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Jordan Decomposition and its Uniqueness of Signed Lattice Measure

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Abstract

In this paper, we define a signed lattice measure on σ-Algebras, as well as give the definition of positive and negative lattice, positive and negative parts of ν and mutually singular lattice measures. Herein, we show that the Hann Decomposition theorem decomposes any space X into a positive lattice A and a negative lattice B such that $A \vee B = X$ and $A \wedge B = \phi$. Also we prove the Jordan Decomposition theorem and its uniqueness of signed lattice measure.

Keywords: Signed Lattice Measure, Hann-Decomposition, Measurable Space

1. INTRODUCTION

In this paper, we illustrate the Jordan Decomposition theorem and its uniqueness of signed lattice measure. We consider a set X be a nonempty arbitrary set and L a lattice subsets of X. All lattices considered throughout the paper will contain ϕ and X.

We define a signed lattice measure on σ-Algebras, and we show that the lattice Hann Decomposition theorem decomposes any set X into a Positive lattice A and a negative lattice B such that $A \vee B = X$ and $A \wedge B = \phi$. We also prove the Jordan Decomposition Theorem and its uniqueness.

In section 2, according to [4][5] the definition of a signed lattice measure on σ-Algebras, as well as give the definition of positive and negative lattice. Also we define positive and negative parts of ν and mutually singular lattice measures. Further, we show that any countable union of positive lattice is a positive lattice as well as if E is a measurable lattice such that$0 < \nu(E) < \infty$, then there is a positive lattice $A \leq E$ with $\nu(A) > 0$[2]. Also we show the Hann Decomposition theorem for signed lattice measure [3]. Finally illustrate the Jordan Decomposition theorem and its uniqueness for a signed lattice measure.

2. PRELIMINARIES

In this section, we shall briefly review the well-known facts about lattice theory due to Birkhoff [1], propose an extension lattice, and investigate its properties. $(L, \Lambda, \vee)$ is called a lattice if it is enclosed under operations $\wedge$ and $\vee$ and satisfies, for any elements x, y, z, in L:

(L1) the commutative law: $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.

(L2) the associative law: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z)=(x \vee y)\vee z$.

(L3) the absorption law: $x \vee (y \wedge x) = x$ and $x \wedge (y \vee x) = x$.

Hereafter, the lattice $(L, \wedge, \vee)$ will often be written as L for simplicity.

A mapping h from a lattice L to another lattice L^1 is called a lattice-homomorphism, if it satisfies

$h (x \wedge y) = h(x) \wedge h(y)$ and $h (x \vee y) = h (x) \vee h (y), \forall x, y \in L$.

If h is a bijection, that is, h is one-to-one and onto, it is called a lattice isomorphism, and in this case, L^1 is said to be lattice-isomorphic to L.

A lattice $(L, \wedge, \vee)$ is called distributive if, for any x, y, z, in L.

(L4) the distributive law holds:

$x \vee (y \wedge z) = (x \vee y) \wedge (y \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (y \wedge z)$.

A lattice L is called complete if, for any subset A of L, L contains the supremum $\vee A$ and the infimum $\wedge A$. If L is complete, then L itself includes the maximum and minimum elements which are often denoted by 1 and 0 or I and O respectively [1].

A distributive lattice is called a Boolean lattice if for any element x in L, there exists a unique complement x^c such that

$x \vee x^c = 1$ (L5) the law of excluded middle

$x \wedge x^c = 0$ (L6) the law of non-contradiction

Let L be a lattice and €: L → L be an operator. Then € is called a lattice complement in L if the following conditions are satisfied.

(L5) and (L6); $\forall\, x \in L, x \vee x^c = 1$ and $x \wedge x^c = 0$,

(L7) the law of contrapositive; $\forall\, x, y \in L, x \leq y$ implies $x^c \geq y^c$,

(L8) the law of double negation; $\forall\, x \in L, (x^c)^c = x$.

Throughout this paper, we consider lattices as complete lattices which obey (L1) - (L8) except for (L6) the law of non-contradiction.

2.1 Definition. Unless otherwise stated, X is the entire set and L is a lattice of any subsets of X. If a lattice L satisfies the following conditions, then it is called a lattice σ-Algebra;

(1) $\forall\, h \in L, h^c \in L$

(2) if $h_n \in L$ for n = 1, 2, 3 ....., then $\bigvee_{n=1}^{\infty} h_n \in L$.

We denote σ(L), as the lattice σ-Algebra generated by L.

2.2 Definition. The ordered pair (X, σ(L)) is said to be lattice measurable space

2.3 Definition. If m: $\sigma(L) \to R \cup \{\infty\}$ satisfies the following properties, then m is called a lattice measure on the lattice σ-Algebra σ(L).

(1) $m(\phi) = m(0) = 0$.

(2) $\forall\, h, g \in \sigma(L)$, s.t. $m(h), m(g) \geq 0$; $h \leq g \Rightarrow m(h) \leq m(g)$.

(3) $\forall\, h, g \in \sigma(L)$: $m(h \vee g) + m(h \wedge g) = m(h) + m(g)$.

(4) If $h_n \subset \sigma(L)$, $n \in N$ such that $h_1 \leq h_2 \leq ... \leq h_n \leq ....$, then $m(\bigvee_{n=1}^{\infty} h_n) = \lim m(h_n)$.

Let m_1 and m_2 be lattice measures define on the same lattice σ-Algebra σ(L). If one of them is finite, the set function $m(E) = m_1(E) - m_2(E)$, $E \in \sigma(L)$ is well defined and countably additive on σ(L). However, it is necessarily nonnegative; it is called a signed lattice measure.

2.4 Definition. By a signed lattice measure on the lattice measurable space (X, σ(L)) we mean $\nu: \sigma(L) \to R \cup \{\infty\}$ or $R \cup \{-\infty\}$, satisfying the following properties :

(1) $\nu(\phi) = \nu(0) = 0$.

(2) $\forall\, h, g \in \sigma(L)$ s.t. $\nu(h), \nu(g) \geq 0$; $h \leq g \Rightarrow \nu(h) \leq \nu(g)$.

$\forall\, h, g \in \sigma(L)$ s.t. $\nu(h), \nu(g) \leq 0$; $h \leq g \Rightarrow \nu(g) \leq \nu(h)$.

(3) $\forall\, h, g \in \sigma(L)$: $\nu(h \vee g) + \nu(h \wedge g) = \nu(h) + \nu(g)$.

(4) if $h_n \subset \sigma(L)$, $n \in N$ such that $h_1 \leq h_2 \leq \ldots \leq h_n \leq \ldots$, then $\nu(\bigvee_{n=1}^{\infty} h_n) = \lim \nu(h_n)$.

This is meant in the sense that if the left-hand side is finite, the limit on the right-hand side is convergent, and if the left-hand side is $\pm\infty$, then the limit on the right-hand side diverges accordingly.

2.5 Definition. A set A is said to be lattice measurable set or measurable lattice, if A belongs to $\sigma(L)$.

2.6 Definition. A is a positive lattice if for any lattice measurable set E in A, $\nu(E) \geq 0$; Similarly, B is a negative lattice if for any lattice measurable set E in B, $\nu(E) \leq 0$.

2.7 Definition. Let $(X, \sigma(L))$ be a lattice measurable space and ν be a signed lattice measure defined on $\sigma(L)$ then there exists a positive set A and a negative set B such that $A \vee B = X$ and $A \wedge B = \phi$. We denote lattice Hann decomposition by {A, B}

2.8 Definition. (**Positive and Negative parts of ν):**
Let $(X, \sigma(L))$ be a lattice measurable space and ν be a signed lattice measure defined on $\sigma(L)$. If we define ν^+ and ν^- on for all $E \in \sigma(L)$ such that $\nu^+(E) = \nu(E \wedge A)$ and $\nu^-(E) = -\nu(E \wedge B)$ with $\nu = \nu^+ - \nu^-$. Then ν^+ and ν^- are called the positive and the negative parts of a signed measure lattice respectively. Where {A, B} be a Hann decomposition on X.

2.9 Definition. (Mutually singular lattice measures):
Let $(X, \sigma(L))$ be a lattice measurable space and ν be a signed lattice measure defined on $\sigma(L)$. If we define two measures ν_1 and ν_2 are said to be mutually singular lattice measures with each other to be denoted by $\nu_1 \perp \nu_2$. If there exists two measurable lattices A and B such that $A \vee B = X$ and $A \wedge B = \phi$ and $\nu_1(A) = \nu_2(B) = 0$.

3. JORDAN-DECOMPOSTION AND ITS UNIQUENESS OF SIGNED LATTICE MEASURE

3.1 Lemma: Every measurable sublattice of a positive lattice is a positive lattice and a countable union of positive lattice is a positive lattice

Proof : Part (i): Let A be a positive lattice and P be a sublattice of A Let E be a

measurable sub lattice of P implies E be a measurable sublattice of A, By definition of a positive lattice $\nu(E) \geq 0$. Therefore P is a positive lattice.
Part (ii): To prove that countable union of positive lattice is a positive lattice.
Let $<A_i>$ be a countable collection of positive lattices such that $A_1 \leq A_2 \leq .... \leq A_n \leq ...$

by the definition4, $\nu(\vee_{i=1}^{\infty} A_i) = \lim \nu(A_i)$. Put $A = \vee_{i=1}^{\infty} A_i$

Now each A_i is a positive lattice implies each A_i is a measurable lattice by the definition3,

$A = \vee_{i=1}^{\infty} A_i$ is measurable. Therefore A is measurable lattice.

To prove that A is positive.

Let E be a measurable sublattice of A that is $E \leq A$ implies $E \leq \vee_{i=1}^{\infty} A_i$

Define $E_n = E \wedge A_n \wedge A_{n-1}^c \wedge$ .... implies $E_n \leq E$ and $E_n \leq A_n$ for all n= 1, 2,3... implies E_n is measurable and E_n is positive lattice therefore $\nu(E_n) \geq 0$ for all n= 1,2,3...also in particular $E_n \leq E_{n+1}$ for all n. Evidently $E = \vee_{n=1}^{\infty} E_n$.

It implies $\nu(E) = \nu(\vee_{n=1}^{\infty} E_n) = \lim \nu(E_n) \geq 0$ implies $\nu(E) \geq 0$. Therefore A is a positive lattice.

3.2 Lemma: Let E be a measurable lattice such that $0 < \nu(E) < \infty$ then there is a positive lattice $A \leq E$ with $\nu(A) > 0$.

Proof: If E is a positive lattice we take A = E. Therefore $\nu(A) > 0$ and A is a positive lattice.
It is given that $0 < \nu(E) < \infty$ Evidently $\nu(E) > 0$ and finite. Suppose E contains a sublattice of negative measures, let n_1 be the smallest positive integer such that there exists a measurable lattice $E_1 < E$ with $\nu(E_1) < \frac{-1}{n_1}$. If $E \wedge E_1^c$ is not already a positive lattice, let n_2 be the least positive integer such that there exists measurable lattice $E_2 < E \wedge E_1^c$ such that $\nu(E_2) < \frac{-1}{n_2}$. Proceeding inductively, if $E \wedge (\vee_{j=1}^{k-1} E_j)^c$ is not already a positive lattice let n_k be the smallest positive integer $> n_{k-1}$ such that there exists a measurable lattice $E_k < E \wedge (\wedge_{j=1}^{k-1} E_j^c)$ such that

$\nu(E_k) < \frac{-1}{n_k}$. Continuing like this let $A = E \wedge (\overset{\infty}{\underset{k=1}{\wedge}} E_k^c)$

implies $A = E - (\overset{\infty}{\underset{k=1}{\vee}} E_k)$ implies $E = A \vee (\overset{\infty}{\underset{k=1}{\vee}} E_k)$ implies $\nu(E) = \nu(A) + \nu(\overset{\infty}{\underset{k=1}{\vee}} E_k)$

implies $\nu(E) = \nu(A) + \lim \nu(E_k)$ ----------------------- (1)

As $\nu(E)$ is finite it implies $\lim \nu(E_k)$ is finite and negative from (1) $\nu(A) = \nu(E) > 0$

Therefore $\nu(A) > 0$. To prove A is a positive lattice, $A = E \wedge (\overset{\infty}{\underset{k=1}{\wedge}} E_k^c)$

Since E is measurable it implies A is measurable. Let P be a measurable sublattice of A.

$P \leq A$ implies $P \leq E - (\overset{\infty}{\underset{k=1}{\vee}} E_k)$ implies $P \leq E$ and $P \not\leq (\overset{\infty}{\underset{k=1}{\vee}} E_k)$

implies $P \leq E$ and $P \not\leq E_k$ for any k. Therefore P is not a sublattice of a negative measure that is A does not have any sublattice of negative measures. Therefore A is a positive lattice. Therefore E has a positive lattice A such that $\nu(A) > 0$.

3.3 THEOREM (Lattice Hann-Decomposition):

Let $(X, \sigma(L))$ be a lattice measurable space and ν be a signed lattice measure defined on $\sigma(L)$ then there exists a positive set A and a negative set B such that $A \vee B = X$ and $A \wedge B = \phi$.

Proof: Without loss of generality $+\infty$ is a maximum value omitted by ν

Define $\lambda = \sup \{\nu(A_i) / A_i$ is a positive lattice$\}$ since $\nu(\phi) = 0$ implies $\lambda \geq 0$

Let $\langle A_i \rangle$ be a sequence of positive lattices such that $\lambda = \lim \nu(A_i)$ Put $A = \overset{\infty}{\underset{i=1}{\vee}} A_i$.

By Lemma1 A is a positive lattice. Therefore $\nu(A) \leq \lambda$----------------------- (1)

Also each $A_i \leq A$ implies $A - A_i \leq A$ implies $\nu(A - A_i) \geq 0$ Evidently $A = A_i \vee (A - A_i)$

implies $\nu(A) = \nu(A_i) + \nu(A - A_i)$ implies $\nu(A) \geq \nu(A_i)$ for all i, implies $\nu(A) \geq \lambda$ ----------------------- (2)

From (1) and (2) $\nu(A) = \lambda$. Put $B = A^c$, to prove that B is a negative lattice.

Let us suppose that B has a positive measure lattice E (say). Now E is a positive lattice and A is a positive lattice implies $A \vee E$ is a positive lattice implies $\nu(A \vee E) \leq \lambda$ implies $\nu(A) + \nu(E) \leq \lambda$ implies $\lambda + \nu(E) \leq \lambda$ implies $\nu(E) \leq 0$. This is a contradiction. Therefore B has no sublattice of positive measures. This implies B is a negative lattice.

Hence by above, we conclude that "Let X be an entire set. Then by above theorem we can find a positive lattice A and a negative lattice B ($=A^c$). By the lattice measurability $A \vee B = X$ and $A \wedge B = \phi$. These characteristics provide the following $X = A \cup B$ and $\phi = A \cap B$".

3.4 THEOREM (Jordan–Decomposition on Signed Lattice Measure):
Let $(X, \sigma(L))$ be a lattice measurable space and ν be a signed lattice measure defined on $\sigma(L)$ then there exists a unique pair of mutually singular lattice measures ν^+ and ν^- with $\nu = \nu^+ - \nu^-$.

Proof: Let {A, B} be a Hann decomposition on X then there exists a positive lattice A and a negative lattice B such that $A \vee B = X$ and $A \wedge B = \phi$
Now for all $E \in \sigma(L)$. Define $\nu^+(E) = \nu(E \wedge A)$ .... (1)
$\nu^-(E) = - \nu(E \wedge B)$.............. (2)
from(1) $\nu^+(B) = \nu(B \wedge A) = \phi$
from(2) $\nu^-(A) = - \nu(A \wedge B) = \phi$
therefore ν^+ and ν^- are mutually singular lattice measures. Also $E=(E\wedge A) \vee (E\wedge B)$
$\nu(E) = \nu(E \wedge A) + \nu(E \wedge B)$, $\nu(E) = \nu^+(E) - \nu^-(E)$ therefore $\nu = \nu^+ - \nu^-$ .... (3)

UNIQUENESS :

Let {A, B} and $\{A^1 B^1\}$ be two lattice Hann decompositions on X
$A \vee A^1 = (A \wedge A^1) \vee (A \Delta A^1)$ .... (4)
where $(A \Delta A^1) = (A - A^1) \vee (A^1 - A)$ .... (5)
$\nu(A \vee A^1) = \nu(A \wedge A^1) + \nu(A \Delta A^1)$ .... (6)
where $\nu(A \Delta A^1) = \nu(A - A^1) + \nu(A^1 - A)$ .... (7)
Clearly $A - A^1 \leq A \wedge B^1$ Now $A - A^1 \leq A$ and $A - A^1 \leq B^1$
It implies $\nu(A - A^1) \geq 0$ and $\nu (A - A^1) \leq 0$ (since A is a positive lattice and B^1 is a negative lattice) therefore $\nu(A - A^1) = 0$ similarly $\nu(A^1 - A) = 0$ therefore from (7) $\nu(A \Delta A^1) = 0$
from (6) $\nu(A \vee A^1) = \nu(A \wedge A^1)$ .... (8)
Also evidently for all $E \in \sigma(L)$
$\nu(E \wedge A \wedge A^1) \leq \nu(E \wedge A) \leq \nu(E \wedge A \wedge A^1)$ .... (9)
$\nu(E \wedge A \wedge A^1) \leq \nu(E \wedge A^1) \leq \nu(E \wedge A \wedge A^1)$ .... (10)
From (9) and (10) we have $\nu(E \wedge A) = \nu(E \wedge A^1)$.
Therefore ν^+ is unique. From (3) $\nu = \nu^+ - \nu^-$ implies ν^- is unique. Therefore there is only one such pair of mutually singular lattice measures.

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