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A Common Random Fixed Point Theorem in Hilbert-Space

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Abstract

The Objective of this paper is to obtain Common Random Fixed Point Theorem for Four Continuous Random Operators defined on a non empty closed subset of a Separable Hilbert-Space.

Keywords: Separable Hilbert Space, Random Operators, Common Random Fixed Point, Rational Inequality.

1. Introduction

In this paper we construct a Sequence of a Separable function and consider its convergence to the common unique Random Fixed Point[1][2][3] of four continuous Random Operator defined on a non empty Closed subset of a Separable Hilbert Space.

In this paper we denotes (P_1, P_2) as Measurable Space consisting of Sets P_1 & P_2, P_2 is subset of P_1, H stands for Separable Hilbert-Space and C is a non empty closed subset of H.

2. Main Result

Let C be a non empty closed subset of a Separable Hilbert-Space H. Let E, F, S and T be four continuous Random Operators defined on C such that
For $t \in P_1$, E(t), F(t), S(t), T(t)
$C \to C$ Satisfy
If ES = SE, FT = TF, $E(H) \subseteq T(H)$, $F(H) \subseteq S(H)$ ……. (1)
And

$$\|Ex - Fy\|^2 \leq \frac{\beta_1 \|Sx - Ex\|^2 [\|Ty - Fy\|^2 + \|Ex - Ty\|^2]}{\|Sx - Ty\|^2 + \|Ex - Ty\|^2} + \frac{\beta_2 \|Ex - Ty\|^2 [\|Sx - Ex\|^2 + \|Ty - Fy\|^2]}{\|Sx - Ty\|^2 + \|Ex - Ty\|^2} + \frac{\beta_3 \|Sx - Ex\|^2 \ \|Ty - Fy\|^2]}{\|Sx - Ty\|^2} + \frac{\beta_4 \|Sx - Ty\|^2 [\|Sx - Ex\|^2 + \|Ty - Fy\|^2]}{1 + \|Ex - Ty\|^2} + \beta_5 \|Ex - Ty\|^2 \quad \text{……. (2)}$$

<u>Proof</u> :
Let the function $g_0 : P_1 \to C$ be arbitrary separable function. From (1). $\exists$ a function $g_1 : P_1 \to C$ such that $T(t, g_1(t)) = E(t, g_0(t))$ for $t \in P_1$ and for this function $g_1 : P_1 \to C$, we can choose another function $g_2 : P_1 \to C$ such that $F(t, g_1(t)) = S(t, g_2(t))$ for $t \in P_1$ and so on.

By using the method of induction we can define a sequence of functions for $t \in P_1$ and $\{y_n(t)\}$ such that

$$y_{2n}(t) = T(t, g_{2n+1}(t)) = E(t, g_{2n}(t)) \quad \text{……. (3)}$$

$$\text{and} \quad y_{2n+1}(t) = S(t, g_{2n+2}(t)) = F(t, g_{2n+1}(t)) \quad \text{……. (4)}$$

for $t \in P_1$ and n = 0, 1, 2, 3, ……..
From (1) and (2) we have for $t \in P_1$

$$\| y_{2n}(t) - y_{2n+1}(t)\|^2 = \| E(t, g_{2n}(t)) - F(t, g_{2n+1}(t)) \|^2$$

$$\leq \frac{\beta_1 \|S(t,g_{2n}(t)) - E(t,g_{2n}(t))\|^2 [\|T(t,g_{2n+1}(t)) - F(t,g_{2n+1}(t))\|^2 + \|E(t,g_{2n}(t)) - T(t,g_{2n+1}(t))\|^2]}{\|S(t,g_{2n}(t)) - T(t,g_{2n+1}(t))\|^2 + \|E(t,g_{2n}(t)) - T(t,g_{2n+1}(t))\|^2}$$

$$+ \frac{\beta_2 \|E(t,g_{2n}(t)) - T(t,g_{2n+1}(t))\|^2 [\|S(t,g_{2n}(t)) - E(t,g_{2n}(t))\|^2 + \|T(t,g_{2n+1}(t)) - F(t,g_{2n+1}(t))\|^2]}{\|S(t,g_{2n}(t)) - T(t,g_{2n+1}(t))\|^2 + \|E(t,g_{2n}(t)) - T(t,g_{2n+1}(t))\|^2}$$

$$+ \frac{\beta_3 \|T(t, g_{2n+1}(t)) - F(t, g_{2n+1}(t))\|^2 \ \|S(t, g_{2n}(t)) - E(t, g_{2n}(t))\|^2}{\|S(t, g_{2n}(t)) - T(t, g_{2n+1}(t))\|^2}$$

$$+ \frac{\beta_4 \|S(t, g_{2n}(t)) - T(t, g_{2n+1}(t))\|^2 [\|S(t, g_{2n}(t)) - E(t, g_{2n}(t))\|^2 + \|T(t, g_{2n+1}(t)) - F(t, g_{2n+1}(t))\|^2]}{1 + \|E(t, g_{2n}(t)) - T(t, g_{2n+1}(t))\|^2}$$

$$+ \beta_5 \ \| S(t, g_{2n}(t)) - T(t, g_{2n+1}(t)) \|^2$$

$$\leq \frac{\beta_1 \|y_{2n-1}(t) - y_{2n}(t)\|^2 [\|y_{2n}(t) - y_{2n+1}(t)\|^2 + \|y_{2n}(t) - y_{2n}(t))\|^2]}{\|y_{2n-1}(t)) - y_{2n}(t))\|^2 + \|y_{2n}(t) - y_{2n}(t))\|^2}$$

$$+ \frac{\beta_2 \|y_{2n}(t) - y_{2n}(t)\|^2 [\|y_{2n-1}(t) - y_{2n}(t)\|^2 + \|y_{2n}(t) - y_{2n+1}(t))\|^2]}{\|y_{2n-1}(t)) - y_{2n}(t))\|^2 + \|y_{2n}(t) - y_{2n}(t))\|^2}$$

$$+ \frac{\beta_3 \|y_{2n}(t) - y_{2n+1}(t)\|^2 \ \|y_{2n-1}(t) - y_{2n}(t)\|^2}{\|y_{2n-1}(t)) - y_{2n}(t))\|^2}$$

$$+ \frac{\beta_4 \|y_{2n-1}(t) - y_{2n}(t)\|^2 [\|y_{2n-1}(t) - y_{2n}(t)\|^2 + \|y_{2n}(t) - y_{2n+1}(t))\|^2]}{1 + \|y_{2n}(t) - y_{2n}(t))\|^2}$$

$$+ \beta_5 \ \| y_{2n-1}(t)) - y_{2n}(t)) \|^2$$

$$\leq \beta_1 \ \| y_{2n}(t)) - y_{2n+1}(t)) \|^2$$

$$+ \beta_2 [0]$$

$$+ \beta_3 \| y_{2n}(t)) - y_{2n+1}(t)) \|^2$$

$$+ \beta_4 \| y_{2n-1}(t)) - y_{2n}(t)) \|^2$$

$$+ \beta_4 \| y_{2n}(t)) - y_{2n+1}(t)) \|^2$$

$$+ \beta_5 \ \| y_{2n-1}(t)) - y_{2n}(t)) \|^2$$

$$\Rightarrow \| y_{2n}(t) - y_{2n+1}(t)\|^2$$

$$\leq (\beta_1 + \beta_3 + \beta_4) \ \| y_{2n}(t)) - y_{2n+1}(t)) \|^2 + (\beta_4 + \beta_5) \| y_{2n-1}(t)) - y_{2n}(t)) \|^2$$

$$\Rightarrow \| y_{2n}(t) - y_{2n+1}(t)\|^2 [1 - (\beta_1 + \beta_3 + \beta_4)] \leq (\beta_4 + \beta_5) \| y_{2n-1}(t)) - y_{2n}(t)) \|^2$$

$$\Rightarrow \| y_{2n}(t) - y_{2n+1}(t)\|^2 \leq \frac{(\beta_4 + \beta_5)}{[1 - (\beta_1 + \beta_3 + \beta_4)]} \| y_{2n-1}(t)) - y_{2n}(t)) \|^2$$

$$\Rightarrow \| y_{2n}(t) - y_{2n+1}(t)\| \leq \left[\frac{(\beta_4 + \beta_5)}{[1 - (\beta_1 + \beta_3 + \beta_4)]}\right]^{1/2} \| y_{2n-1}(t)) - y_{2n}(t)) \|$$

Taking $Q = \left[\frac{(\beta_4 + \beta_5)}{[1 - (\beta_1 + \beta_3 + \beta_4)]}\right]^{1/2}$

$$\Rightarrow \| y_{2n}(t) - y_{2n+1}(t)\| \leq Q \| y_{2n-1}(t)) - y_{2n}(t)) \|$$

Replacing 2n by n

$$\Rightarrow \| y_n(t) - y_{n+1}(t)\| \leq Q \| y_{n-1}(t)) - y_n(t)) \|$$

On further reducing

$\Rightarrow \| y_n(t) - y_{n+1}(t)\| \leq Q^n \| y_0(t)) - y_1(t)) \|$
for $t \in P_1$
Now we shall prove that for $t \in P_1$ $\{y_n(t)\}$ is a Cauchy Sequence.
For this every positive integer we have
$\Rightarrow \| y_n(t) - y_{n+k}(t)\| = \| y_n(t) - y_{n+1}(t) + y_{n+1}(t) - y_{n+2}(t) + \ldots\ldots y_{n+k-1}(t) - y_{n+k}(t)\|$
$\leq [Q^n + Q^{n-1} + \ldots\ldots Q^{n+k-1}] \| y_0(t)) - y_1(t)) \|$
$\leq [1 + Q + Q^2 + \ldots\ldots Q^{k-1}]\ Q^n \| y_0(t)) - y_1(t)) \|$
$\leq \dfrac{Q^n}{1-Q} \| y_0(t)) - y_1(t)) \|$
$\Rightarrow \| y_n(t) - y_{n+k}(t)\| \to 0$ as $n \to \infty$ $t \in P_1$ .... (5)

Hence from Equation it follows that for $t \in P_1$,$\{y_n(t)\}$ is a Cauchy Sequence & hence is Convergent in closed subset C of a Hilbert Space H.
For $t \in P_1$, Let $\{y_n(t)\} \to y(t)$ as $n \to \infty$
Again as closeness of C given that g is a function from C to C.
And Consequently the sub sequence E $(t, g_{2n}(t))$, F $(t, g_{2n+1}(t))$, T $(t, g_{2n+1}(t))$ and S $(t, g_{2n+2}(t))$ of $\{y_n(t)\}$ for $t \in P_1$ also converges to the y(t)
Continuity of E, F, S and T give
$E[t, S(t, g_n(t))] \to E(t, y(t))$
$S[t, E(t, g_n(t))] \to S(t, y(t))$
$F[t, T(t, g_n(t))] \to F(t, y(t))$
&
$T[t, F(t, g_n(t))] \to T(t, y(t))$
And
$E(t, y(t)) = S(t, y(t))$
$F(t, y(t)) = T(t, y(t))$ for $t \in P_1$
From equation (1) existence of Random Fixed Point consider for $t \in P_1$.

Existence of Random Fixed Point for $t \in P_1$.

$$\| E(t, y(t))-y(t) \|^2 = \| E(t, y(t))-y_{2n+1}(t) + y_{2n+1}(t) - y(t) \|^2$$
$$\leq 2\| E(t, y(t))- F(t, g_{2n+1}(t)) \|^2 + 2 \| y_{2n+1}(t) - y(t) \|^2$$

(By Parallelogram law $\| x^2 + y^2 \| \leq 2 \| x^2 \| + 2 \| y^2 \|$)
$\leq$
2
$$\frac{\beta_1 \|S(t,y(t)) - E(t,y(t))\|^2 [\|T(t,g_{2n+1}(t)) - F(t,g_{2n+1}(t))\|^2 + \|E(t,y(t)) - T(t,g_{2n+1}(t))\|^2]}{\|S(t,y(t)) - T(t,g_{2n+1}(t))\|^2 + \|E(t,y(t)) - T(t,g_{2n+1}(t))\|^2}$$

+2
$$\frac{\beta_2 \|E(t,y(t)) - T(t,g_{2n+1}(t))\|^2 [\|S(t,y(t)) - E(t,y(t))\|^2 + \|T(t,g_{2n+1}(t)) - F(t,g_{2n+1}(t))\|^2]}{\|S(t,y(t)) - T(t,g_{2n+1}(t))\|^2 + \|E(t,y(t)) - T(t,g_{2n+1}(t))\|^2}$$

$$+\ 2\,\frac{\beta_3 \| T(t, g_{2n+1}(t)) - F(t, g_{2n+1}(t)) \|^2 \ \| S(t, y(t)) - E(t, y(t)) \|^2}{\| S(t, y(t)) - T(t, g_{2n+1}(t)) \|^2}$$

$$+2\,\frac{\beta_4 \| S(t, y(t)) - E(t, y(t)) \|^2 + \| T(t, g_{2n+1}(t)) - F(t, g_{2n+1}(t)) \|^2]}{[\ 1 + \| E(t, y(t)) - T(t, g_{2n+1}(t)) \|^2]}$$

$$+\ 2\beta_5\ \| S(t, y(t)) - T(t, g_{2n+1}(t)) \|^2 + 2\beta_5\ \| y_{2n+1}(t) - y(t) \|^2 \qquad \ldots\ldots (6)$$

$$\leq\ 2\,\frac{\beta_1 \| S(t, y(t)) - E(t, y(t)) \|^2 [\| y(t)) - y(t)) \|^2 + \| E(t, y(t)) - y(t)) \|^2]}{\| S(t, y(t)) - y(t)) \|^2 + \| E(t, y(t)) - y(t)) \|^2}$$

$$+2\,\frac{\beta_2 \| E(t, y(t)) - y(t)) \|^2 [\| S(t, y(t)) - E(t, y(t)) \|^2 + \| y(t) - y(t) \|^2]}{\| S(t, y(t)) - y(t) \|^2 + \| E(t, y(t)) - y(t) \|^2}$$

$$+\ 2\,\frac{\beta_3 \| y(t) - y(t) \|^2 \ \| S(t, y(t)) - E(t, y(t)) \|^2}{\| S(t, y(t) - y(t) \|^2}$$

$$+2\,\frac{\beta_4 \| S(t, y(t)) - E(t, y(t)) \|^2 + \| y(t) - y(t) \|^2]}{[\ 1 + \| E(t, y(t) - y(t) \|^2]}$$

$$+\ 2\beta_5\ \| S(t, y(t)) - y(t) \|^2 + 2\beta_5\ \| y(t) - y(t) \|^2$$

Therefore for for $t \in P_1$.

$$\| E(t, y(t)) - y(t) \|^2 \leq 2\beta_5\ \| E(t, y(t)) - y(t) \|^2$$

$$(\ 1 - 2\beta_5\ ) \leq \ \| E(t, y(t)) - y(t) \|^2 \leq 0$$

$$\| E(t, y(t)) - y(t) \|^2 = 0 \quad \beta_5 < ½$$

$$E(t, y(t)) = y(t) \ \ t \in P_1$$

From equation no (6) $\qquad E(t, y(t)) = y(t) = S(t, y(t))$

In an exactly similar way ,we can prove that for all $t \in P_1$

$$F(t, y(t)) = y(t) = T(t, y(t))$$

Again if A :P_1 x C→C is a continuous random operator on a non empty closed subset C of a separable Hilbert Space H[3] then for any measurable function f : P_1 →C .The function

h(t) = A(t,f(t)) is also measurable [4].

UNIQUENESS

Let h: P_1 →C be another random fixed point common to E, F , T & S that is for $t \in P_1$

$$\| g(t) - h(t) \|^2 = \| E(t, g(t)) - F(t, h(t)) \|^2$$

$$\leq$$

$$\frac{\beta_1 \| S(t, g(t)) - E(t, g(t)) \|^2 [\| T(t, h(t)) - F(t, h(t)) \|^2 + \| E(t, g(t)) - T(t, h(t)) \|^2]}{\| S(t, g(t)) - T(t, h(t)) \|^2 + \| E(t, g(t)) - T(t, h(t)) \|^2}$$

$$+\ \frac{\beta_2 \| E(t, g(t)) - T(t, h(t)) \|^2 [\| S(t, g(t)) - E(t, g(t)) \|^2 + \| T(t, h(t)) - F(t, h(t))) \|^2]}{\| S(t, g(t)) - T(t, h(t)) \|^2 + \| E(t, g(t)) - T(t, h(t)) \|^2}$$

$$+ \frac{\beta_3 \| S(t, g(t)) - E(t, g(t)) \|^2 \,.\, \| T(t, h(t)) - F(t, h(t)) \|^2}{\| S(t, g(t) - T(t, h(t)) \|^2}$$

$$+ \frac{\beta_4 \| S(t, g(t)) - E(t, g(t)) \|^2 + \| T(t, h(t)) - F(t, h(t)) \|^2]}{[\, 1 + \| E(t, g(t) - T(t, h(t) \|^2]}$$

$$+ \beta_5 \; \| S(t, g(t)) - T(t, h(t)) \|^2$$

$$\| g(t) - h(t) \|^2 \le \beta_5 \; \| g(t) - h(t) \|^2$$

$$(1 - \beta_5) \; \| g(t) - h(t) \|^2 \le 0 \text{ where } \beta_5 < ½$$

$$g(t) = h(t) \quad \text{for } t \in P_1$$

This completes the proof of the theorem

REFERENCES

[1] B.E. Rhoades, Iteration to obtain random solutions and fixed points of operators in uniformly convex Banach spaces, *Sochow Journal of mathematics*, **27(4)** (2001), 401 - 404.

[2] B. Fisher, Common fixed point and constant mapping satisfying a rational inequality, *Math. Sem. Kobe Univ*, **6**(1978), 29 - 35.

[3] Binayak S. Choudhary, A common unique fixed point theorem for two random operators in Hilbert space, *IJMMS* **32(3)**(2002), 177 - 182.

[4] C.J. Himmelberg, Measurable relations, *Fund Math*, **87** (1975), 53 - 72.

[5] S.S. Pagey, Shalu Srivastava and Smita Nair , Common fixed point theorem for rational inequality in a quasi 2-metric space, *Jour. Pure Math.*,**22**(2005), 99 - 104.

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