An Iterative Method with Ninth-Order Convergence for Solving Nonlinear Equations

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Abstract
In this paper, we first present a fifth-order iterative method, which is a variant of the double-Newton’s method. Based on this new method, we propose a ninth-order iterative method. In contrast to the double-Newton’s method, the ninth-order method only needs one additional function evaluation per iteration, but the order of convergence increases five units. Numerical examples are given to show the efficiency of the presented methods.

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1 Introduction

We consider the iterative methods for finding a simple root $\alpha$ of a nonlinear equation $f(x) = 0$, where $f : I \subseteq R \rightarrow R$ for an open interval $I$ is a scalar function.

The well-known and widely used method is the classical Newton’s method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (1)$$

which converges quadratically in some neighborhood of $\alpha$[1].

In recent years, many modifications of Newton’s method with at least cubic convergence have been proposed, see[2-14]and references therein. Especially, based on some famous fourth-order methods, such as the Jarratt method
and the King’s method, some iterative methods with seventh-order or eighth-order have been developed in [15-18]. Many numerical applications use high precision in their computation, so higher-order numerical methods are important[19].

In this paper, we consider the double-Newton’s method

\[
\begin{align*}
    y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
    x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)},
\end{align*}
\]

which has fourth-order convergence[12]. First, we present a variant of the double-Newton’s method

\[
\begin{align*}
    y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
    x_{n+1} &= y_n - \left[1 + \left(\frac{f(y_n)}{f(x_n)}\right)^2\right] \frac{f(y_n)}{f'(y_n)},
\end{align*}
\]

which is a variant of the double-Newton’s method.

Theorem 2.1 Let \( \alpha \) be a simple zero of sufficiently differentiable function \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) for an open interval \( I \). If \( x_0 \) is sufficiently close to \( \alpha \), then the method defined by (3) is of fifth-order and satisfies the error equation

\[
e_{n+1} = 4c_2^4e_n^5 - 2c_2^2c_3e_n^5 + O(e_n^6),
\]

where \( e_n = x_n - \alpha \) and \( c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)} \).

**Proof** Using Taylor expansion, we have

\[
    f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)],
\]

\[
    f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + O(e_n^3)].
\]

Furthermore, we can get

\[
    \frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4),
\]

\[
    f(y_n) = f'(\alpha)[c_2e_n^2 - (2c_2^2 - 2c_3)e_n^3 + O(e_n^4)],
\]

\[
    \frac{f(y_n)}{f(x_n)} = c_2e_n + (2c_3 - 3c_2)e_n^2 + O(e_n^3)
\]
and

$$f'(y_n) = f'(\alpha)[1 + 2c_2e_n^2 - 4c_2(c_2 - c_3)e_n^3 + O(e_n^4)]. \quad (9)$$

From (7-9), we obtain

$$e_{n+1} = d_n - \frac{1+(c_2e_n^2+c_1-3c_2)e_n^2+O(e_n^2)^2}{1+2c_2e_n^2-4c_2(c_2-c_3)e_n^3+O(e_n^3)}[d_n + c_2d_n^2 + O(e_n^6)]$$

$$= d_n - [1 - c_2e_n^2 - 2c_3e_n^3 + O(e_n^4)][d_n + c_2d_n^2 + O(e_n^6)] \quad (10)$$

$$= 4c_2^4e_n^5 - 2c_2^2c_3e_n^5 + O(e_n^8),$$

where $d_n = y_n - \alpha = c_2e_n^2 - (2c_2^2 - 2c_3)e_n^3 + O(e_n^4)$.

This means the method defined by (3) is of fifth-order. That completes the proof.

Based on the new method (3), we can construct a three-step iterative method

$$\begin{align*}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= y_n - \left[1 + \frac{f'(y_n)^2}{f'(x_n)}\right] \frac{f(y_n)}{f'(y_n)}, \\
x_{n+1} &= z_n - \left[1 + 2\left(\frac{f'(y_n)}{f'(x_n)}\right)^2 + 2\frac{f(z_n)}{f'(x_n)}\frac{f'(y_n)}{f'(y_n)}\right].
\end{align*} \quad (11)$$

For the method (11), we have the following convergence result.

**Theorem 2.2** Let $\alpha$ be a simple zero of sufficiently differentiable function $f : I \subseteq R \to R$ for an open interval $I$. If $x_0$ is sufficiently close to $\alpha$, then the method defined by (11) is of ninth-order.

**Proof** Using Taylor expansion, (7) and (10), we have

$$f(z_n) = f'(\alpha)[z_n - \alpha + O((z_n - \alpha)^{2})] = f'(\alpha)[z_n - \alpha + O(e_n^{10})] \quad (12)$$

and

$$\frac{f(z_n)}{f'(y_n)} = 4c_2^4e_n^3 - 2c_2c_3e_n^3 + O(e_n^{4}). \quad (13)$$

By (8-9) and (12-13), we obtain

$$e_{n+1} = z_n - \alpha - \frac{1+2(c_2e_n^2+c_1-3c_2)e_n^2+O(e_n^2)^2}{1+2c_2e_n^2-4c_2(c_2-c_3)e_n^3+O(e_n^3)}[z_n - \alpha + O(e_n^{10})]$$

$$= z_n - \alpha - \left[1 + O(e_n^{4})\right][z_n - \alpha + O(e_n^{10})] = O(e_n^{9}).$$

This means the method defined by (11) is of ninth-order. That completes the proof.

### 3 Numerical examples

In this section, we employ the new methods defined by (3) and (11) to solve some nonlinear equations and compare them with Newton’s method (NM) and the double-Newton’s method (DNM). Displayed in Table 1 are the number of
Table 1: Comparison of various iterative methods

<table>
<thead>
<tr>
<th>( f(x_0) )</th>
<th>( x_0 )</th>
<th>IT</th>
<th>NFE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>NM</td>
<td>DNM</td>
</tr>
<tr>
<td>( f_1 )</td>
<td>-1</td>
<td>24</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>( f_2 )</td>
<td>1.2</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>( f_3 )</td>
<td>3.5</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>19</td>
<td>10</td>
</tr>
<tr>
<td>( f_4 )</td>
<td>1.6</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>( f_5 )</td>
<td>0.5</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>( f_6 )</td>
<td>-1</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>-3</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>( f_7 )</td>
<td>0</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 2: Newton’s method for solving \( f_7(x) = 0, \ x_0 = 1.5 \)

| \( n \) | \( x_n \) | \( |f(x_n)| \) |
|---------|----------|---------------|
| 1       | 1.047978478152371 | 0.8003764211641961 |
| 2       | 0.8284482173647322  | 0.131859457704464  |
| 3       | 0.7756136816823298  | 0.0061698862580058  |
| 4       | 0.7728898515480687  | 0.0000155336663091  |
| 5       | 0.7728829591932177  | 9.9181134302505959E-11 |
| 6       | 0.7728829591492101  | 4.0434052750244913E-21 |
| 7       | 0.7728829591492101  | 6.7202229017632478E-42 |
| 8       | 0.7728829591492101  | 1.8563355779020206E-83 |
Table 3: The double-Newton’s method for solving $f_7(x) = 0$, $x_0 = 1.5$

| $n$ | $x_n$ | $|f(x_n)|$ |
|-----|-------|----------|
| 1   | 0.8284482173647322 | 0.1318594577704464 |
| 2   | 0.7728898515480687 | 0.0000155336663091 |
| 3   | 0.7728829591492101 | 4.0434052750244913E-21 |
| 4   | 0.7728829591492101 | 1.8563355779020206E-83 |

Table 4: Eq.(3) for solving $f_7(x) = 0$, $x_0 = 1.5$

| $n$ | $x_n$ | $|f(x_n)|$ |
|-----|-------|----------|
| 1   | 0.8142907772453919 | 0.0969779854243526 |
| 2   | 0.7728831833696511 | 5.0533117233062243E-7 |
| 3   | 0.7728829591492101 | 2.7151084137118892E-33 |
| 4   | 0.7728829591492101 | 1.2157475219488642E-164 |

Table 5: Eq.(11) for solving $f_7(x) = 0$, $x_0 = 1.5$

| $n$ | $x_n$ | $|f(x_n)|$ |
|-----|-------|----------|
| 1   | 0.7778117097548697 | 0.0111588924490578 |
| 2   | 0.7728831833696511 | 8.7548707643361337E-21 |
| 3   | 0.7728829591492101 | 1.0257291342665512E-183 |
iterations (IT) and the number of function evaluations (NFE) required such that $|f(x_n)| < 10^{-15}$.

We use the following functions:

$$f_1(x) = x^3 + 4x^2 - 10, \alpha = 1.36523001341409688791373,$$

$$f_2(x) = x^5 + x^4 + 4x^2 - 20, \alpha = 1.46627907386472267070587,$$

$$f_3(x) = e^{x^2 + 7x - 30} - 1, \alpha = 3,$$

$$f_4(x) = (\sin x)^2 - x^2 + 1, \alpha = 1.40449164821534111524670,$$

$$f_5(x) = e^x \sin x + \ln(x^2 + 1), \alpha = 0,$$

$$f_6(x) = x^3 - \sin^2 x + 3\cos x + 5, \alpha = -1.58268704575206986540081,$$

$$f_7(x) = x^3 - e^{-x}, \alpha = 0.772882959149210124749629.$$

The computational results presented in Table 1 show that, the presented methods converge more rapidly than Newton’s method and the double-Newton’s method, and require the less NFE. Therefore, the new methods (3) and (11) have better convergence efficiency.

We also consider high-precision calculation and take $f_7(x) = 0$ for example. Iterative results obtained by Newton’s method, the double-Newton’s method, the methods defined by (3) and (11) are shown in Tables 2-5 respectively. From Table 5, we can see $|f(x_{n+1})| \approx |f(x_n)|^9$. Thus, the superiority of the method (11) is more obvious for high-precision computation.

References


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