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Irregular Boundary Area Computation

by Quantic Hermite Polynomial

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Abstract

In this study, a new method is derived for computing irregular boundary area by quantic hermite polynomial. Finally, some numerical examples presented to show the efficiency of the theoretical results.

Keywords: Irregular Area, Quantic Hermite Polynomials, Taylor's Method

1. INTRODUCTION

The term "area" in the context of surveying refers to the area of a tract of a land projected upon the horizontal plane, and not to the actual area of a land surface [3].It is often necessary to compute the area of a tract of a land which may be regular or irregular in shape (Fig.1).Land is ordinarily bought and sold on the basis of cost per unit area [5]. Some rules were used recently by a surveyors and engineers for computation of irregular boundary of a tract, such as trapezoid and Simpson's one third rule. There is no limitation in trapezoidal rule , this rule can be applied for any number of ordinates and, it can be applied while boundaries between the ends of ordinates are assumed to be straight, area by this rule is equal to the product of the common interval h and the sum of intermediate ordinates plus average of the first and last ordinates. In Simpson's rule ,the boundary between the ends of ordinates are assumed to form an arc of a parabola, hence Simpson's rule is some times called the parabolic rule, this rule can be applicable only when the number of divisions is even (the number of ordinate is odd),and the boundary between the ordinates is considered to be an arc of a parabola [3]. Simpson's one third rules were extended for unequal intervals by [6] .Presented a method that employed different polynomial functions using salient boundary points [1]. A new trial done here in this paper for deriving a new formula for calculating irregular boundary area by using quantic hermite polynomial , then we showed numerically that our formula results better than the used methods such as Simpson's and Trapezoidal rules .

2. QUANTIC HERMITE POLYNOMAIL

Consider an irregular boundary for which the offsets $y_0, y_1, ..., y_n$ are measured at $x_0, x_1, ..., x_n$ where n is the number of intervals (Fig.1).The corresponding intervals are $h_i = x_{i+1} - x_i$,$i = 0,1,2,...,n-1$ .The actual boundary for each interval is approximated by a QH polynomial ,which is defined class of Hermite polynomials [2,4]. The QH polynomial is defined in below, followed by area for a single QH, the composite QH formula for unequal intervals, and the special case of equal intervals.

Consider the interval (x_i, x_{i+1}) in (Fig.3). A quantic Hermite polynomial for this interval is a five-degree polynomial passing through the two points (x_i, y_i) and (x_{i+1}, y_{i+1}) and satisfying the first and second derivatives at x_i and x_{i+1}.The QH polynomial is:

$$Q_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 + e_i(x - x_i)^4 + f_i(x - x_i)^5 \quad (1)$$

This satisfies the six conditions

$$Q_i(x_i) = y_i, Q_i(x_{i+1}) = y_{i+1}, Q_i^{(k)}(x_i) = y_i^{(k)} \text{ And}$$
$$Q_i^{(k)}(x_{i+1}) = y_{i+1}^{(k)}, i = 0,1,2,...,n-1 \text{ and } k = 1,2. \quad (2)$$

Clearly, in definition of $Q_i(x)$ we have six unknowns which are a_i, b_i, c_i, d_i, e_i and f_i

To find each of these, we must use all conditions in (2) as:

at $x = x_i$,$Q_i(x_i) = y_i \rightarrow a_i = y_i$ (3)

at $x = x_{i+1}$,$Q_i(x_{i+1}) = y_{i+1} \rightarrow a_i + h_i b_i + h_i^2 c_i + h_i^3 d_i + h_i^4 e_i + h_i^5 f_i = y_{i+1}$ (4)

at $x = x_i$,$Q_i'(x_i) = y_i' \rightarrow b_i = y_i'$ (5)

at $x = x_{i+1}$,$Q_i'(x_{i+1}) = y_{i+1}' \rightarrow b_i + 2h_i c_i + 3h_i^2 d_i + 4h_i^3 e_i + 5h_i^4 f_i = y_{i+1}'$ (6)

at $x = x_i$,$Q_i''(x_i) = y_i'' \rightarrow c_i = \frac{y_i''}{2}$ (7)

at $x = x_{i+1}$,$Q_i''(x_{i+1}) = y_{i+1}'' \rightarrow 2c_i + 6h_i d_i + 12h_i^2 e_i + 20h_i^3 f_i = y_{i+1}''$ (8)

Solving (3-8) and we get

$$d_i = 10h_i^{-3}(y_{i+1} - y_i) - 2h_i^{-2}(2y_{i+1}' + 3y_i') + \frac{h_i^{-1}}{2}(y_{i+1}'' - 3y_i'') \tag{9}$$

$$e_i = h_i^{-1}\left\{15h_i^{-3}(y_i - y_{i+1}) + h_i^{-2}(7y_{i+1}' + 8y_i') + \frac{h_i^{-1}}{2}(3y_i'' - 2y_{i+1}'')\right\} \tag{10}$$

$$f_i = \frac{h_i^{-2}}{2}\left\{12h_i^{-3}(y_{i+1} - y_i) - 6h_i^{-2}(y_{i+1}' + y_i') + h_i^{-1}(y_{i+1}'' - y_i'')\right\} \tag{11}$$

Esq. (3), (5), (7), (9), and (10) and (11) give the six unknowns.

3. SINGLE QUANTIC HERMIT AREA

The area under the single quantic polynomial between x_i and x_{i+1} is obtained by integrating (1)

$$A_i = \int_{x_i}^{x_{i+1}} Q_i(x)dx = \int_{x_i}^{x_{i+1}} [a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 e_i(x - x_i)^4 +$$

$$+ f_i(x - x_i)^5]dx = a_i h_i + \frac{b_i h_i^2}{2} + \frac{c_i h_i^3}{3} + \frac{d_i h_i^4}{4} + \frac{e_i h_i^5}{5} + \frac{f_i h_i^6}{6} \tag{12}$$

Substituting for a_i, b_i, c_i, d_i, e_i and f_i from (3), (5), (9), (10) and (11) into (12) and integrating, then

$$A_i = \frac{h_i}{2}(y_{i+1} + y_i) + \frac{h_i^2}{10}(y_i' - y_{i+1}') + \frac{h_i^3}{120}(y_{i+1}'' + y_i'') \tag{13}$$

4. COMPOSITE QH AREA FOR UNEQUAL INTERVALS

The total area from x_0 to x_n, A_u is the sum of the single quantic areas given by (13). That is

$$A_u = \sum_{i=0}^{n-1} A_i \tag{14}$$

Substituting for A_i from (13) and collecting similar terms, then

$$A_u = [\frac{1}{2}(h_0 y_0 + h_{n-1}y_n) + \sum_{i=1}^{n-1}(h_{i-1} + h_i)y_i] + \frac{1}{10}(h_0^2 y_0' - h_{n-1}^2 y_n') + \frac{1}{10}\sum_{i=1}^{n-1}(h_{i-1}^2 - h_i^2)y_i'$$

$$+ \frac{1}{120}(h_0^3 y_0'' + h_{n-1}^3 y_n'') + \frac{1}{120}\sum_{i=1}^{n-1}(h_{i-1}^3 + h_i^3)y_i'' \tag{15}$$

Eq. (15) is the quantic hermit formula for unequal intervals .This formula, which involves only a minor modification to the trapezoidal rule is known as the corrected trapezoidal rule. The first term of (15) is the trapezoidal rule. The unknown first and second derivatives y_i' and

y_i'' for $i = 0,1,...,n$ are estimated numerically using known offsets .Employing Taylor's theorem of order two each of these obtained as follows:

$$y_0' = \frac{(h_0 + h_1)^2 y_1 - h_0^2 y_2 - h_1(2h_0 + h_1) y_0}{h_0 h_1 (h_0 + h_1)} \tag{16}$$

$$y_i' = \frac{h_{i-1}^2 y_{i+1} + (h_i^2 - h_{i-1}^2) y_i - h_i^2 y_{i-1}}{h_{i-1} h_i (h_{i-1} + h_i)}, \quad i = 1,2,...,n-1 \tag{17}$$

$$y_n' = \frac{h_{n-2}(h_{n-2} + 2h_{n-1}) y_n - (h_{n-2} + h_{n-1})^2 y_{n-1} + h_{n-1}^2 y_{n-2}}{h_{n-2} h_{n-1} (h_{n-2} + h_{n-1})} \tag{18}$$

$$y_0'' = \frac{2[h_0 y_2 - (h_0 + h_1) y_1 + h_1 y_0]}{h_0 h_1 (h_0 + h_1)}, \tag{19}$$

$$y_i'' = \frac{2[h_{i-1} y_{i+1} - (h_{i-1} + h_i) y_i + h_i y_{i-1}]}{h_{i-1} h_i (h_{i-1} + h_i)}, i = 1,2,...,n-1 \tag{20}$$

and

$$y_n'' = \frac{2[h_{n-2} y_n - (h_{n-2} + h_{n-1}) y_{n-1} + h_{n-1} y_{n-2}]}{h_{n-2} h_{n-1} (h_{n-2} + h_{n-1})} \tag{21}$$

An algorithm for area computation using (15)- (21) is in Appendix I ,II, and III .

5. SPECIAL CASE OF EQUAL INTERVALS

The preceding relationships can obviously be applied to unequal or equal intervals. For equal intervals, however, the relationships can be simplified and a single formula for computing the area can be obtained. Area computation with such formula would be more efficient .Assuming that the interval equal h, then after all simplifies(i.e. using (16)-(21) into (15)) Equ.(15) becomes

$$A_e = h[\sum_{i=1}^{n-1} y_i + \frac{1}{24}(9y_0 + 4y_1 - y_2 + 9y_n + 4y_{n-1} - y_{n-2})] \tag{22}$$

6. Verification

Obtained equation (22) tested on some functions, the results tabulated in table 1 and table 2, as follow:

6.1: Algebraic function:

Table 1: comparison between exact integration values and value by Quantic formula

S	Function	Quantic value	Exact value	MATLAB Algorithm
1	Y=f(x)= $\int_0^1 (x^5 - 2x^4 + x^3 - 3x^2 + x + 7)dx$	6.5167	$\frac{391}{60}$	Syms x Y=int (x ^ 5 − 2 * x ^ 4 + x ^ 3 − 3 * x ^ 2 + x + 7, 0, 1)
2	Y=f(x)= $\int_0^1 (x^5 + x^3 + 7x + 2)dx$	5.917	$\frac{71}{12}$	Syms x Y=int(x ^ 5 + x ^ 3 + 7 * x + 2, 0, 1)
3	Y=f(x)= $\int_0^1 (2x^5 + 3x^4 + 7x^2 + 1)dx$	4.27	$\frac{64}{15}$	Syms x Y=int(2 * x ^ 5 + 3 * x ^ 4 + 7 * x ^ 2 + 1, 0, 1)

6.2: Trigonometric function:

Table 2: comparesion between exact integration values and value by Quantic formula

S	Function	Qunatic value	Exact Value	MATLAB Algorithm
1	Y=f(x)= $\int_0^{\pi} \sin(x)dx$	2	2	Syms x Int(sin(x),0,pi)
2	Y=f(x)= $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} Cos(x)dx$	2	2	Syms x Int(cos(x),0.5*pi,1.5*pi)

6.3: Practical field data from surveying text book [3]:

A comparison was done between different methods by using existing data in [3, p.215], and the results were tabulated in table 3. as follow:

Table 3: Comparison between different methods

S	Methods	Area	Error %
1	Exact**	193.697	-
2	Trapezoidal rule	188.000	- 2.94 %
3	Simpson one third rule	196.667	+1.533%
4	Quantic Hermite	194.125	+0.221%

* Only equal intervals case tested here because derived equations for unequal intervals is little complex for applying by surveyor, and never can be use without programmed handheld calculator .It is write some time outside boundary condition need to choose unequal intervals to obtained a better result but this case can be treated by selection offsets at regular interval and at closer distance between offsets

** Exact area was calculated by MATLAB, through entering the data into EXCEL spread sheet, drawing best curve which can pass the offsets end, then calculating the coefficients for Quantic Hermite polynomial, and calculating the exact area through taking integration (see algorithm for MATLAB in appendix II).

7. CONCLUTION

In this paper ,we conclude that the Quantic Hermite polynomial introduced in this paper performs better results than the Trapezoid and Simpson's method for calculating irregular boundary area when $n \geq 6$.

8. APPENDIX I , II AND III:

APPENDIX I.

Algorithm for computing irregular boundary area using quantic hermit method.

* Step 1: Input (a) Number of intervals, $n (n \geq 6)$; (b) the offsets, $y_i, i = 0,1,2,...,n$; and (c) the intervals, $h_i, i = 0,1,2,...,n-1$ (or h if the intervals are equal).Set i=1 and S=0

* Step 2: if the intervals are equal, go to step 10. Otherwise, continue.

* Step 3: Compute the first derivatives at the first and last offsets, y'_0 and y'_n ,

$$y'_0 = \frac{(h_0+h_1)^2 y_1 - h_0^2 y_2 - h_1(2h_0+h_1)y_0}{h_0 h_1 (h_0+h_1)} \tag{23}$$

$$y'_n = \frac{h_{n-2}(h_{n-2}+2h_{n-1})y_n - (h_{n-2}+h_{n-1})^2 y_{n-1} + h_{n-1}^2 y_{n-2}}{h_{n-2}h_{n-1}(h_{n-2}+h_{n-1})} \tag{24}$$

* Step 4: Compute the first derivative at offsets i , y'_i

$$y'_i = \frac{h_{i-1}^2 y_{i+1} + (h_i^2 - h_{i-1}^2)y_i - h_i^2 y_{i-1}}{h_{i-1}h_i(h_{i-1}+h_i)} \tag{25}$$

*Step 5: Compute the second derivatives at the first and last offsets, y''_0 and y''_n ,

$$y''_0 = \frac{2[h_0 y_2 - (h_0+h_1)y_1 + h_1 y_0]}{h_0 h_1 (h_0+h_1)} \tag{26}$$

$$y''_n = \frac{2[h_{n-2}y_n - (h_{n-2}+h_{n-1})y_{n-1} + h_{n-1}y_{n-2}]}{h_{n-2}h_{n-1}(h_{n-2}+h_{n-1})} \tag{27}$$

* Step 6: Compute the second derivative at offsets i , y''_i

$$y''_i = \frac{2[h_{i-1}y_{i+1} - (h_{i-1}+h_i)y_i + h_i y_{i-1}]}{h_{i-1}h_i(h_{i-1}+h_i)} \tag{28}$$

*Step 7: Compute the term inside the summation (equation 15), S_i

$$S_i = (h_{i-1}+h_i)y_i + \frac{1}{10}(h_{i-1}^2 - h_i^2)y'_i + \frac{1}{120}(h_{i-1}^3 + h_i^3)y''_i \tag{29}$$

Set the cumulative term, *S, equal* to the current value plus S_i ; $\leftarrow S + S_i$.

*Step 8: Set $i \leftarrow i+1$. if $i = n$,go to step 9.Otherwise, go to step 4

*Step 9: Compute the required area, A_u ,and go to step 11

$$A_u = \frac{1}{2}(h_0 y_0 + h_{n-1}y_n) + \frac{1}{10}(h_0^2 y'_0 - h_{n-1}^2 y'_n) + \frac{1}{120}(h_0^3 y''_0 + h_{n-1}^3 y''_n) + S \tag{30}$$

*Step 10: Compute the required area for equal intervals, A_e

$$A_e = h[\sum_{i=1}^{n-1} y_i + \frac{1}{24}(9y_0 + 4y_1 - y_2 + 9y_n + 4y_{n-1} - y_{n-2})] \tag{31}$$

*Step 11: Output (A_u or A_e)

APPENDIX II.

Algorithm for calculation of exact area for the practical field data by using EXCEL & MATLAB

Step 1:

Enter data into EXCEL spread sheet

Step 2:

Startmatlab (start MATLAB)

putmatrix (send data to MATLAB)

Step 3:
Define variable name in MATLAB
(defines (x) & y variables)
Step4:
Evalstring (execute the MATLAB command)
cftool (open best curve fit window)
Step 5:
Fitting (Draw new fit by selecting 5^{th} degree polynomials (draw best curve which can pass offsets end)
Step 6:
Analysis (Calculate area under the drawn best curve by taking integration)

APPENDIX III.

MATLAB output relating best curve equation and its coefficients
linear model Poly5:
f(x) = p1*x^5 + p2*x^4 + p3*x^3 + p4*x^2 + p5*x + p6
Coefficients (with 95% confidence bounds):
p1 = 1.125e-007 (-1.915e-006, 2.14e-006)
p2 = -1.756e-005 (-0.0003227, 0.0002876)
p3 = 0.0009131 (-0.01525, 0.01708)
p4 = -0.02237 (-0.3763, 0.3316)
p5 = 0.3892 (-2.381, 3.16)
p6 = 0.01374 (-5.292, 5.32)
Goodness of fit:
SSE: 0.1746
R-square: 0.9928
Adjusted R-square: 0.9571
RMSE: 0.4178

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Notations:

The following symbols are used in this paper
a,b = limit of integration
n = total number of intervals
h= length of one equal interval
A_i= Area of region for interval i
A = total area
y_i= offsets at points
f''_o , f''_n ,f''_i=second derivatives at points 0 ,n, i respective
$Q_i(x)$ = ordinate of 5th degree (Quantic) polynomial i at point x

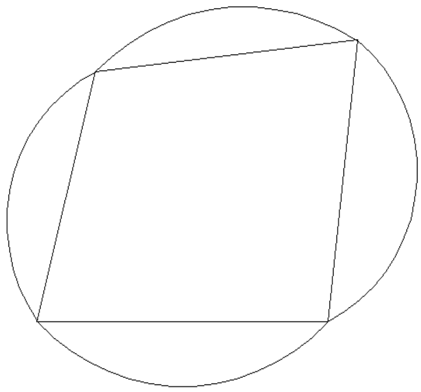


Fig-1-,irregular boundary tract

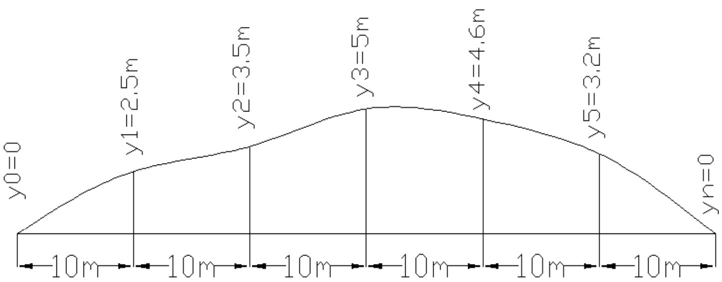
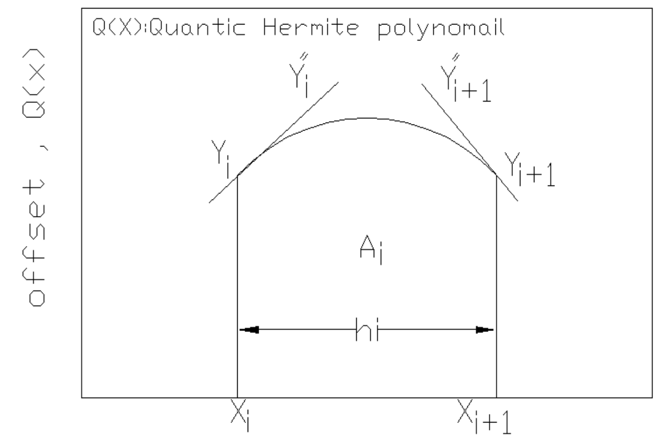


Fig-2-,practical data from text book[3]



Single Quantic Hermite Area

(Fig-3-)

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