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Refinement of Generalized Jacobi (RGJ) Method for Solving System of Linear Equations

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Abstract

In this paper Refinement of Generalized Jacobi (RGJ) method for solving systems of linear algebraic equations is proposed and its convergence is discussed. Few numerical examples are considered to show the efficiency of the Refinement of Generalized Jacobi method over generalized Jacobi method.

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1 Introduction

In many applications, one faces with the problem of solving large sparse linear systems of the form

$$Au = b \tag{1.1}$$

where A is a given non-singular real matrix of order N and b is a given N-dimensional real vector. As it is discussed by F.Naeimi Dafchahi [4], Ibrahim B.Kalambi [5] and Davod K. Salkuyeh [2], that Jacobi method is easier method to use for determination of the N-dimensional solution vector u of (1.1)but slow to converge. Davod K. Salkuyeh [2], introduced generalized method of Jacobi, which is more efficient than conventional Jacobi method. On the other hand F.Naeimi Dafchahi [4] developed the method called Refinement of Jacobi method and mentioned that the refinement Jacobi is as fast as successive over relaxation (SOR) method.

Consider linear system of equations (1.1) and the splitting made by Davod K. Salkuyeh [2] as

$$A = T_m - E_m - F_m \tag{1.2}$$

where $T_m = (a_{ij})$ be a banded matrix with band length 2m+1 is defined as
$$t_{ij} = \begin{cases} a_{ij}, \mid j-i \mid \leq m \\ 0 \qquad\qquad \mathit{otherwise} \end{cases}$$
where $-E_m$ and $-F_m$ are strictly lower and strictly upper triangular parts of $A - T_m$ respectively and they are defined as follows:

$$T_m = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m+1} & & \\ \vdots & \ddots & & \ddots & \\ a_{m+1} & & \ddots & & a_{n-m,n} \\ & \ddots & & \ddots & \vdots \\ & & a_{n,n-m} & \cdots & a_{n,n} \end{pmatrix}$$

$$E_m = \begin{pmatrix} & & & \\ & & & \\ -a_{m+2,1} & & & \\ \vdots & \ddots & & \\ -a_{n,1} & \cdots & -a_{n-m-1,n} & \end{pmatrix}$$

$$F_m = \begin{pmatrix} & -a_{1,m+2} & \cdots & -a_{1,n} \\ & & \ddots & \vdots \\ & & & -a_{n-m-1,n} \\ & & & \\ & & & \end{pmatrix}$$

Then the generalized Jacobi method for solving equation (1.1) is defined as

$$u^{n+1} = T_m^{-1}(E_m + F_m)u^n + T_m^{-1}b, \quad n = 0, 1, 2, ... \tag{1.3}$$

where $B_{GJ}^m = T_m^{-1}(E_m + F_m)$ is generalized Jacobi iteration matrix and $T_m^{-1}b$ its corresponding iteration vector. In this paper, the refinement of generalized Jacobi method is considered in section 2 and its convergence properties are discussed in section 3. Few numerical examples are considered in the concluding section.

2 Refinement of Generalized Jacobi method(RGJ)

Since the rate of convergence of stationary iterative process depends on spectral radius of the iterative matrix, any reasonable modification of iterative matrix that will reduce the spectral radius and increases the rate of convergence of that method. Assume $u^{(1)}$ be an initial approximation for solution of the linear system of equation (1.1), and $b_i^{(1)} = \sum_{j=1}^{N} a_{ij} u_j^{(1)}, i = 1(1)N$. After solution of n-steps of (1.3), we have $u^{(n+1)} = (u_1^{(n+1)}, u_2^{(n+1)}, ..., u_N^{(n+1)})$. So we have again $b_i^{(n+1)} = \sum_{j=1}^{N} a_{ij} u_j^{(n+1)}, i = 1(1)N$. Now we refine this obtained solution as $b_i^{(n+1)} \longrightarrow b_i$. Here we again assume $\overline{u}^{(n+1)} = (\overline{u}_1^{(n+1)}, \overline{u}_2^{(n+1)}, ..., \overline{u}_N^{(n+1)})$ be good approximation for solution of linear system(1.1).
$i.e\ \overline{u}^{(n+1)} \longrightarrow u,\ where\ u\ is\ the\ exact\ solution\ for$ (1.1). and

$$b_i = \sum_{j=1}^{N} a_{ij} \overline{u}_j^{(n+1)}, i = 1(1)N$$

Since all $\overline{u}_j^{(n+1)}$ are unknown, we define it as follows.

$$\overline{u}^{(n+1)} = u^{(n+1)} + b^{(n+1)} - b.$$

Now the Refinement of Generalized Jacobi in matrix form , by Considering equation (1.1) and splitting (1.2), is defined as

$$\begin{aligned}
(T_m - E_m - F_m)u &= b \\
T_m u &= (E_m + F_m)u + b \\
T_m u &= (T_m - A)u + b \\
T_m u &= T_m u + (b - Au) \\
u &= u + T_m^{-1}(b - Au)
\end{aligned}$$

From the above we obtain the iterative refinement formula in matrix form as

$$\overline{u}^{(n+1)} = u^{(n+1)} + T_m^{-1}(b - Au^{(n+1)})$$

From (1.3) we have

$$\begin{aligned}
u^{(n+1)} &= T_m^{-1}(E_m + F_m)u^n + T_m^{-1}b + T_m^{-1}(b - A[T_m^{-1}(E_m + F_m)u^n \\
&\quad + T_m^{-1}b]) \\
&= [T_m^{-1}(E_m + F_m)]^2 u^n + [I + T_m^{-1}(E_m + F_m)]T_m^{-1}b \qquad (2.1)
\end{aligned}$$

We shall call the matrix $\overline{B}_{GJ}^{m} = [(T_m - E_m)^{-1}F_m]^2$ as the refinement of generalized Jacobi iteration matrix and $[I+T_m^{-1}(E_m+F_m)]T_m^{-1}b$, as the refinement of generalized Jacobi vector.

3 Condition on the convergence of RGJ method

Definition 1. Let $A = (a_{ij})$ be a real matrix, then A is an M-matrix if $a_{ij} \leq 0 for i \neq j$, A is non-singular and if $A^{-1} \geq 0$.

Definition 2. An NxN matrix $A = (a_{ij})$ is said to be strictly diagonally dominant (SDD) if $\sum_{j \neq i, j=1}^{N} |a_{ij}| < |a_{ii}|, i = 1(1)N$.

Theorem 3.1. Let A be SDD matrix. Then for any natural number $m \leq N$ the GJ method convergent for any initial guess $u^{(0)}$.

Proof:- See, Davod K. Salkuyeh [2]

Theorem 3.2. Let $A = (a_{ij})$ be an M-matrix. Then for a given natural number $m \leq N$, generalized Jacobi methods are convergent for any initial guess $u^{(0)}$.

Proof:- See, Davod K. Salkuyeh [2]

Theorem 3.3. If A is strictly diagonally dominant matrix, then the refinement of generalized Jacobi method converges for any choice of the initial approximation $u^{(0)}$.

Proof:-Assuming u is the real solution of (1.1), as A is SDD by Theorem 3.1, generalized Jacobi method is convergent.
Let $u^{(n+1)} \longrightarrow u$ (exact solution). Then we have

$$||\overline{u}^{(n+1)} - u||_\infty \leq ||u^{(n+1)} - u||_\infty + ||(T_m - E_m)^{-1}||_\infty ||(b - Au^{(n+1)})||_\infty$$

From the fact $||u^{(n+1)} - u||_\infty \longrightarrow 0$, we have $||(b - Au^{(n+1)})||_\infty \longrightarrow 0$. Therefore , $||\overline{u}^{(n+1)} - u||_\infty \longrightarrow 0$. Hence refinement of generalized Jacobi method is convergent.

Theorem 3.4. Let $A = (a_{ij})$ be an M-matrix. Then for a given natural number $m \leq N$, the refinement of generalized Jacobi method converges for any choice of initial approximation $u^{(0)}$.

Proof:-It follows from Theorem 3.2 and the proof of Theorem 3.3 above.

Theorem 3.5. If A is SDD matrix, then $||B_{GJ}^m||_\infty < 1$.

Proof:- By Theorem 3.1 above and Convergence Theorem of Gourdin and Boumarat [1] , $||B_{GJ}^m||_\infty < 1$.

Theorem 3.6. i) If A is SDD matrix, then $||\overline{B}_{GJ}^m||_\infty \le ||B_{GJ}^m||_\infty < 1$.
ii) If A is an M-matrix, then $||\overline{B}_{GJ}^m||_\infty \le ||B_{GJ}^m||_\infty < 1$.

Proof:-i) By the convergence theorem of Gourdin and Boumarat [1] , we have $\rho(B_{GJ}^m) \le ||B_{GJ}^m||_\infty < 1$.
Where $\rho(B_{GJ}^m)$ is the spectral radius of generalized Jacobi method.

Again by Theorem 3.5, we have

$$||T_m^{-1}(E_m + F_m)||_\infty < 1. \tag{3.1}$$

As a result of equation (3.1), we have

$$||[T_m^{-1}(E_m + F_m)]^2||_\infty = ||T_m^{-1}(E_m + F_m||_\infty^2 \le ||T_m^{-1}(E_m + F_m)||_\infty < 1.$$

Therefore, $||\overline{B}_{GJ}^m||_\infty \le ||B_{GJ}^m||_\infty < 1$
ii) Similar to (i) above. We can have the proof the theorem.

Remark 1. We observe that the iterative matrix of refinement of generalized Jacobi is the square of generalized Jacobi iterative matrix. *i.e.* $\overline{B}_{GJ}^m = [B_{GJ}^m]^2$. As it can be easily realized $\rho(\overline{B}_{GJ}^m) = [\rho(B_{GJ}^m)]^2$, where $\rho(\overline{B}_{GJ}^m)$ is the spectral radius of RGJ iterative matrix and $[\rho(B_{GJ}^m)]^2$ is the spectral radius of GJ iterative matrix. As GJ method converges $[\rho(B_{GJ}^m)] < 1$, then
$\rho(\overline{B}_{GJ}^m) < \rho(B_{GJ}^m)$. Thus, if the GJ and RGJ method converge, the RGJ method converges faster than the GJ method.

4 Numerical Examples

Here we consider two examples to illustrate the theory developed in this paper.The result shows that refinement of generalized Jacobi method converges faster than Generalized Jacobi method, as in the two examples below the refinement of Generalized Jacobi reduces the number of iteration approximately by 50 percent.
1. Consider the system of equation considered by Young [3]

$$\begin{pmatrix} 4 & 0 & -1 & -1 \\ 0 & 4 & -1 & -1 \\ -1 & -1 & 4 & 0 \\ -1 & -1 & 0 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ 100 \\ 0 \end{pmatrix}$$

The solution of the above system is obtained and tabulated below by using the methods GJ and RGJ taking the initial approximations for u's as all zeros, with an accuracy of 0.5×10^{-4}.

Table I: RGJ and GJ when $m = 1$

	RGJ				GJ			
i	$x_1^{(i)}$	$x_2^{(i)}$	$x_3^{(i)}$	$x_4^{(i)}$	$x_1^{(i)}$	$x_2^{(i)}$	$x_3^{(i)}$	$x_4^{(i)}$
1	31.6667	8.3333	33.3333	7.9167	25.0000	6.6667	26.6667	0.0000
2	36.4097	11.6875	36.7500	11.5503	31.6667	8.3333	33.3333	7.9167
...	...	...	...	...	...	...	...	...
8	37.4999	12.4999	37.4999	12.4999	37.4608	12.4705	37.4719	12.4633
9	37.5000	12.5000	37.5000	12.4999	37.4838	12.4876	37.4871	12.4828
...					...	...	...	...
15					37.4999	12.4999	37.4999	12.4999
16					37.4999	12.4999	37.4999	12.4999

Table II: RGJ and GJ when $m = 2$

	RGJ				GJ			
i	$x_1^{(i)}$	$x_2^{(i)}$	$x_3^{(i)}$	$x_4^{(i)}$	$x_1^{(i)}$	$x_2^{(i)}$	$x_3^{(i)}$	$x_4^{(i)}$
1	34.7748	12.0533	36.7070	11.5061	33.9713	9.5693	35.8852	2.3900
2	37.3017	12.4659	37.4418	12.4216	34.7748	12.0531	36.7069	11.5038
...	...	...	...	...	...	...	...	...
5	37.4999	12.4999	37.4999	12.4999	37.4774	12.4841	37.4904	12.4440
6	37.4999	12.5000	37.4999	12.4999	37.4849	12.4972	37.4956	12.4913
...					...	...	...	...
11					37.4993	12.4998	37.4998	12.4975
12					37.4993	12.4998	37.4998	12.4975

2. Consider 2-cyclic matrix, which arises from discretization of the poisson's equations $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f$, on the uint square as considered by Dafchahi [4]. Now consider Au = b, where $u = (u_1, ..., u_6)^T$ and
b = (1, 0, 0, 0, 0, 0) or

$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The solution of the above system is obtained and tabulated below by using the methods GJ and RGJ taking the initial approximations for u's as all zeros, with an accuracy of 0.5×10^{-6}.

Table III: refinement of Generalized Jacobi when $m = 1$

	Refinement of Generalized Jacobi					
i	$u_1^{(i)}$	$u_2^{(i)}$	$u_3^{(i)}$	$u_4^{(i)}$	$u_5^{(i)}$	$u_6^{(i)}$
1	0.267857	0.071429	0.017857	0.077168	0.040816	0.014668
2	0.291705	0.089650	0.026079	0.085004	0.048313	0.018598
...	...	...	...	...	...	...
6	0.294823	0.093166	0.028156	0.086127	0.049688	0.019461
7	0.294823	0.093166	0.028157	0.086127	0.049688	0.019461

Table IV: Generalized Jacobi when $m = 1$

	Generalized Jacobi					
i	$u_1^{(i)}$	$u_2^{(i)}$	$u_3^{(i)}$	$u_4^{(i)}$	$u_5^{(i)}$	$u_6^{(i)}$
1	0.267857	0.071429	0.017857	0.000000	0.000000	0.000000
2	0.267857	0.071429	0.017857	0.077168	0.040816	0.014668
...	...	...	...	...	...	...
11	0.294824	0.093166	0.028156	0.086125	0.049685	0.019458
12	0.294824	0.093166	0.028156	0.086125	0.049685	0.019458

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