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On Absolute Cesáro Summability Factors of Infinite Series

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Abstract

In this paper a general theorem concerning the $\varphi-|C,1;\delta|_k$ summability factors of infinite series has been proved.

Keywords: Summability factors, Infinite series

1 INTRODUCTION

Let (φ_n) be a sequence of positive real numbers and $\sum a_n$ be a given infinite series with the sequence of partial sum (s_n). By (t_n), we denote the n-th $(C,1)$ means of the sequence (na_n). The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if [1]

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty \tag{1.1}$$

and it is said to be summable $\varphi - |C,1|_k$, $k \geq 1$, if [5]

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |t_n|^k < \infty \tag{1.2}$$

and it is also summable for $\varphi - |C,1;\delta|_k$, $k \geq 1$, $\delta \geq 0$if [2]

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} |t_n|^k < \infty \tag{1.3}$$

If we take $\varphi = n$, $\delta = 0$, then $\varphi - |C,1;\delta|_k$ summability reduces to $|C,1|_k$ summability.

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Concerning the $\varphi - |C,1|_k$ summability factor, Özarslan [4] has proved the following theorem.

Theorem 2.1. Let (φ_n) be a sequence of positive real numbers If

$$\lambda_m = O(1) \text{ as } m \to \infty \tag{2.1}$$

$$\sum_{n=1}^{m} n \log n \left|\Delta^2 \lambda_n\right| = O(1) \tag{2.2}$$

$$\sum_{v=1}^{m} \frac{\varphi_v^{k-1}}{v^k} |t_v|^k = O(\log m) \text{ as } m \to \infty \tag{2.3}$$

$$\sum_{n=v}^{m} \frac{\varphi_n^{k-1}}{n^{k+1}} = O\left(\frac{\varphi_v^{k-1}}{v^k}\right) \tag{2.4}$$

then the series $\sum a_n \lambda_n$ is summable $\varphi - |C,1|_k$, $k \geq 1$.

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The aim of this paper is to generalize Theorem 2.1.

Theorem 3.1. Let (φ_n) be a sequence of positive real numbers and the condition (2.1)-(2.2) of Theorem 2.1 are satisfied. If

$$\sum_{v=1}^{m} \frac{\varphi_v^{k-1}}{v^{k-\delta k}} |t_v|^k = O(\log m.\mu_m) \text{ as } m \to \infty \tag{3.1}$$

$$\sum_{n=v}^{m} \frac{\varphi_n^{k-1}}{n^{k+1-\delta k}} = O\left(\frac{\varphi_v^{k-1}}{v^{k-\delta k}}\right) \tag{3.2}$$

$$n \log n \mu_n \Delta\left(\frac{1}{\mu_n}\right) = O(1) \tag{3.3}$$

then the series $\sum \frac{a_n \lambda_n}{\mu_n}$ is summable $\varphi - |C,1;\delta|_k$, $k \geq 1$, $\delta \geq 0$ and μ_n is a positive non-decreasing sequence.

4 PROOF OF THE THEOREM 3.1

Let T_n be the n-th $(C,1)$ mean of the sequence $(na_n\lambda_n)$. Applying Able's transformation, we get that

$$\begin{aligned}
T_n &= \frac{1}{n+1}\sum_{v=1}^{n}\frac{va_v\lambda_v}{\mu_v} \\
&= \frac{1}{n+1}\sum_{v=1}^{n-1}\Delta\left(\frac{\lambda_v}{\mu_v}\right)\sum_{r=0}^{v}ra_r + \frac{\lambda_n}{(n+1)\mu_n}\sum_{r=0}^{n}ra_r \\
&= \frac{1}{n+1}\sum_{v=1}^{n-1}\frac{1}{\mu_v}\Delta\lambda_v\sum_{r=0}^{v}ra_r + \frac{1}{n+1}\sum_{v=1}^{n-1}\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)\sum_{r=0}^{v}ra_r \\
&+ \frac{\lambda_n}{(n+1)\mu_n}\sum_{r=0}^{n}ra_r \\
&= \frac{1}{n+1}\sum_{v=1}^{n-1}\frac{\Delta\lambda_v}{\mu_v}(v+1)t_v + \frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)\lambda_{v+1}t_v\Delta\left(\frac{1}{\mu_v}\right) + \\
&+ \frac{\lambda_n t_n}{\mu_n} \\
&= T_{n,1}+T_{n,2}+T_{n,3}, \text{ (say)}.
\end{aligned}$$

Since $|T_{n,1}+T_{n,2}+T_{n,3}|^k < 3^k\left(|T_{n,1}|^k+|T_{n,2}|^k+|T_{n,3}|^k\right)$.

In order to complete the proof of the Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^{k-\delta k}}|T_{n,r}|^k < \infty, \text{ for } r = 1,2,3$$

Now, when $k \geq 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'} = 1$, we get that

$$\sum_{n=2}^{m} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left|T_{n,1}\right|^k =$$
$$= \sum_{n=2}^{m} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left|\frac{1}{n+1} \sum_{v=1}^{n-1} \frac{\Delta\lambda_v}{\mu_v}(v+1)t_v\right|^k$$
$$= O(1) \sum_{n=2}^{m} \frac{\varphi_n^{k-1}}{n^{2k-\delta k}} \left\{\sum_{v=1}^{n-1} v\frac{|\Delta\lambda_v|}{\mu_v} |t_v|\right\}^k$$
$$= O(1) \sum_{n=2}^{m} \frac{\varphi_n^{k-1}}{n^{2k-\delta k}} \sum_{v=1}^{n-1} \frac{v\,|\Delta\lambda_v|}{\mu_v} |t_v|^k \left\{\sum_{v=1}^{n-1} v\frac{|\Delta\lambda_v|}{\mu_v}\right\}^{k-1}$$
$$= O(1) \sum_{n=2}^{m} \frac{\varphi_n^{k-1}}{n^{k-\delta k-1}} \left\{\sum_{v=1}^{n-1} v\frac{|\Delta\lambda_v|}{\mu_v} |t_v|^k\right\}$$
$$= O(1) \sum_{v=1}^{m} \frac{v\,|\Delta\lambda_v|\,|t_v|^k}{\mu_v} \left\{\sum_{n=v}^{m} \frac{\varphi_n^{k-1}}{n^{k+1-\delta k}}\right\}$$
$$= O(1) \sum_{v=1}^{m} v\frac{|\Delta\lambda_v|}{\mu_v} \frac{\varphi_v^{k-1}}{v^{k-\delta k}} |t_v|^k$$
$$= O(1) \sum_{v=1}^{m-1} \left|\Delta\left(\frac{v\,|\Delta\lambda_v|}{\mu_v}\right)\right| \sum_{r=1}^{v} \frac{\varphi_r^{k-1}}{r^{k-\delta k}} |t_r|^k + m\frac{\Delta\lambda_m}{\mu_m} \sum_{r=1}^{v} \frac{\varphi_r^{k-1}}{r^{k-\delta k}} |t_r|^k$$
$$= O(1) \sum_{v=1}^{m-1} \frac{|\Delta\lambda_v|}{\mu_v} \log v.\mu_v + \sum_{v=1}^{m-1} (v+1)\Delta\left(\frac{|\Delta\lambda_v|}{\mu_v}\right) \log v.\mu_v +$$
$$+ m\frac{|\Delta\lambda_m|}{\mu_m} \log m.\mu_m$$
$$= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| \log v + \sum_{v=1}^{m-1} (v+1)\frac{1}{\mu_v} \left|\Delta^2\lambda_v\right| \log v.\mu_v +$$
$$+ \sum_{v=1}^{m-1} (v+1) \left|\Delta\lambda_{v+1}\right| \Delta\left(\frac{1}{\mu_v}\right) \log v.\mu_v + m \left|\Delta\lambda_m\right| \log m$$
$$= O(1)$$

by virtue of the hypotheses of Theorem 3.1.

$$\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^{k-\delta k}}\left|T_{n,2}\right|^k =$$

$$= \sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{k-\delta k}}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}\lambda_{v+1}(v+1)t_v\Delta\left(\frac{1}{\mu_v}\right)\right|^k$$

$$= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{2k-\delta k}}\left\{\sum_{v=1}^{n-1}v\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)|t_v|\right\}^k$$

$$= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{2k-\delta k}}\sum_{v=1}^{n-1}v\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)|t_v|^k \times$$

$$\times\left\{\sum_{v=1}^{n-1}v\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{k-\delta k+1}}\left\{\sum_{v=1}^{n-1}v\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)|t_v|^k\right\}$$

$$= O(1)\sum_{v=1}^{m-1}\left|\Delta\{v\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)\right|\sum_{r=1}^{v}\frac{\varphi_r^{k-1}}{r^{k-\delta k}}|t_r|^k +$$

$$+ m\lambda_{m+1}\Delta\left(\frac{1}{\mu_m}\right)\sum_{r=1}^{m}\frac{\varphi_r^{k-1}}{r^{k-\delta k}}|t_r|^k$$

$$= O(1)\sum_{v=1}^{m-1}\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)\mu_v\log v +$$

$$+ \sum_{v=1}^{m-1}\Delta\left\{\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)\right\}\log v.\mu_v + m\lambda_{m+1}\Delta\left(\frac{1}{\mu_m}\right)\log m.\mu_m$$

$$= O(1)\sum_{v=1}^{m-1}\lambda_{v+1}\log v + \sum_{v=1}^{m-1}(v+1)\Delta\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)\log v.\mu_v +$$

$$+ \sum_{v=1}^{m-1}(v+1)\lambda_{v+2}\Delta^2\left(\frac{1}{\mu_v}\right)\log v.\mu_v + m\lambda_{m+1}\log m$$

$$= O(1) \text{ as } m\to\infty$$

by virtue of the hypotheses of the Theorem 3.1, Finally

$$\begin{aligned}
&\sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left|T_{n,3}\right|^k = \\
&= \sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left|\frac{\lambda_n t_n}{\mu_n}\right|^k \\
&= O(1) \sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left|t_n\right|^k \left|\sum_{v=n}^{\infty} \Delta\left(\frac{\lambda_v}{\mu_v}\right)\right| \\
&= O(1) \sum_{v=1}^{\infty} \left|\Delta\left(\frac{\lambda_v}{\mu_v}\right)\right| \sum_{n=1}^{v} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left|t_n\right|^k \\
&= O(1) \sum_{v=1}^{\infty} \frac{1}{\mu_v} \Delta\lambda_v \log v.\mu_v + \sum_{v=1}^{\infty} \lambda_{v+1} \Delta\left(\frac{1}{\mu_v}\right) \log v.\mu_v \\
&= O(1) \quad \text{as } m \to \infty
\end{aligned}$$

by virtue of the hypotheses of the Theorem 3.1.

Therefore we get that $\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left|T_{n,r}\right|^k = O(1)$, as $m \to \infty$ for $r = 1, 2, 3$

This complete proof of Theorem 3.1.

References

[1] FLETT, T. M. : On an extension of absolute summability and some theorems of Littlewood and Paley; Proc. Lond. Math. Soc. 7, (1957), 113-141.

[2] FLETT, T. M. : Some more theorems concerning the absolute summability of Fourier series; Proc. Lond. Math. Soc. 8, (1958), 357-387.

[3] MAZHAR, S. M. : On $|C, 1|_k$ summability factors of infinite series; Indian J. Math. 14, (1972), 45-48.

[4] ÖZARSLAN, H. S. : On absolute Cesáro summability factors of infinite series; Com. in Math. Anal. 3, 1, (2007), 53-56.

[5] SEYHAN, H. : The absolute summability Methods; Ph.D. Thesis, Kayseri (1995), 1-57.

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