Galios Connections and Operators

Yong Chan Kim

Department of Mathematics
Gangneung-Wonju National University
Gangneung, Gangwondo 210-702, Korea
yck@gwnu.ac.kr

Young Sun Kim

Department of Applied Mathematics, Pai Chai University
Dae Jeon, 302-735, Korea
yskim@pcu.ac.kr

Abstract

In this paper, we investigate the properties of isotone (antitone) Galois connections on a complete residuated lattice. In particular, we study the relations between isotone (antitone) Galois connections and implicative closure (open) operators.

Mathematics Subject Classification: 03E72, 03G10, 06A15, 06F07

Keywords: Complete residuated lattice, isotone (antitone) Galois connection, implicative closure (open) operators

1 Introduction

Galois connections on fuzzy sets introduced by Bělohlávek [1] are important mathematical tools [1-4]. Recently, Bělohlávek [1-3] investigate the properties of fuzzy relations and similarities on a residual lattice which supports part of foundation of theoretic computer science.

In this paper, we investigate the properties of isotone (antitone) Galois connection on on a complete residuated lattice. In particular, we study the relations between isotone (antitone) Galois connection and implicative closure (open) operators.
2 Preliminaries

Definition 2.1 [1-7,9] A triple \((L, \leq, \odot)\) is called a complete residuated lattice iff it satisfies the following properties:

(L1) \(L = (L, \leq, 1, 0)\) is a complete lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(L2) \((L, \odot, 1)\) is a commutative monoid;

(L3) \(\odot\) is distributive over arbitrary joins, i.e.

\[
(\bigvee_{i \in \Gamma} a_i) \odot b = \bigvee_{i \in \Gamma} (a_i \odot b).
\]

Example 2.2 [1-7,9] (1) Each frame \((L, \leq, \wedge)\) is a complete residuated lattice.

(2) The unit interval with a left-continuous t-norm \(t\), \([0, 1], \leq, t\), is a complete residuated lattice.

(3) Define a binary operation \(\odot\) on \([0, 1]\) by \(x \odot y = \max\{0, x + y - 1\}\). Then \(([0, 1], \leq, \odot)\) is a complete residuated lattice.

Let \((L, \leq, \odot)\) be a complete residuated lattice. A order reversing map \(\ast : L \to L\) defined by \(a \ast = a \to 0\) is called a strong negation if \(a \ast \ast = a\) for each \(a \in L\).

In this paper, we assume \((L, \leq, \odot, \ast)\) is a complete residuated lattice with a strong negation \(\ast\).

Lemma 2.3 [1-7,9] For each \(x, y, z, x_i, y_i \in L\), we have the following properties.

(1) If \(y \leq z\), \((x \odot y) \leq (x \odot z)\), \((x \oplus y) \leq (x \oplus z)\), \(x \to y \leq x \to z\) and \(z \to x \leq y \to x\).

(2) \(x \odot y \leq x \land y \leq x \lor y \leq x \oplus y\).

(3) \(x \to (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma} (x \to y_i)\).

(4) \(\bigvee_{i \in \Gamma} x_i \to y = \bigvee_{i \in \Gamma} (x_i \to y)\).

(5) \(x \to y = y^\ast \to x^\ast\).

(6) \(\bigwedge_{i \in \Gamma} y_i^\ast = (\bigvee_{i \in \Gamma} y_i)^\ast\) and \(\bigvee_{i \in \Gamma} y_i^\ast = (\bigwedge_{i \in \Gamma} y_i)^\ast\).

(7) \(x \oplus (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma} (x \oplus y_i)\).

(8) \((x \odot y) \to z = x \to (y \to z) = y \to (x \to z)\).

(9) \(x \odot y = (x \to y^\ast)^\ast\), \(x \oplus y = x^\ast \to y\).

Let \(U\) be a set. A function \(R : U \times U \to L\) is called:

(R1) reflexive if \(R(x, x) = 1\) for all \(x \in U\),

(R2) symmetric if \(R(x, y) = R(y, x)\), for all \(x, y \in U\),

(R3) transitive if \(R(x, y) \odot R(y, z) \leq R(x, z)\), for all \(x, y, z \in U\).

If \(R\) satisfies (R1) and (R2), \(R\) is a quasi-equivalence relation. If a quasi-equivalence relation satisfies (R2), \(R\) is an equivalence relation.
Definition 2.4 [5] Let $U$ and $V$ be two sets. Let $\omega^-, \phi^-, \xi^- : L^U \to L^V$ and $\omega^-, \phi^-, \xi^- : L^V \to L^U$ be operators.

1. The pair $(\phi^-, \phi^-)$ is called an isotone Galois connection between $U$ and $V$ if for $A \in L^U$ and $B \in L^V$, $\phi^-(A) \leq B$ if $A \leq \phi^-(B)$. Moreover, the pair $(\xi^-, \xi^-)$ is called an isotone Galois connection between $U$ and $V$ if for $A \in L^U$ and $B \in L^V$, $\xi^-(B) \leq A$ if $B \leq \xi^-(A)$.

2. The pair $(\omega^-, \omega^-)$ is called antitone Galois connection between $U$ and $V$ if for $A \in L^U$ and $B \in L^V$, $B \leq \omega^-(A)$ if $A \leq \omega^-(B)$.

Theorem 2.5 [7] Let $\phi^- : L^U \to L^V$ and $\phi^- : L^V \to L^U$ be operators. Let $(\phi^-, \phi^-)$ be an isotone Galois connection between $U$ and $V$. For each $A, A_i \in L^U$ and $B, B_j \in L^V$, the following properties hold:

1. $A \leq \phi^-(\phi^-(A))$ and $\phi^-(\phi^-(B)) \leq B$.

2. If $A_1 \leq A_2$, then $\phi^-(A_1) \leq \phi^-(A_2)$. Moreover, if $B_1 \leq B_2$, then $\phi^-(B_1) \leq \phi^-(B_2)$.

3. $\phi^-(\phi^-(\phi^-(B))) = \phi^-(B)$ and $\phi^-(\phi^-(\phi^-(A))) = \phi^-(A)$.

4. $\phi^-(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} \phi^-(A_i)$ and $\phi^-(\bigwedge_{j \in J} B_j) = \bigwedge_{j \in J} \phi^-(B_j)$.

Theorem 2.6 [7] Let $\omega^- : L^U \to L^V$ and $\omega^- : L^V \to L^U$ be operators. Let $(\omega^-, \omega^-)$ be an antitone Galois connection between $U$ and $V$. For each $A, A_i \in L^U$ and $B, B_j \in L^V$, the following properties hold:

1. $A \leq \omega^-(\omega^-(A))$ and $B \leq \omega^-(\omega^-(B))$.

2. If $A_1 \leq A_2$, then $\omega^-(A_1) \geq \omega^-(A_2)$. Moreover, if $B_1 \leq B_2$, then $\omega^-(B_1) \geq \omega^-(B_2)$.

3. $\omega^-(\omega^-(\omega^-(B))) = \omega^-(B)$ and $\omega^-(\omega^-(\omega^-(A))) = \omega^-(A)$.

4. $\omega^-(\bigvee_{i \in I} A_i) = \bigwedge_{i \in I} \omega^-(A_i)$ and $\omega^-(\bigwedge_{j \in J} B_j) = \bigvee_{j \in J} \omega^-(B_j)$.

3 Galios connections and operators

Put $[A, B] = \bigwedge_{x \in U}(A(x) \to B(x))$ for all $A, B \in L^U$.

Definition 3.1 (1) A map $G : L^U \to L^V$ is an isotone map if for all $A, B \in L^U$, $[A, B] \leq [G(A), G(B)]$.

(2) A map $G : L^U \to L^V$ is an antitone map if for all $A, B \in L^U$, $[A, B] \leq [G(B), G(A)]$.

Definition 3.2 (2) A map $C : L^U \to L^U$ is called an implicative closure operator if it satisfies the following conditions:

1. $A \leq C(A)$, for all $A \in L^U$.

2. $C(C(A)) = C(A)$, for all $A \in L^U$.

3. $C$ is an isotone map.
A map \( I : L^U \to L^U \) is called an implicative open operator if it satisfies the following conditions:

(I1) \( I(A) \leq A \), for all \( A \in L^U \).

(I2) \( I(I(A)) = I(A) \), for all \( A \in L^U \).

(I3) \( I \) is an isotone map.

**Theorem 3.3** Let \( G : L^U \to L^V \) and \( H : L^V \to L^U \) be two maps.

(1) A pair \((G, H)\) is an isotone Galois connection with two isotone maps \( G \) and \( H \) iff for all \( A \in L^U \) and \( B \in L^V \), \([G(A), B] = [A, H(B)]\).

(2) A pair \((G, H)\) of two antitone maps \( G : L^U \to L^V \) and \( H : L^V \to L^U \) is an antitone Galois connection with two antitone maps \( G \) and \( H \) iff for all \( A \in L^U \) and \( B \in L^V \), \([A, H(B)] = [B, G(A)]\).

**Proof.** (1) Let \((G, H)\) be an isotone Galois connection. Then \( A \leq H(G(A)) \) and \( G(H(B)) \leq B \). Hence

\[
[G(A), B] \leq [H(G(A)), H(B)] \leq [A, H(B)]
\]

\[
[A, H(B)] \leq [G(A), G(H(B))] \leq [G(A), B]
\]

Conversely, since \([G(A), B] = [A, H(B)]\) for all \( A \in L^U \) and \( B \in L^V \), put \( B = G(A) \), then \( 1 = [G(A), G(A)] = [A, H(G(A))] \). So, \( A \leq H(G(A)) \). Put \( A = H(B) \), we similarly obtain \( G(H(B)) \leq B \).

\[
[G(A), G(B)] = [A, H(G(B))] \geq [A, B]
\]

\[
[H(A), H(B)] = [G(H(A)), B] \geq [A, B]
\]

(2) Let \((G, H)\) be an antitone Galois connection. Then \( A \leq H(G(A)) \) and \( B \leq G(H(B)) \). Hence

\[
[B, G(A)] \leq [H(G(A)), H(B)] \leq [A, H(B)]
\]

\[
[A, H(B)] \leq [G(H(B)), G(A)] \leq [B, G(A)]
\]

Conversely,

\[
[G(A), G(B)] = [B, H(G(A))] \geq [B, A]
\]

\[
[H(A), H(B)] = [B, G(H(A))] \geq [B, A]
\]

**Theorem 3.4** Let \( G : L^U \to L^V \) and \( H : L^V \to L^U \) be isotone maps with an isotone Galois connection \((G, H)\). Then the following statements hold:

(1) \( H \circ G \) is an implicative closure operator.

(2) \( G \circ H \) is an implicative open operator.
Proof. (1) By Theorem 2.5, \( A \leq H(G(A)) \) and \( H(G(A)) = H(G(H(G(A)))) \) for all \( A \in L^U \). Since \( G \) and \( H \) is an isotone maps, \( [A, B] \leq [H(G(A)), H(G(B))] \).

(2) It is similarly proved as in (1)

Corollary 3.5 Let \( G : L^U \rightarrow L^V \) and \( H : L^V \rightarrow L^U \) be antitone maps with an antitone Galois connection \((G, H)\). Then \( H \circ G \) and \( G \circ H \) are implicative closure operators.

Definition 3.6 For each \( A \in L^U \) and \( B \in L^V \) and \( R \in L^{U \times V} \), we define:

(1) \( \phi_R, \xi_R, \omega_R, \theta_R, \tau_R : L^U \rightarrow L^V \) is defined as:
\[
\phi_R(A)(y) = \bigvee_{x \in U} (R(x, y) \odot A(x)), \quad \xi_R(A)(y) = \bigwedge_{x \in U} (R(x, y) \rightarrow A(x))
\]
\[
\omega_R(A)(y) = \bigwedge_{x \in U} (A(x) \rightarrow R(x, y)), \quad \theta_R(A)(y) = \bigvee_{x \in U} (A(x) \oplus R(x, y))
\]
\[
\tau_R(A)(y) = (\omega_R)^*(A)(y).
\]

(2) \( \phi_R, \xi_R, \omega_R, \theta_R, \tau_R : L^V \rightarrow L^U \) is defined as:
\[
\phi_R(B)(x) = \bigwedge_{y \in V} (R(x, y) \rightarrow B(y)), \quad \xi_R(B)(x) = \bigvee_{y \in V} (R(x, y) \odot B(y))
\]
\[
\omega_R(B)(x) = \bigvee_{y \in V} (B(y) \rightarrow R(x, y)), \quad \theta_R(B)(x) = \bigwedge_{y \in V} (B(y) \oplus R(x, y))
\]
\[
\tau_R(B)(x) = (\omega_R)^*(B)(x).
\]

Theorem 3.7 In above definition, we obtain the following properties.

(1) \( \phi_R(B) = \theta_R(B) = \omega_R(B^*) \), for all \( B \in L^V \).

(2) \( \xi_R(A) = \theta_R^B(A) = \omega_R^B(A^*) \), for all \( A \in L^U \).

(3) \( (\xi_R(B))^* = \omega_R(B) = \theta_R(B^*) = \phi_R(B^*) \), for all \( B \in L^V \).

(4) \( (\phi_R(A))^* = \omega_R(A^*) = \xi_R^B(A^*) \), for all \( A \in L^U \).

(5) \( \theta_R(B) = \phi_R^B(B) = \omega_R^B(B^*) \), for all \( B \in L^V \).

(6) \( \theta_R(A) = \xi_R(A) = \omega_R^A(A^*) \), for all \( A \in L^U \).

(7) \( \theta_R(B) = (\phi_R(B^*))^* = \xi_R(B) \), for all \( B \in L^V \).

(8) \( \theta_R(A) = \phi_R^A(A) = (\xi_R(A^*))^* \), for all \( A \in L^U \).

Proof. (1)
\[
\phi_R(B)(x) = \bigwedge_{y \in V} (R(x, y) \rightarrow B(y))
\]
\[
= \bigwedge_{y \in V} (R^*(x, y) \oplus B(y)) = \theta_R^B(B)(x)
\]
\[
= \bigwedge_{y \in V} (B(y)^* \rightarrow R^*(x, y)) = \omega_R^B(B^*)(x)
\]
\[ (\xi_R^-(B))(x) = \bigwedge_{y \in V} (R(x, y) \odot B(y))^* = \bigwedge_{y \in V} (R^*(x, y) \oplus B^*(y)) = \theta^-_R(B^*)(x) = \bigwedge_{y \in V} (B(y) \rightarrow R^*(x, y)) = \omega^-_R(B)(x) = \bigwedge_{y \in V} (R(x, y) \rightarrow B^*(y)) = \phi^-_R(B^*)(x) \]

Other cases are similarly proved.

**Theorem 3.8**

(1) $\phi^-_R$ and $\phi^-_R$ are isotone maps with an isotone Galois connection $(\phi^-_R, \phi^-_R)$.

(2) $\phi^-_R \circ \phi^-_R$ is an implicative closure operator.

(3) $\phi^-_R \circ \phi^-_R$ is an implicative open operator.

**Proof.** (1) By Theorem 3.3(1), we only show the following statement.

\[ [\phi^-_R(A), B] = \bigwedge_{x \in U} (\phi^-_R(A)(x) \rightarrow B(x)) = \bigwedge_{x \in U} \left( \bigvee_{y \in V} (R(x, y) \odot A(y)) \rightarrow B(x) \right) = \bigwedge_{x \in U} \bigwedge_{y \in V} \left( (R(x, y) \odot A(y)) \rightarrow B(x) \right) = \bigwedge_{x \in U} \bigwedge_{y \in V} \left( A(y) \rightarrow (R(x, y) \rightarrow B(x)) \right) = \bigwedge_{y \in V} \left( A(y) \rightarrow \phi^-_R(B)(y) \right) = [A, \phi^-_R(B)] \]

(2) and (3) follow from Theorem 3.4.

**Theorem 3.9** Let $\phi^-_R, \phi^-_R: L^U \rightarrow L^V$ be maps in above theorem with $U = V$ and $R$ an quasi-equivalence relation. Then the following statements hold.

(1) $\phi^-_R$ is an implicative open operator.

(2) $\phi^-_R$ is an implicative closure operator.

(3) $\phi^-_R \circ \phi^-_R = \phi^-_R$.

(4) $\phi^-_R \circ \phi^-_R = \phi^-_R$.

**Proof.** (1) Since $R$ is an quasi-equivalence relation, $R(x, x) \rightarrow A(x) = A(x)$ and $R \circ R = R$. Hence $\phi^-_R(A) \leq A$. 
\[ \phi_R^{-1}(\phi_R^{-1}(A))(x) = \bigwedge_{y \in U} (R(x, y) \rightarrow \phi_R^{-1}(A)(y)) \]
\[ = \bigwedge_{y \in U} (R(x, y) \rightarrow \bigwedge_{z \in U} (R(y, z) \rightarrow A(z))) \]
\[ = \bigwedge_{y \in U} \big( (R(x, y) \odot R(y, z)) \rightarrow A(z) \big) \]
\[ = \bigwedge_{x \in U} \big( (R \circ R)(x, z) \rightarrow A(z) \big) \]
\[ = \bigwedge_{x \in U} \big( R(x, z) \rightarrow A(z) \big) = \phi_R^{-1}(A)(x). \]

By Theorem 3.8, \( \phi_R^{-1} \) is an isotone map.

(2) Since \( R \) is an quasi-equivalence relation, \( R(x, x) \odot A(x) = A(x) \) and \( R \circ R = R \). Hence \( \phi_R^{-1}(A) \geq A \).

\[ \phi_R^{-1}(\phi_R^{-1}(A))(y) = \bigvee_{x \in U} (\phi_R^{-1}(A)(x) \odot R(x, y)) \]
\[ = \bigvee_{x \in U} \big( \bigvee_{z \in U} (A(z) \odot R(z, x)) \odot R(x, y) \big) \]
\[ = \bigvee_{x \in U} \big( A(z) \odot \bigvee_{x \in U} (R(z, x) \odot R(x, y)) \big) \]
\[ = \bigvee_{x \in U} \big( A(z) \odot (R \circ R)(z, y) \big) \]
\[ = \bigvee_{x \in U} \big( A(z) \odot R(z, y) \big) = \phi_R^{-1}(y). \]

(3) By (1), since \( \phi_R^{-1} \geq \phi_R^{-1} \circ \phi_R^{-1} \), we only show \( \phi_R^{-1} \leq \phi_R^{-1} \circ \phi_R^{-1} \) from:
\[ 1 = [\phi_R^{-1}(A), \phi_R^{-1}(A)] = [\phi_R^{-1}(\phi_R^{-1}(A)), \phi_R^{-1}(A)] = [\phi_R^{-1}(A), \phi_R^{-1}(\phi_R^{-1}(A))]. \]

(4) By (2), since \( \phi_R^{-1} \circ \phi_R^{-1} \geq \phi_R^{-1} \), we only show \( \phi_R^{-1} \circ \phi_R^{-1} \leq \phi_R^{-1} \) from:
\[ 1 = [\phi_R^{-1}(B), \phi_R^{-1}(B)] = [\phi_R^{-1}(B), \phi_R^{-1}(\phi_R^{-1}(B))] = [\phi_R^{-1}(\phi_R^{-1}(B)), \phi_R^{-1}(B)]. \]

**Theorem 3.10**

(1) \( \omega_R^{-1} \) and \( \omega_R^{-1} \) are antitone maps with an antitone Galois connection \( \omega_R^{-1}, \omega_R^{-1} \).

(2) \( \omega_R^{-1} \circ \omega_R^{-1} \) and \( \omega_R^{-1} \circ \omega_R^{-1} \) are implicative closure operators.

(3) \( \omega_R^{-1} \circ \omega_R^{-1} = \xi_R^{-1} \circ \xi_R^{-1} \) and \( \omega_R^{-1} \circ \omega_R^{-1} = \phi_R^{-1} \circ \phi_R^{-1} \).

**Proof.** (1) By Theorem 3.3(2), we only show the following statement.
\[ [B, \omega_R^{-1}(A)] = \bigwedge_{x \in U} (B(x) \rightarrow \omega_R^{-1}(A)(x)) \]
\[ = \bigwedge_{x \in U} (B(x) \rightarrow \bigwedge_{y \in W} (A(y) \rightarrow R(x, y))) \]
\[ = \bigwedge_{y \in W} (A(y) \rightarrow \bigwedge_{z \in U} (B(x) \rightarrow R(x, y))) \]
\[ = \bigwedge_{y \in W} (A(y) \rightarrow \omega_R^{-1}(B)(y)) = [A, \omega_R^{-1}(B)]. \]

(3) For each \( A \in L^U \) and \( x \in U \),
\[ \omega_R^{-1}(\omega_R^{-1})(A)(x) = \bigwedge_{y \in U} (\omega_R^{-1}(A)(y) \rightarrow R(x, y)) \]
\[ = \bigwedge_{y \in U} \big( \bigwedge_{z \in U} (A(z) \rightarrow R(z, y)) \rightarrow R(x, y) \big) \]
\[ = \bigwedge_{y \in U} \big( R(x, y)^* \rightarrow \bigwedge_{z \in U} (A(z) \rightarrow R(z, y)^*) \big) \]
\[ = \bigwedge_{y \in U} \big( R(x, y)^* \rightarrow \phi_R^{-1}(A)(y) \big) \]
\[ = \phi_R^{-1}(\phi_R^{-1}(A))(x). \]
Other case and (2) are similarly proved from Corollary 3.5.

**Theorem 3.11** (1) $\tau_R^-$ and $\theta_R^-$ are isotone maps with an isotone Galois connection $(\tau_R^-, \theta_R^-)$.
(2) $\theta_R^- \circ \tau_R^-$ is an implicative closure operator.
(3) $\tau_R^- \circ \theta_R^-$ is an implicative open operator.
(4) $\theta_R^- \circ \tau_R^- = \omega_R^- \circ \omega_R^-$.
(5) $\tau_R^- \circ \theta_R^- = \phi_R^- \circ \phi_R^-$. 

**Proof.** (1) By Theorem 3.3(1), we only show the following statement.

$$[\tau_R^-(A), B] = \bigwedge_{y \in V} (\tau_R^-(A)(y) \to B(y))$$

$$= \bigwedge_{y \in V} ((\omega_R^-)^*(A)(y) \to B(y))$$

$$= \bigwedge_{y \in V} (B^*(y) \to \omega_R^-(A)(y))$$

$$= \bigwedge_{y \in V} (B^*(y) \to \bigwedge_{x \in U}(A(x) \to R(x, y)))$$

$$= \bigwedge_{y \in V} (\bigwedge_{x \in U}(B^*(y) \to (R^*(x, y) \to A^*(x))))$$

$$= \bigwedge_{y \in V} (\bigwedge_{x \in U}(B^* \circ R^*(x, y) \to A^*(x)))$$

$$= \bigwedge_{x \in U}(A(x) \to \bigwedge_{y \in V}(B^*(y) \circ R^*(x, y)))$$

$$= \bigwedge_{x \in U}(A(x) \to \bigwedge_{y \in V}(B(y) \circ R(x, y)))$$

$$= \bigwedge_{x \in U}(A(x) \to \theta_R^-(B)(x)) = [A, \theta_R^-(B)]$$

(2) and (3) follow from Theorem 3.4.

(4)

$$\theta_R^- (\tau_R^-(A))(x) = \bigwedge_{y \in U} (\tau_R^-(A)(y) \oplus R(x, y))$$

$$= \bigwedge_{y \in U} ((\omega_R^-)^*(A)(y) \oplus R(x, y))$$

$$= \bigwedge_{y \in U} (\omega_R^-(A)(y) \to R(x, y)))$$

$$= \omega_R^- (\omega_R^-(A))(x)$$

(5)

$$\tau_R^- (\theta_R^-(B))(y) = \left( \bigwedge_{x \in U}(\theta_R^-(B)(x) \to R(x, y)) \right)^*$$

$$= \bigvee_{x \in U}(\theta_R^-(B)^*(x) \oplus R(x, y))^*$$

$$= \bigvee_{x \in U}(\theta_R^-(B)(x) \circ R^*(x, y))$$

$$= \phi_R^-(\theta_R^-(B))(y) = \phi_R^-(\phi_R^-(B))(y)$$

**Theorem 3.12** Let $\theta_R^-, \tau_R^- : L^V \to L^U$ be maps in above theorem with $V = U$ and $R^*$ is a quasi-equivalence relation. Then the following statements hold.

(1) $\theta_R^-$ is an implicative open operator.
(2) $\tau_R^-$ is an implicative closure operator.
(3) $\theta_R^- \circ \tau_R^- = \tau_R^-$. 
(4) $\tau_R^- \circ \theta_R^- = \theta_R^-$. 

**Proof.** (1) Since $R^*$ is a quasi-equivalence relation, $A(x) \oplus R(x, x) = A(x)$, $\theta_R^{-1}(A) \leq A$ and $R \oplus R = R$.

\[
\theta_R^{-1}(\theta_R^{-1}(A))(x) = \bigwedge_{y \in U}(\theta_R^{-1}(A)(y) \oplus R(x, y)) \\
= \bigwedge_{y \in U} \big( \bigwedge_{z \in U}(A(z) \oplus R(y, z)) \oplus R(x, y) \big) \\
= \bigwedge_{z \in U} \big( A(z) \oplus \bigwedge_{y \in U}(R(y, z) \oplus R(x, y)) \big) \\
= \bigwedge_{z \in U} \big( A(z) \oplus (R \oplus R)(x, z) \big) \\
= \theta_R^{-1}(A)(x).
\]

(2)

\[
(\tau_R^{-1}(\tau_R^{-1}(A)))(x) = \bigwedge_{x \in U}(\tau_R^{-1}(x) \rightarrow R(x, y)) \\
= \bigwedge_{x \in U} \big( R^*(x, y) \rightarrow \bigwedge_{z \in U}(A(z) \rightarrow R(z, x)) \big) \\
= \bigwedge_{x \in U} \bigwedge_{z \in U} \big( A(z) \rightarrow (R^*(x, y) \rightarrow R(z, x)) \big) \\
= \bigwedge_{z \in U} \big( A(z) \rightarrow \bigwedge_{x \in U}(R(x, y) \oplus R(z, x)) \big) \\
= \bigwedge_{z \in U} \big( A(z) \rightarrow (R \oplus R)(z, y) \big) \\
= \bigwedge_{z \in U} \big( A(z) \rightarrow R(z, y) \big) = \omega_R^{-1}(A)(x).
\]

(3) and (4) are similarly proved as in Theorem 3.9.

**References**


[7] Y.C. Kim, J.W. Park, Join preserving maps and various concepts, (submit to)


Received: October, 2009