Completeness of a Nearly Bi-Quasi Metric Space

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Abstract

The paper is intended to study the characterization of the completeness property of a nearly bi-quasi-metric space that leads to obtain some important results on fixed point theorems.

Mathematics Subject Classification: 47H10, 54H25

Keywords: fixed point, 2-metric space, completeness

1 Introduction

The concept of bi-quasi metric space was first introduced by J.C. Kelly[3]. But till now completeness is not introduced in the problem of completion of a bi-quasi metric space. J. D. Weston [4] introduced the concept of characterization of completeness of a metric space. In order to obtain some important results of fixed point theorems, the concept of completeness has been introduced in a nearly bi-quasi metric space. In earlier papers fixed point theorems were proved by using iterative procedure in metric space or quasi metric space setting. In this paper iterative procedure has been replaced by introducing a suitable function and order relation in a nearly bi-quasi metric space setting. Some important analogous results like [1], [2], [5] has been obtained in this setting. Before going to main results let us introduce some important definitions.
2 Preliminaries

Definition 2.1 A quasi metric \( p \) on a non empty set \( X \) is a non negative real valued function \( p(\cdot, \cdot) \) on \( X \times X \) satisfying

(i) \( p(x, y) = 0 \) if and only if \( x = y; x, y \in X \)

(ii) \( p(x, z) \leq p(x, y) + p(y, z) \) for all \( x, y, z \in X \)

Definition 2.2 If \( p \) is a quasi metric on \( X \) then \((X, p)\) is called a quasi-metric space.

Example 2.1 Let \( R \) be the space of real numbers and \( p : R \times R \to R \) be defined as follows

\[
p(x, y) = \begin{cases} 
\min[1, |x - y|], & \text{if } x \leq y \\
1, & \text{if } x > y
\end{cases}
\]

Then \( p \) is a quasi metric on \( R \) without being a metric because \( p(0, \frac{1}{2}) = \frac{1}{2} \) but \( p(\frac{1}{2}, 0) = 1 \).

Let \( p \) be a quasi metric defined on \( X \). Let \( x \in X \) and \( r > 0 \). Then \( p-B(x, r) = \{y \in X : p(x, y) < r\} \neq \emptyset \) and is called a \( p \)-open ball centred at \( x \) having radius \( r \). The set \( \{y \in X : p(x, y) \leq r\} \) is called a \( p \)-closed ball in \( X \) centred at \( x \) having radius \( r \). Let \( \wp = \{p-B(x, r) : x \in X \text{ and } r > 0\} \) i.e. \( \wp \) is the collection of all \( p \)-open balls in \( X \). Clearly \( \wp \) is a base for a topology on \( X \). It is to be noted that every quasi metric space \((X, p)\) is \( T_1 \) but not \( T_2 \). (see [3]) Let \( p(\cdot, \cdot) \) be a quasi metric on \( X \) and let \( q(\cdot, \cdot) \) be defined by \( q(x, y) = p(y, x) \) for all \( x, y \in X \). One can easily verify that \( q(\cdot, \cdot) \) is also a quasi metric on \( X \) which is called the conjugate of \( p(\cdot, \cdot) \).

Definition 2.3 A sequence \( \{x_n\} \) in a quasi-metric space \((X, p)\) is called a \( p \)-Cauchy sequence if given \( \epsilon > 0 \) there exists an integer \( m \) such that \( p(x_r, x_s) < \epsilon \), whenever \( r > s > m \).

Definition 2.4 A sequence \( \{x_n\} \) in \((X, p)\) is said to \( p \)-converge at \( x_0 \in X \) if given \( \epsilon > 0 \) there exists an integer \( m \) such that \( p(x_0, x_n) < \epsilon \), whenever \( n > m \).

Definition 2.5 A quasi-metric space \((X, p)\) is said to be \( p \)-complete if every \( p \)-Cauchy sequence is \( p \)-convergent in \((X, p)\).

Similarly one can define the \( q \)-Cauchy, \( q \)-convergence, \( q \)-completeness in \( X \).

Definition 2.6 The triplet \((X, p, q)\) is called a bi-quasi metric space.
Definition 2.7 A bi-quasi metric space \((X, p, q)\) is called a nearly bi-quasi-metric space if the following condition is satisfied:

\[ p(x, y) + q(x, y) > 0 \quad \text{for} \quad x \neq y. \]

Definition 2.8 A point \(x_0 \in X\) is called a \((p, q)\) point for \(h\) if for every point \(x(\neq x_0) \in X\),

\[ h(x_0) - h(x) < \frac{1}{2} [p(x_0, x) + q(x_0, x)]. \]

Definition 2.9 A sequence \(\{x_n\}\) in a nearly bi-quasi-metric space \((X, p, q)\) is said to be \((p, q)\) cauchy sequence if it is both \(p\)-cauchy and \(q\)-cauchy with respect to respective \(p\)-metric and \(q\)-metric on \(X\).

Definition 2.10 A nearly bi-quasi metric space \((X, p, q)\) is said to be complete if every \((p, q)\) cauchy sequence \(\{x_n\}\) in \(X\) there exist a point \(\xi \in X\) such that \(p(\xi, x_n) \to 0\) as and \(n \to \infty\) and \(q(\xi, x_n) \to 0\) as and \(n \to \infty\).

Definition 2.11 A nearly bi-quasi metric space \((X, p, q)\) is said to be strongly complete iff every sequence \(\{x_n\}\) in \(X\) that is both \(p\)-Cauchy and \(q\)-cauchy converges to the same point of \(X\) with respect to \(p\) and \(q\).

Definition 2.12 A function \(h : (X, p, q) \to R\) is called \(p\) lower semi continuous at \(x \in X\) iff for every \(\epsilon > 0\), there exists \(\delta > 0\) such that \(h(x) - \epsilon < h(y)\) for all \(y \in X\) with \(p(x, y) < \delta\). Similarly one can define \(q\) lower semi continuity of \(h\) at \(x \in X\).

Definition 2.13 A function \(h : (X, p, q) \to R\) is called \((p, q)\) lower semi continuous \((l. s. c.)\) at \(x \in X\) iff for every \(\epsilon > 0\), there exists \(\delta > 0\) such that \(h(x) - \epsilon < h(y)\) for all \(y \in X\) with \(p(x, y) < \delta\) and \(q(x, y) < \delta\).

Remark 2.1 A \(p(q)\) lower semi continuous function is \((p, q)\) lower semi continuous function.

3 Main Results

Theorem 3.1 If \((X, p, q)\) is a nearly bi-quasi metric space which is strongly complete, then any \((p, q)\) lower semi-continuous function \(h : X \to R\) which is bounded below has \((p, q)\) point.
Proof : Suppose \((X, p, q)\) is strongly complete and let \(h : X \rightarrow R\) be a \((p, q)\) lower semi continuous function which is bounded below. Let us choose \(\{x_n\}\) in the following way. For each \(n\), let \(\alpha_n = \inf \{h(x) : h(x_n) - h(x) \geq \frac{1}{2} [p(x_n, x) + q(x_n, x)] > 0\}\) 

Let \(x_{n+1}\) be a point in \(X\) such that 

\[
\begin{align*}
  h(x_n) - h(x_{n+1}) &\geq \frac{1}{2} [p(x_n, x_{n+1}) + q(x_n, x_{n+1})] \quad (1) \\
  \text{and } h(x_{n+1}) &< \alpha_n + n^{-1} \quad (2)
\end{align*}
\]

If \(x_n\) is a \((p, q)\) point for \(h\), then \(x_{n+1}\) must be \(x_n\); otherwise from (1) it follows that \(\{h(x_n)\}\) is a monotonically decreasing sequence and for \(m \geq n\), we have 

\[
\begin{align*}
  h(x_n) - h(x_m) &= h(x_n) - h(x_{n+1}) + h(x_{n+1}) - h(x_{n+2}) + \ldots \\
  &\quad + h(x_{m-1}) - h(x_m) \\
  &\geq \frac{1}{2} [p(x_n, x_m) + q(x_n, x_m)] \\
  \text{i.e } h(x_n) - h(x_m) &\geq \frac{1}{2} [p(x_n, x_m) + q(x_n, x_m)] \quad (3)
\end{align*}
\]

Since \(h\) is bounded below, \(\{h(x_n)\}\) is convergent and hence Cauchy with respect to usual metric of reals. By (3) \(\{x_n\}\) is both \(p\)-cauchy as well as \(q\)-cauchy in \(X\), and by strong completeness of \(X\), let \(p - \lim x_n = q - \lim x_n = x_0\).

Now \(h(x_n) - h(x_0) \geq \frac{1}{2} [p(x_n, x_0) + q(x_n, x_0)]\) for every \(n\); 

(4)

Otherwise, if for some \(n\), \(h(x_n) - h(x_0) \leq \frac{1}{2} [p(x_n, x_0) + q(x_n, x_0)] - \epsilon\)

where \(\epsilon > 0\), then \((p, q)\) lower semicontinuity of \(h\) at \(x_0\) gives for 

\[
\epsilon_1 = \frac{1}{2} [p(x_n, x_0) + q(x_n, x_0)] - \epsilon - h(x_n) + h(x_0) > 0
\]

there exists a positive \(\delta = \delta(\epsilon_1)\) satisfying \(h(x_0) - \epsilon_1 < h(x)\) or \(h(x_n) - h(x) < \frac{1}{2} [p(x_n, x_0) + q(x_n, x_0)] - \epsilon\) for all \(x \in X\) with \(p(x_0, x) < \delta\) and \(q(x_0, x) < \delta\). As \(p - \lim x_n = q - \lim x_n = x_0\), for large \(m\)

\[
\begin{align*}
  h(x_n) - h(x_m) &< \frac{1}{2} [p(x_n, x_0) + q(x_n, x_0)] - \epsilon \\
  &\leq \frac{1}{2} [p(x_n, x_m) + q(x_n, x_m)] + \frac{1}{2} [p(x_m, x_0) + q(x_m, x_0)] - \epsilon
\end{align*}
\]

Choosing \(\delta = \epsilon\) we get 

\[
\begin{align*}
  h(x_n) - h(x_m) &< \frac{1}{2} [p(x_n, x_m) + q(x_n, x_m)]
\end{align*}
\]
which contradicts (3). If \( x_0 \) is not a \((p, q)\) point for \( h \), then for some \( x \neq x_0 \), we have

\[
h(x_0) - h(x) \geq \frac{1}{2} [p(x_0, x) + q(x_0, x)] > 0 \tag{5}
\]
as \((X, p, q)\) is a nearly bi-quasi metric space. Replacing \( n \) by \( n + 1 \) in (4) and by (2) we have \( h(x) \leq h(x) + h(x_{n+1}) - h(x_0) < \alpha_n + n^{-1} + h(x) - h(x_0) \). Hence by (5) \( h(x) < \alpha_n + n^{-1} - \frac{1}{2} [p(x_0, x) + q(x_0, x)] \). Thus one can choose \( n \) so that \( h(x) < \alpha_n \). From (4) and (5) we get \( h(x_n) \geq h(x_0) > h(x) \). This implies \( x_n \neq x \) that and hence \( p(x_n, x) + q(x_n, x) > 0 \). Moreover (4) and (5) give

\[
h(x_n) - h(x) \geq \frac{1}{2} [p(x_n, x_0) + q(x_n, x_0) + p(x_0, x) + q(x_0, x)]
\geq \frac{1}{2} [p(x_n, x) + q(x_n, x)] > 0
\]

Now the definition of \( \alpha_n \) gives \( h(x) \geq \alpha_n \), which is a contradiction. So \( x_0 \) is a \((p, q)\) point for \( h \).

**Remark 3.2** If \((X, p, q)\) is not complete, then there is a \((p, q)\) uniformly continuous function \( X \to R \) which is bounded below having no \((p, q)\) point.

**Proof:** Let \((X, p, q)\) be not complete, then there is a \( p \)-cauchy as well as \( q \)-cauchy sequence \( \{x_n\} \) which does not converge with respect to \( p \) and \( q \). For \( x \in X \), put \( \beta_n = p(x, x_n) \). Then \( \{\beta_n\} \) is a Cauchy sequence of reals, because \( \beta_m - \beta_n = p(x, x_m) - p(x, x_n) \leq p(x, x_n) < \epsilon \), for large \( m \) and \( n \) with \( n \geq m \). Similarly \( \beta_n - \beta_m \leq p(x_m, x_n) < \epsilon \) for large \( m \) and \( n \) with \( m \geq n \). So \( |\beta_m - \beta_n| < \epsilon \) for large \( m \) and \( n \). Clearly \( \lim_{n} p(x, x_n) \) and by similar argument \( \lim_{n} q(x, x_n) \) exist. Let \( h(x) = \lim_{n} p(x, x_n) + \lim_{n} q(x, x_n) \). If \( x_0 \in X \), then

\[
h(x_0) - h(x) = \lim_{n} p(x_0, x_n) + \lim_{n} q(x_0, x_n) - \lim_{n} p(x, x_n) - \lim_{n} q(x, x_n)
= \lim_{n} [p(x_0, x_n) - p(x, x_n)] + \lim_{n} [q(x_0, x_n) - q(x, x_n)]
\leq p(x_0, x) + q(x_0, x)
\]

So for \( \epsilon > 0 \), there exists \( \delta < \frac{\epsilon}{2} \) such that \( h(x_0) - h(x) < \epsilon \) whenever \( p(x_0, x) < \delta \) and \( q(x_0, x) < \delta \) and similarly \( h(x_0) - h(x) < \epsilon \) whenever \( p(x_0, x) < \delta \) and \( q(x_0, x) < \delta \). Thus \( |h(x_0) - h(x)| < \epsilon \) for all \( x, x_0 \in X \) with \( p(x_0, x) < \delta \) and \( q(x_0, x) < \delta \). Hence \( h \) is \((p, q)\) uniformly continuous. Now

\[
h(x_0) + h(x) = \lim_{n} p(x_0, x_n) + \lim_{n} q(x_0, x_n) + \lim_{n} p(x, x_n) + \lim_{n} q(x, x_n)
\geq p(x_0, x) + q(x_0, x)
\]
Also

\[ h(x_0) - h(x) = \frac{1}{2} [h(x_0) + h(x)] + \frac{1}{2} [h(x_0) - 3h(x)] \]
\[ \geq \frac{1}{2} [p(x_0, x) + q(x_0, x)] + \frac{1}{2} [h(x_0) - 3h(x)] \]

Since \( h(x_m) = \lim_{n \to \infty} p(x_m, x_n) + \lim_{n \to \infty} q(x_m, x_n) \). So \( h(x_m) \to 0 \) as \( m \to \infty \) and hence \( 3h(x) < h(x_0) \) if \( x = x_m \) and for large \( m \) we have \( h(x_0) - h(x) \geq \frac{1}{2} [p(x_0, x) + q(x_0, x)] \), showing that \( x_0 \) is not a \( (p, q) \) point for \( h \). Thus \( h \) has no \( (p, q) \) point in \( X \).

**Remark 3.3** If \( p = q \) = a metric \( d \), we get the result of J.D.Watson[4].

**Definition 3.2** A function \( h : (X, p, q) \to R \) is called \( p \) lower semi continuous at \( x \in X \) if and only if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( h(x) - \epsilon < h(y) \) for all \( y \in X \) with \( p(x, y) < \delta \). Similarly one can define \( q \) lower semi continuity of \( h \) at \( x \in X \).

### 4 Some fixed point theorems

Let \( p, q, \) and \( h \) are given. A relation ‘\(<<\)’ can be defined on \( X \) as follows: \( x << y \) if and only if \( h(y) - h(x) \geq \frac{1}{2} [p(y, x) + q(y, x)] \). One can verify that ‘\(<<\)’ is an order relation on \( X \).

**Definition 4.1** A point \( x_0 \in X \) is said to be a minimal point with respect to ‘\(<<\)’ if and only if \( x << x_0 \) implies \( x = x_0 \).

**Remark 4.1** A point of \( X \) is a \( (p, q) \) point for \( h \) if and only if it is a minimal point with respect to ‘\(<<\)’.

**Remark 4.2** If a function \( f : X \to X \) is such that it may be possible to choose \( p, q \) and \( h \) so that the relation ‘\(<\)’ has the property \( f(x) \neq x \) implies \( f(x) << x \), then any \( (p, q) \) point for \( h \) is a fixed point for \( f \).

**Proof:** If \( x_0 \) is a \( (p, q) \) point for \( h \), then \( h(x_0) - h(x) < \frac{1}{2} [p(x_0, x) + q(x_0, x)] \) for all \( x \in X \) and if \( f(x_0) \neq x_0 \), then \( f(x_0) << x_0 \) which implies that \( h(x_0) - h(f(x_0)) \geq \frac{1}{2} [p(x_0, f(x_0)) + q(x_0, f(x_0))] \) which is a contradiction. So \( f(x_0) = x_0 \).

**Theorem 4.2** Let \( (X, p, q) \) be a strongly complete nearly bi-quasi metric space and let \( f : X \to X \) be such that the following conditions are satisfied:

\[ p(f(x), f(y)) \leq aq(x, f(x)) + bq(y, f(y)) + cp(x, y) \]

where \( a, b, c \geq 0 \) and \( a + b + c < 1 \). Also let \( p(x, f(x)) \) and \( q(x, f(x)) \) are \( (p, q) \) lower semicontinuous functions. Then \( f \) has a fixed point in \( X \).
Proof: Let \( h(x) = \beta_1 p(x, f(x)) + \beta_2 q(x, f(x)); \) \( x \in X \) where \( \beta_1 = \frac{1}{2(1-c)} \) and \( \beta_2 = \frac{1 + b^2 + bc + ca - c^2}{2(1-c)(1-a-b-c)} \). Obviously, \( \beta_1 > 0 \). Also \( a < 1 \) implies \( a_2 < a \) which further gives \( 1 - c - a^2 > 1 - c - a > 0 \). Thus \( 1 + b^2 + bc + ca - c^2 > 0 \). So \( \beta_2 > 0 \). Now \( h(x) \) is \((p,q)\) lower semi continuous function which is bounded below and hence \( h \) has a \((p,q)\) point, by Theorem(3.1). Let \( f(x) \neq x \). Now

\[
\begin{align*}
h(x) - h(f(x)) &\geq \beta_1 p(x, f(x)) + \beta_2 q(x, f(x)) - \beta_1 aq(x, f(x)) \\
&\quad - \beta_1 b(q(f(x), f^2(x)) - \beta_1 c(p(x, f(x)) - \beta_2 q(f(x), f^2(x)) \\
&= (\beta_1 - \beta_1 c)p(x, f(x)) + (\beta_2 - \beta_1 a)q(x, f(x)) \\
&\quad - (\beta_2 + \beta_1 b)q(f(x), f^2(x)) \\
&\geq \beta_1 (1-c)p(x, f(x)) + (\beta_2 - \beta_1 a)q(x, f(x)) \\
&\quad - (\beta_2 + \beta_1 b)\left(\frac{b + c}{1 - a}\right)q(x, f(x)) \\
&= \beta_1 (1-c)p(x, f(x)) + \left\{\beta_2 \left(1 - \frac{b + c}{1 - a}\right)\right\}q(x, f(x)) \\
&\quad - \beta_1 \left(a + \frac{b(b + c)}{1 - a}\right)q(x, f(x)) \\
&= \frac{1}{2} p(x, f(x)) + \frac{(1 - a + ca - c)}{2(1-c)(1-a)}q(x, f(x)) \\
&= \frac{1}{2} [p(x, f(x)) + q(x, f(x))]
\end{align*}
\]

Thus \( f \) satisfies the condition \( f(x) \neq x \) implies \( f(x) \ll x \). By remark 4.2 \((p,q)\) point for \( h \) is fixed point for \( f \) in \( X \).

Remark 4.3 If \( p = q = d \), we get the result of C. S. Wong [2].

**Theorem 4.3** Let \((X, p, q)\) be a strongly complete nearly bi-quasi metric space and let \( f : X \rightarrow X \) be such that the following conditions are satisfied:

\[
p(f(x), f(y)) \leq \alpha [p(x, f(y)) + q(y, f(x))]
\]

where \( 0 \leq \alpha \leq \frac{1}{2} \) and \( x, y \in X \). Also let \( p(x, f(x)) \) and \( q(x, f(x)) \) are \((p,q)\) lower semicontinuous functions. Then \( f \) has a fixed point in \( X \).

**Proof**: Put \( h(x) = \frac{(1-a)}{2(1-2a)} [p(x, f(x)) + q(x, f(x))] \) as \( x \in X \), proceeding as before, one proves that \( f \) has a fixed point in \( X \).

**Theorem 4.4** Let \((X, p, q)\) be a strongly complete nearly bi-quasi metric space and let \( f : X \rightarrow X \) be such that the following conditions are satisfied:

\[
p(f(x), f(y)) \leq \alpha [q(x, f(x)) + p(y, f(y))] + \beta p(x, y) \\
+ \gamma \max \{p(x, f(y)), q(y, f(x))\}
\]
for all $x, y \in X$ and $\alpha, \beta, \gamma \geq 0$ with $2\alpha + \beta + 2\gamma < 1$. Also let $p(x, f(x))$ and $q(x, f(x))$ are $(p, q)$ lower semicontinuous functions. Then $f$ has a fixed point in $X$.

**Proof**: Take $h(x) = a [p(x, f(x)) + q(x, f(x))], \ x \in X$, where $a = \frac{(1-\alpha-\gamma)}{2(1-2\alpha-\beta-2\gamma)}$. Then as before one can prove that $f$ has a fixed point in $X$.

**Remark 4.4** If $p = q = d$, we get the result of Kannan [5].

**Acknowledgement.** The first author acknowledges the UGC for the financial support sanctioning the Minor Research Project to the author. The authors also thank Professor A.P. Baisnab, F.N. A. Sc. for his kind help and encouragement in the preparation of the manuscript.

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Received: June, 2009