) its corresponding iteration vector. In this paper, the refinement of generalized Jacobi method is considered in section 2 and its convergence properties are discussed in section 3. Few numerical examples are considered in the concluding section.
2 Refinement of Generalized Jacobi method (RGJ)

Since the rate of convergence of stationary iterative process depends on spectral radius of the iterative matrix, any reasonable modification of iterative matrix that will reduce the spectral radius and increases the rate of convergence of that method. Assume \( u^{(1)} \) be an initial approximation for solution of the linear system of equation (1.1), and \( b^{(1)}_i = \sum_{j=1}^{N} a_{ij} u_j^{(1)}, i = 1(1)N \). After solution of n-steps of (1.3), we have \( u^{(n+1)} = (u_1^{(n+1)}, u_2^{(n+1)}, \ldots, u_N^{(n+1)}) \). So we have again \( b^{(n+1)}_i = \sum_{j=1}^{N} a_{ij} u_j^{(n+1)}, i = 1(1)N \). Now we refine this obtained solution as \( b^{(n+1)}_i \rightarrow b_i \). Here we again assume \( \tilde{u}^{(n+1)} = (\tilde{u}_1^{(n+1)}, \tilde{u}_2^{(n+1)}, \ldots, \tilde{u}_N^{(n+1)}) \) be good approximation for solution of linear system (1.1).

i.e \( \tilde{u}^{(n+1)} \rightarrow u \), where \( u \) is the exact solution for (1.1). and

\[
b_i = \sum_{j=1}^{N} a_{ij} \tilde{u}_j^{(n+1)}, i = 1(1)N
\]

Since all \( \tilde{u}_j^{(n+1)} \) are unknown, we define it as follows.

\[
\tilde{u}^{(n+1)} = u^{(n+1)} + b^{(n+1)} - b,
\]

Now the Refinement of Generalized Jacobi in matrix form, by Considering equation (1.1) and splitting (1.2), is defined as

\[
(T_m - E_m - F_m)u = b \\
T_m u = (E_m + F_m)u + b \\
T_m u = (T_m - A)u + b \\
T_m u = T_m u + (b - Au) \\
u = u + T_m^{-1}(b - Au)
\]

From the above we obtain the iterative refinement formula in matrix form as

\[
\tilde{u}^{(n+1)} = u^{(n+1)} + T_m^{-1}(b - Au^{(n+1)})
\]

From (1.3) we have

\[
u^{(n+1)} = T_m^{-1}(E_m + F_m)u^{n} + T_m^{-1}b + T_m^{-1}(b - A[T_m^{-1}(E_m + F_m)u^{n} + T_m^{-1}b]) \\
= [T_m^{-1}(E_m + F_m)]^2u^n + [I + T_m^{-1}(E_m + F_m)]T_m^{-1}b
\]

We shall call the matrix \( \overline{B}_{GJ}^m = [(T_m - E_m)^{-1}F_m]^2 \) as the refinement of gen-
eralized Jacobi iteration matrix and \((I + T_m^{-1}(E_m + F_m))T_m^{-1}b\), as the refinement of generalized Jacobi vector.

3 Condition on the convergence of RGJ method

**Definition 1.** Let \(A = (a_{ij})\) be a real matrix, then \(A\) is an M-matrix if \(a_{ij} \leq 0\) for \(i \neq j\), \(A\) is non-singular and if \(A^{-1} \geq 0\).

**Definition 2.** An \(N\times N\) matrix \(A = (a_{ij})\) is said to be strictly diagonally dominant (SDD) if \(\sum_{j \neq i, j=1}^{N} |a_{ij}| < |a_{ii}|, i = 1(1)N\).

**Theorem 3.1.** Let \(A\) be SDD matrix. Then for any natural number \(m \leq N\) the GJ method convergent for any initial guess \(u^{(0)}\).

Proof:- See, Davod K. Salkuyeh [2]

**Theorem 3.2.** Let \(A = (a_{ij})\) be an M-matrix. Then for a given natural number \(m \leq N\), generalized Jacobi methods are convergent for any initial guess \(u^{(0)}\).

Proof:- See, Davod K. Salkuyeh [2]

**Theorem 3.3.** If \(A\) is strictly diagonally dominant matrix, then the refinement of generalized Jacobi method converges for any choice of the initial approximation \(u^{(0)}\).

Proof:- Assuming \(u\) is the real solution of (1.1), as \(A\) is SDD by Theorem 3.1, generalized Jacobi method is convergent. Let \(u^{(n+1)} \rightarrow u\) (exact solution). Then we have

\[ ||\pi^{(n+1)} - u||_{\infty} \leq ||u^{(n+1)} - u||_{\infty} + ||(T_m - E_m)^{-1}||_{\infty}||b - Au^{(n+1)}||_{\infty} \]

From the fact \(||u^{(n+1)} - u||_{\infty} \rightarrow 0\), we have \(||(b - Au^{(n+1)})||_{\infty} \rightarrow 0\). Therefore, \(||\pi^{(n+1)} - u||_{\infty} \rightarrow 0\). Hence refinement of generalized Jacobi method is convergent.

**Theorem 3.4.** Let \(A = (a_{ij})\) be an M-matrix. Then for a given natural number \(m \leq N\), the refinement of generalized Jacobi method converges for any choice of initial approximation \(u^{(0)}\).

Proof:- It follows from Theorem 3.2 and the proof of Theorem 3.3 above.

**Theorem 3.5.** If \(A\) is SDD matrix, then \(||B_{GJ}^m||_{\infty} < 1\).
Proof:- By Theorem 3.1 above and Convergence Theorem of Gourdin and Boumarat [1], \( ||B_{GJ}^m||_\infty < 1 \).

**Theorem 3.6.**

i) If A is SDD matrix, then \( ||\overline{B}_{GJ}^m||_\infty \leq ||B_{GJ}^m||_\infty < 1 \).

ii) If A is an M-matrix, then \( ||\overline{B}_{GJ}^m||_\infty \leq ||B_{GJ}^m||_\infty < 1 \).

Proof:-i) By the convergence theorem of Gourdin and Boumarat [1], we have

\[ \rho(B_{GJ}^m) \leq ||B_{GJ}^m||_\infty < 1 \]

Where \( \rho(B_{GJ}^m) \) is the spectral radius of generalized Jacobi method.

Again by Theorem 3.5, we have

\[ ||T_m^{-1}(E_m + F_m)||_\infty < 1 \].

(3.1)

As a result of equation (3.1), we have

\[ ||(T_m^{-1}(E_m + F_m))^2||_\infty = ||T_m^{-1}(E_m + F_m)||_\infty^2 \leq ||T_m^{-1}(E_m + F_m)||_\infty < 1 \].

Therefore, \( ||\overline{B}_{GJ}^m||_\infty \leq ||B_{GJ}^m||_\infty < 1 \)

ii) Similar to (i) above. We can have the proof the theorem.

**Remark 1.** We observe that the iterative matrix of refinement of generalized Jacobi is the square of generalized Jacobi iterative matrix. i.e. \( \overline{B}_{GJ}^m = [B_{GJ}^m]^2 \).

As it can be easily realized \( \rho(\overline{B}_{GJ}^m) = [\rho(B_{GJ}^m)]^2 \), where \( \rho(\overline{B}_{GJ}^m) \) is the spectral radius of RGJ iterative matrix and \( [\rho(B_{GJ}^m)]^2 \) is the spectral radius of GJ iterative matrix. As GJ method converges \( [\rho(B_{GJ}^m)] < 1 \), then \( \rho(\overline{B}_{GJ}^m) < \rho(B_{GJ}^m) \). Thus, if the GJ and RGJ method converge, the RGJ method converges faster than the GJ method.

### 4 Numerical Examples

Here we consider two examples to illustrate the theory developed in this paper. The result shows that refinement of generalized Jacobi method converges faster than Generalized Jacobi method, as in the two examples below the refinement of Generalized Jacobi reduces the number of iteration approximately by 50 percent.

1. Consider the system of equation considered by Young [3]

\[
\begin{pmatrix}
4 & 0 & -1 & -1 \\
0 & 4 & -1 & -1 \\
-1 & -1 & 4 & 0 \\
-1 & -1 & 0 & 4
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{pmatrix}
=
\begin{pmatrix}100 \\
0 \\
100 \\
0
\end{pmatrix}
\]

The solution of the above system is obtained and tabulated below by us-
ing the methods GJ and RGJ taking the initial approximations for u’s as all zeros, with an accuracy of $0.5 \times 10^{-4}$.

Table I: RGJ and GJ when $m = 1$

<table>
<thead>
<tr>
<th>i</th>
<th>RGJ $x_1^{(i)}$</th>
<th>RB $x_2^{(i)}$</th>
<th>$x_3^{(i)}$</th>
<th>$x_4^{(i)}$</th>
<th>GJ $x_1^{(i)}$</th>
<th>$x_2^{(i)}$</th>
<th>$x_3^{(i)}$</th>
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Table II: RGJ and GJ when $m = 2$

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<th>RB $x_2^{(i)}$</th>
<th>$x_3^{(i)}$</th>
<th>$x_4^{(i)}$</th>
<th>GJ $x_1^{(i)}$</th>
<th>$x_2^{(i)}$</th>
<th>$x_3^{(i)}$</th>
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<td>37.4998</td>
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2. Consider 2-cyclic matrix, which arises from discretization of the poisson's equations $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f$, on the unit square as considered by Dafchahi [4]. Now consider $Au = b$, where $u = (u_1, ..., u_6)^T$ and $b = (1, 0, 0, 0, 0, 0)$ or

$$
\begin{pmatrix}
4 & -1 & 0 & -1 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 \\
-1 & 0 & 0 & 4 & -1 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & -1 & 0 & -1 & 4
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6
\end{pmatrix}
= 
\begin{pmatrix}1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
$$

The solution of the above system is obtained and tabulated below by using the methods GJ and RGJ taking the initial approximations for $u$'s as all zeros, with an accuracy of $0.5 \times 10^{-6}$.

Table III: refinement of Generalized Jacobi when $m = 1$

<table>
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<tr>
<th>$i$</th>
<th>$u_1^{(i)}$</th>
<th>$u_2^{(i)}$</th>
<th>$u_3^{(i)}$</th>
<th>$u_4^{(i)}$</th>
<th>$u_5^{(i)}$</th>
<th>$u_6^{(i)}$</th>
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Table IV: Generalized Jacobi when $m = 1$

<table>
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<tr>
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<th>$u_1^{(i)}$</th>
<th>$u_2^{(i)}$</th>
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References


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