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An Iterative Method with Ninth-Order Convergence for Solving Nonlinear Equations

Zhongyong Hu[1,2], Liu Guocai[1] and Li Tian[1]

[1] School of Mathematics and System Sciences
Taishan College, 271000 Tai'an, China

[2] School of Mathematical Sciences
Dalian University of Technology
116024 Dalian, China

Abstract

In this paper, we first present a fifth-order iterative method, which is a variant of the double-Newton's method. Based on this new method, we propose a ninth-order iterative method. In contrast to the double-Newton's method, the ninth-order method only needs one additional function evaluation per iteration, but the order of convergence increases five units. Numerical examples are given to show the efficiency of the presented methods.

Mathematics Subject Classification: 49M99; 65H05

Keywords: high-precision calculation; Newton's method; ninth-order convergence; nonlinear equations

1 Introduction

We consider the the iterative methods for finding a simple root α of a nonlinear equation$f(x) = 0$, where $f : I \subseteq R \longrightarrow R$ for an open interval I is a scalar function.

The well-known and widely used method is the classical Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

which converges quadratically in some neighborhood of α[1].

In recent years, many modifications of Newton's method with at least cubic convergence have been proposed, see[2-14]and references therein. Especially, based on some famous fourth-order methods, such as the Jarratt method and the King's method, some iterative methods with seventh-order or eighth-order have been developed in [15-18]. Many numerical applications use high precision in their computation, so higher-order numerical methods are important[19].

In this paper, we consider the double-Newton's method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}, \end{cases} \tag{2}$$

which has fourth-order convergence[12]. First, we present a variant of the double-Newton's method with fifth-order convergence. Based on the new method, a ninth-order iterative method is proposed. Finally, numerical examples are given to show the performance.

2 Convergence analysis

Now, we consider the iteration scheme

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - [1 + (\frac{f(y_n)}{f(x_n)})^2]\frac{f(y_n)}{f'(y_n)}, \end{cases} \tag{3}$$

which is a variant of the double-Newton's method.

Theorem 2.1 *Let α be a simple zero of sufficiently differentiable function $f : I \subseteq R \longrightarrow R$ for an open interval I. If x_0 is sufficiently close to α, then the method defined by (3) is of fifth-order and satisfies the error equation*

$$e_{n+1} = 4c_2^4 e_n^5 - 2c_2^2 c_3 e_n^5 + O(e_n^6),$$

where $e_n = x_n - \alpha$ and $c_k = f^{(k)}(\alpha)/k!f'(\alpha)$.

Proof Using Taylor expansion, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)], \tag{4}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + O(e_n^3)]. \tag{5}$$

Furthermore, we can get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4), \tag{6}$$

$$f(y_n) = f'(\alpha)[c_2e_n^2 - (2c_2^2 - 2c_3)e_n^3 + O(e_n^4)], \tag{7}$$

$$\frac{f(y_n)}{f(x_n)} = c_2e_n + (2c_3 - 3c_2^2)e_n^2 + O(e_n^3) \tag{8}$$

and

$$f'(y_n) = f'(\alpha)[1 + 2c_2e_n^2 - 4c_2(c_2^2 - c_3)e_n^3 + O(e_n^4)]. \tag{9}$$

From (7-9), we obtain

$$\begin{aligned} e_{n+1} &= d_n - \tfrac{1+(c_2e_n+(2c_3-3c_2^2)e_n^2+O(e_n^3))^2}{1+2c_2e_n^2-4c_2(c_2^2-c_3)e_n^3+O(e_n^4)}[d_n + c_2d_n^2 + O(e_n^6)] \\ &= d_n - [1 - c_2e_n^2 - 2c_3e_n^3 + O(e_n^4)][d_n + c_2d_n^2 + O(e_n^6)] \\ &= 4c_2^4e_n^5 - 2c_2^2c_3e_n^5 + O(e_n^6), \end{aligned} \tag{10}$$

where $d_n = y_n - \alpha = c_2e_n^2 - (2c_2^2 - 2c_3)e_n^3 + O(e_n^4)$.

This means the method defined by (3) is of fifth-order. That completes the proof.

Based on the new method (3), we can construct a three-step iterative method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - [1 + (\frac{f(y_n)}{f(x_n)})^2]\frac{f(y_n)}{f'(y_n)}, \\ x_{n+1} = z_n - [1 + 2(\frac{f(y_n)}{f(x_n)})^2 + 2\frac{f(z_n)}{f(y_n)}]\frac{f(z_n)}{f'(y_n)}. \end{cases} \tag{11}$$

For the method (11), we have the following convergence result.

Theorem 2.2 *Let α be a simple zero of sufficiently differentiable function $f : I \subseteq R \longrightarrow R$ for an open interval I. If x_0 is sufficiently close to α, then the method defined by (11) is of ninth-order.*

Proof Using Taylor expansion, (7) and (10), we have

$$f(z_n) = f'(\alpha)[z_n - \alpha + O((z_n - \alpha)^2)] = f'(\alpha)[z_n - \alpha + O(e_n^{10})] \tag{12}$$

and

$$\frac{f(z_n)}{f(y_n)} = 4c_2^3e_n^3 - 2c_2c_3e_n^3 + O(e_n^4). \tag{13}$$

By (8-9) and (12-13), we obtain

$$\begin{aligned} e_{n+1} &= z_n - \alpha - \tfrac{1+2(c_2e_n+(2c_3-3c_2^2)e_n^2+O(e_n^3))^2+2(4c_2^3e_n^3-2c_2c_3e_n^3+O(e_n^4))}{1+2c_2e_n^2-4c_2(c_2^2-c_3)e_n^3+O(e_n^4)}[z_n - \alpha + O(e_n^{10})] \\ &= z_n - \alpha - [1 + O(e_n^4)][z_n - \alpha + O(e_n^{10})] = O(e_n^9). \end{aligned}$$

This means the method defined by (11) is of ninth-order. That completes the proof.

3 Numerical examples

In this section, we employ the new methods defined by (3) and (11) to solve some nonlinear equations and compare them with Newton's method (NM) and the double-Newton's method (DNM). Displayed in Table 1 are the number of

Table 1: Comparison of various iterative methods

$f(x_0)$	x_0	IT				NFE			
		NM	DNM	Eq.(3)	Eq.(11)	NM	DNM	Eq.(3)	Eq.(11)
f_1	-1	24	12	15	7	48	48	45	35
	1	5	3	3	2	10	12	12	10
f_2	1.2	5	3	3	2	10	12	12	10
	2	6	3	3	2	12	12	12	10
f_3	3.5	12	6	6	4	24	24	24	20
	4	19	10	9	7	38	40	36	35
f_4	1.6	5	3	2	2	10	12	8	10
	2.5	6	3	3	2	12	12	12	10
f_5	0.5	6	3	3	2	12	12	12	10
	2	6	3	3	2	12	12	12	10
f_6	-1	5	3	3	2	10	12	12	10
	-3	6	3	3	2	12	12	12	10
f_7	0	6	3	3	2	12	12	12	10
	1.5	6	3	3	2	12	12	12	10

Table 2: Newton's method for solving $f_7(x) = 0$, $x_0 = 1.5$

n	x_n	$\|f(x_n)\|$
1	1.0479978478152371	0.8003764211641961
2	0.8284482173647322	0.1318594577704464
3	0.7756136816823298	0.0061698862580058
4	0.7728898515480687	0.0000155336663091
5	0.7728829591932177	9.9181134302505959E-11
6	0.7728829591492101	4.0434052750244913E-21
7	0.7728829591492101	6.7202229017632478E-42
8	0.7728829591492101	1.8563355779020206E-83

Table 3: The double-Newton's method for solving $f_7(x) = 0$, $x_0 = 1.5$

n	x_n	$\lvert f(x_n)\rvert$
1	0.8284482173647322	0.1318594577704464
2	0.7728898515480687	0.0000155336663091
3	0.7728829591492101	4.0434052750244913E-21
4	0.7728829591492101	1.8563355779020206E-83

Table 4: Eq.(3)for solving $f_7(x) = 0$, $x_0 = 1.5$

n	x_n	$\lvert f(x_n)\rvert$
1	0.8142907772453919	0.0969779854243526
2	0.7728831833696511	5.0533117233062243E -7
3	0.7728829591492101	2.7151084137118892E-33
4	0.7728829591492101	1.2157475219488642E-164

Table 5: Eq.(11)for solving $f_7(x) = 0$, $x_0 = 1.5$

n	x_n	$\lvert f(x_n)\rvert$
1	0.7778117097548697	0.0111588924490578
2	0.7728831833696511	8.7548707643361337E-21
3	0.7728829591492101	1.0257291342665512E-183

iterations (IT) and the number of function evaluations (NFE) required such that $|f(x_n)| < 10^{-15}$.

We use the following functions:

$f_1(x) = x^3 + 4x^2 - 10, \alpha = 1.365230013414096887913 73,$
$f_2(x) = x^5 + x^4 + 4x^2 - 20, \alpha = 1.46627907386472267070587,$
$f_3(x) = e^{x^2+7x-30} - 1, \alpha = 3,$
$f_4(x) = (sinx)^2 - x^2 + 1, \alpha = 1.40449164821534111524670,$
$f_5(x) = e^x sinx + ln(x^2 + 1), \alpha = 0,$
$f_6(x) = x^3 - sin^2x + 3cosx + 5, \alpha = -1.58268704575206986540081,$
$f_7(x) = x^3 - e^{-x}, \alpha = 0.77288295914921012474 9629.$

The computational results presented in Table 1 show that, the presented methods converge more rapidly than Newton's method and the double-Newton's method, and require the less NFE. Therefore, the new methods (3) and (11) have better convergence efficiency.

We also consider high-precision calculation and take $f_7(x) = 0$ for example. Iterative results obtained by Newton's method, the double-Newton's method, the methods defined by (3) and (11) are shown in Tables 2-5 respectively. From Table 5, we can see $|f(x_{n+1})| \approx |f(x_n)|^9$. Thus, the superiority of the method (11) is more obvious for high-precision computation.

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