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A Common Fixed Point Theorems in Menger(PQM) Spaces with Using Property (E.A)

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Abstract

The main purpose of this paper is to define Menger (PQM) space and the nation of weakly compatible and define a new property and prove a common fixed point theorem for four self maps in menger(PQM) space with using the notion of property (E.A).

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1 Introduction and Preliminaries

The concept of probabilistic metric space was first introduced and studied by Menger [6], which is a generalization of the metric space and also the study of this space was expanded rapidly with the pioneering works of Schweizer and Sklar [11, 12]. The theory of probabilistic space is of fundamental importance in probabilistic functional analysis. In 1986, Jungck [3] introduced the notion of compatible mappings. This concept was frequently used to prove existence theorems in common fixed point theory. However, the study of common fixed points of non compatible mappings is also very interesting. Research along this direction has recently been initiated by Pant [8, 9]. The aim of this paper is to define a new property, and prove a common fixed point theorem for four self maps in menger(PQM) space with using the notion of property (E.A). In this section we explain some definitions that will be used later.

Definition 1.1 [12]*A mapping* $T : [0,1] \times [0,1] \to [0,1]$*, is t-norm if is satisfying the following conditions:*
(*i*) *T is commutative and associative;*
(*ii*) $T(a,1) = a$ *for all* $a \in [0,1]$*;*
(*iii*) $T(a,b) \leq T(c,d)$ *whenever* $a \leq c$ *and* $b \leq d$ *and* $a,b,c,d \in [0,1]$.
The following are the basic t-norms:
(*i*) $T_M(x,y) = \min\{x,y\}$
(*ii*) $T_P(x,y) = x.y$
(*iii*) $T_L(x,y) = \max\{x+y-1,0\}$.

Each t-norm T can be extended [5] (by associativity) in a unique way taking for $(x_1,...,x_n) \in [0,1]^n$ $(n \in N)$ the values $T^1(x_1,x_2) = T(x_1,x_2)$, $T^n(x_1,...x_{n+1}) = T(T^{n-1}(x_1,...,x_n),x_{n+1})$ for $n \geq 2$ and $x_i \in [0,1]$, for all $i \in \{1,2,...,n+1\}$

Definition 1.2 [7,10] *A Menger(PQM) space is a triple* (X,F,T) *where* X *is a nonempty set,* T *is a continuous t-norm and* F *is a mapping from* $X \times X$ *in* L *(L is set of all distribution function) For* $(p,q) \in X \times X$*, the distribution function* $F(p,q)$ *is denoted by* $F_{p,q}$*. Then the following conditions hold, for all* $p,q,r \in X$.

$$(PQM_1) \quad F_{p,q}(t) = F_{q,p}(t) = \varepsilon_0(t) \quad \text{for all} \quad t > 0 \quad \text{if and only if} \quad p = q$$
$$(PQM_2) \quad F_{p,q}(t+s) \geq T\Big(F_{p,r}(t), F_{r,q}(s)\Big) \quad \text{for all} \quad p,q,r \in X \quad \text{and} \quad t,s \geq 0$$

Definition 1.3 [7,10]. *Let* (X,F,T) *be a Menger(PQM) space.*

(i) A sequence $\{x_n\}$ *in* X *is said to be convergent to* x *in* X*, if for every* $\varepsilon > 0$, $\lambda > 0$*, there exists positive integer* N *such that* $F_{x_n,x}(\varepsilon) > 1-\lambda$ *whenever* $n \geq N$*. we write* $x_n \to x$ *as* $n \to \infty$ *or* $\lim_{n\to\infty} x_n = x$
(ii) A sequence $\{x_n\}$ *in* X *is called cauchy sequence, if for every* $\varepsilon > 0$, $\lambda > 0$*, there exists positive integer* N *such that* $F_{x_n,x_m}(\varepsilon) > 1-\lambda$ *whenever* $n \geq n \geq m \geq N$.
(iii) A Menger(PQM) space (X,F,T) *is said to be complete if and only if every cauchy sequence in* X *is convergent to a point in* X.

Definition 1.4 [2] *Let* (X,F,T) *be a Menger space such that the t-norm* T *is continuous and* A,S *be mappings from* X *into itself. Then,* A *and* S *are said to be compatible if*

$$\lim_{n\to\infty} FASx_n, SAx_n = 1$$

for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$$

for some $z \in X$.

Definition 1.5 *Two self mappings A and S are said to be weakly compatible if they commute at their coincidence points that is $Ax = Sx$. for some $x \in X$ implies $ASx = SAx$.*

Definition 1.6 *Let A and S be two self mappings of a Menger space (X, F, T). We say that A and S satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.*

Example 1.7 [1] *Let $X = [0, +\infty)$. Define $A, S : X \to X$ by*

$$Ax = \frac{x}{4} \quad and \quad Sx = \frac{3x}{4}, \quad \forall x \in X.$$

Consider the sequence $x_n = \frac{1}{n}$. Clearly

$$\lim_{n\to\infty} x_n = Ax_n = lim_{n\to\infty} x_n = Sx_n = 0.$$

Then S and A satisfy (E, A).

Example 1.8 [1] *Let $X = [2, +\infty)$. Define $A, S : X \to X$ by*

$$Ax = x + 1 \quad and \quad Sx = 2x + 1, \quad \forall x \in x.$$

Suppose that the property (E, A) holds. Then, there exists in X a sequence $\{x_n\}$ satisfying

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z \text{ for some } z \in X.$$

Therefor

$$\lim_{n\to\infty} x_n = z - 1 \quad and \quad \lim_{n\to\infty} x_n = \frac{z-1}{2}.$$

Thus, $z = 1$, which is a contradiction since $1 \notin X$. Hence A and S do not satisfy $(E.A)$.

2 Main Results

Lemma 2.1 *Let (X, F, T) be a Menger(PQM)space. If there exists $k \in (0, 1)$ such that*

$$F_{p,q}(kt) \geq F_{p,q}(t)$$

for all $p, q \in X$ and $t > 0$ then $p = q$.

Theorem 2.2 *Let (X, F, T) be a Menger (PQM) space with $T(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$. Let A, B, S and L be mappings of X into itself such that*

(i) $AX \subset LX$ and $BX \subset SX$,

(ii) (A, S) or (B, L) satisfies the property $(E.A)$,

(iii) there exists a number $k \in (0, 1)$ such that

$$FAu, Bv(kx) \geq \min\{FSu, Lv(x), FSu, Bv(x), FLv, Bv(x), FAu, Su(x), FAu, Lv(x)\},$$

for all $u, v \in X$

(iv) (A, S) and (B, L) are weakly compatible,

(v) one of AX, BX, SX or LX is a closed subset of X.

Then A, B, S and L have a unique common fixed point in X.

Proof. Suppose that (B, L) satisfies the property $(E.A)$. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Lx_n = z$ for some $z \in X$.

Since $BX \subset SX$, there exists in X a sequence $\{y_n\}$ such that $Bx_n = Sy_n$. Hence $\lim_{n\to\infty} Sy_n = z$. Let us show that $\lim_{n\to\infty} Ay_n = z$.

$$\begin{aligned} &FAy_n, Bx_n(kx) \\ &\geq \min[FSy_n, Lx_n(x), FSy_n, Bx_n(x), FLx_n, Bx_n(x), FAy_n, Sy_n(x), FAy_n, Lx_n(x)] \\ &\geq \min[FBx_n, Lx_n(x), FLx_n, Bx_n(x), FAy_n, Bx_n(x), FAy_n, Lx_n(x)] \\ &\geq FAy_n, Bx_n(x) \end{aligned}$$

Therefor with the Lemma (2.1) $Ay_n = Bx_n$. Letting $n \to \infty$, we obtain $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Ay_n = z$. Suppose SX is a closed subset of X. then $z = Su$ for some $u \in X$.Subsequently, we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Lx_n = \lim_{n\to\infty} Sy_n = Su$$

we have

$$\begin{aligned} &FAu, Bx_n(kx) \\ &\geq \min[FSu, Lx_n(x), FSu, Bx_n(x), FLx_n, Bx_n(x), FAu, Su(x), FAu, Lx_n(x)] \end{aligned}$$

Letting $n \to \infty$, we obtain

$$FAu, Su(kx) \geq FAu, Su(x),$$

There for with the Lemma (2.1) we have $Au = Su$. The weak compatibility of A and S implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSu$.

On the other hand, since $AX \subset TX$, there exists a point $v \in X$ such that $Au = Lv$. We claim that $Lv = Bv$.
we have

$F Au, Bv(kx)$

$$
\begin{aligned}
&\geq \min[FSu, Lv(x), FSu, Bv(x), FLv, Bv(x), FAu, Su(x), FAu, Lv(x)] \\
&\geq FAu, Bv(x).
\end{aligned}
$$

Therefor, with the Lemma (2.1) we have $Au = Bv$. Thus $Au = Su = Lv = Bv$.

The weak compatibility of B and L implies that $BLv = LBv$ and $LLv = LBv = BLv = BBv$. Let us show that Au is a common fixed point of A, B, S and L.we have

$M(Au, AAu, kt) = M(AAu, Bv, kt)$

$$
\begin{aligned}
&\geq \min[FSAu, Lv(x), FSAu, Bv(x), FLv, Bv(x), FAAu, SAu(x), FAAu, Lv(x)] \\
&\geq FAAu, Av(x).
\end{aligned}
$$

Therefore, we have $Au = AAu = SAu$ and Au is a common fixed point of A and S.
Similarly, we can prove that Bv is a common fixed point of B and L .Since $Au = Bv$, we conclude that Au is a common fixed point of A, B, S and L. The proof is similar when LX is assumed to be a closed subset of X. The cases in which AX or BX is closed subset of X are similar to the cases in which LX or SX, respectively, is closed since $AX \subset LX$ and $BX \subset SX$. If $Au = Bu = Su = Lu = u$ and $Av = Bv = Sv = Lv = v$, we have

$Fu, v(kx) = FAu, Bv(kx)$

$$
\begin{aligned}
&\geq \min[FSu, Lv(x), FSu, Bv(x), FLv, Bv(x), FAu, Su(x), FAu, Lv(x)] \\
&\geq Fu, v(x).
\end{aligned}
$$

Thus we have $u = v$ and the common fixed point is unique. This completes the proof of the theorem.

for three mapping , we have the following result:

Corollary 2.3 *Let (X, F, T) be a Menger(PQM) space with $T(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$. Let A, B and S be mappings of X into itself such that*
(i) $AX \subset SX$ and $BX \subset SX$,

(ii) (A, S) *or* (B, S) *satisfies the property* $(E.A)$,
(iii) there exists a number $k \in (0, 1)$ *such that*

$$FAx, By(x) \geq \min\{FSx, Sy(x), FSx, By(x), FSy, By(x), FAu, Su(x)\}$$

for all $x, y \in X$
(iv) (A, S) *and* (B, S) *are weakly compatible,*
(v) one of AX, BX *and* SX *is a closed subset of* X.
Then A, B *and* S *have a unique common fixed point in* X.

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