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# Lightlike Submanifolds of Indefinite Kenmotsu Manifolds

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**Abstract.** In this paper we introduce the notion of lightlike submanifolds of an indefinite Kenmotsu manifold. We have studied the invariant, contact CR, contact screen Cauchy-Riemann (contact SCR) lightlike submanifolds of an indefinite Kenmotsu manifold. We give the condition under which lightlike submanifold of an indefinite Kenmotsu manifold is minimal. We have also studied totally contact umbilical lightlike submanifolds. Examples of lightlike submanifold of an indefinite Kenmotsu manifold have also been given.

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## Introduction

In the theory of submanifolds of semi-Riemannian manifolds it is interesting to study the geometry of lightlike submanifolds due to the fact that the intersection of normal vector bundle and the tangent bundle is non-trivial making it more interesting and remarkably different from the study of non-degenerate submanifolds. The geometry of lightlike submanifolds of indefinite Kaehler manifolds was studied by Duggal and Bejancu [6] and a general notion of lightlike submanifolds of indefinite Sasakian manifolds was introduced by Duggal and Sahin [8]. However, a general notion of lightlike submanifolds of indefinite Kenmotsu manifolds has not been introduced as yet.

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In section 1, we have collected the formulae and information which are useful in subsequent sections. In section 2, we have studied the invariant lightlike submanifolds of indefinite Kenmotsu manifold. In Section 3, we introduced the notion of contact CR- lightlike submanifolds of indefinite Kenmotsu manifold and have given an example of contact CR-lightlike submanifold of $R_2^9$. In section 4, we have studied contact Screen Cauchy-Riemann lightlike submanifolds of indefinite Kenmotsu manifold and have given an example of contact SCR lightlike submanifold in $R_2^9$. In section 5, we have studied the minimal lightlike submanifolds of invariant, contact CR, contact SCR lightlike submanifold of indefinite Kenmotsu manifold and have given an example of minimal lightlike submanifold in $R_4^{11}$.

## 1. Preliminaries

An odd-dimensional semi-Riemannian manifold $\overline{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors ($\phi$, $V$, $\eta$, $\bar{g}$), where $\phi$ is a (1, 1) tensor field, $V$ a vector field, $\eta$ a 1-form and $\bar{g}$ is the semi-Riemannian metric on $\overline{M}$ satisfying

$$\phi^2 X = -X + \eta(X)\, V, \qquad \eta \circ \phi = 0, \qquad \phi V = 0 \quad \eta(V) = 1$$
$$\bar{g}\,(\phi X, \phi Y) = \bar{g}\,(X, Y) - \eta(X)\,\eta(Y), \qquad \bar{g}\,(X, V) = \eta(X) \tag{1.1}$$

for any $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$.
An indefinite almost contact metric manifold $\overline{M}$ is called an indefinite Kenmotsu manifold if [5],

$$(\overline{\nabla}_X \phi)Y = -\bar{g}\,(\phi X, Y)V + \eta(Y)\phi X, \quad \text{and} \quad \overline{\nabla}_X V = -X + \eta(X)\, V \tag{1.2}$$

for any $X, Y \in T\overline{M}$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$.
A submanifold $M^m$ immersed in a semi-Riemannian manifold $(\overline{M}^{m+k}, \bar{g})$ is called a lightlike submanifold if it admits a degenerate metric $g$ induced from $\bar{g}$ whose radical distribution Rad($TM$) is of rank $r$, where $1 \le r \le m$. Now, Rad($TM$) $= TM \cap TM^{\perp}$, where

$$TM^{\perp} = \bigcup_{x \in M} \left\{ u \in T_x\overline{M} : \bar{g}(u, v) = 0, \forall v \in T_x M \right\} \tag{1.3}$$

Let $S(TM)$ be *a screen distribution* which is a semi-Riemannian complementary distribution of Rad($TM$) in $TM$, that is, $TM = \text{Rad}(TM) \perp S(TM)$.
We consider *a screen transversal vector bundle* $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of Rad($TM$) in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of Rad($TM$), there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\bar{g}\,(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}\,(N_i, N_j) = 0$, it follows that there exists a *lightlike transversal vector bundle* $ltr(TM)$ locally spanned by $\{N_i\}$ $\left(cf.[6], page 144\right)$. Let tr($TM$) be complementary (but not orthogonal) vector bundle to $TM$ in $T\overline{M}|_M$. Then

$$\left.\begin{aligned} &tr(TM) = l\,tr(TM) \perp S(TM^{\perp}) \\ &T\overline{M}\big|_M = S(TM) \perp \left[Rad(TM) \oplus l\,tr(TM)\right] \perp S(TM^{\perp}) \end{aligned}\right\} \tag{1.4}$$

A submanifold ($M$, $g$, $S(TM)$, $S(TM^{\perp})$) of $\overline{M}$ is said to be

(1) $r$-lightlike if $r < \min\{m, k\}$;

(2) Coisotropic if $r = k < m$, $S(TM^{\perp}) = \{0\}$;

(3) Isotropic if $r = m < k$, $S(TM) = \{0\}$;

(4) Totally lightlike if $r = m = k$, $S(TM) = \{0\} = S(TM^{\perp})$.

Let $\overline{\nabla}$, $\nabla$ and $\nabla^t$ denote the linear connections on $\overline{M}$, $M$ and vector bundle tr($TM$), respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall X, Y \in \Gamma(TM), \tag{1.5}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U, \quad \forall U \in \Gamma(tr(TM)), \tag{1.6}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$, respectively and $A_U$ is the shape operator of $M$ with respect to $U$. Moreover, according to decomposition (1.4), $h^l$, $h^s$ are $\Gamma(ltr(TM)-$valued and $\Gamma(S(TM^{\perp})-$valued *lightlike second fundamental form* and *screen second fundamental form* of $M$ respectively, then

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \quad \forall X, Y \in \Gamma(TM), \tag{1.7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X,N), \quad N \in \Gamma(l\,tr(TM)), \tag{1.8}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X,W), \quad W \in \Gamma(S(TM^{\perp})), \tag{1.9}$$

where $D^l(X,W)$, $D^s(X,N)$ are the projections of $\nabla^t$ on $\Gamma(ltr(TM)$ and $\Gamma(S(TM^{\perp})$, respectively and $\nabla^l$, $\nabla^s$ are linear connections on $\Gamma(ltr(TM)$ and $\Gamma(S(TM^{\perp})$, respectively. We call $\nabla^l$, $\nabla^s$ the lightlike and screen transversal connections on $M$, and $A_N$, $A_W$ are shape operators on $M$ with respect to $N$ and $W$, respectively.

Using (1.5) and (1.7)~(1.9), we obtain

$$\overline{g}(h^s(X,Y),W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{1.10}$$

$$\overline{g}(D^s(X,N),W) = \overline{g}(N, A_W X) \tag{1.11}$$

Let $\overline{P}$ denote the projection of $TM$ on $S(TM)$ and $\nabla^*$, $\nabla^{*t}$ denote the linear connections on $S(TM)$ and $Rad(TM)$, respectively. Then from the decomposition of tangent bundle of lightlike submanifold, we have

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \tag{1.12}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi \ , \tag{1.13}$$

for $X$, $Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}TM)$, where $h^*$, $A^*$ are the second fundamental forms of distributions $S(TM)$ and $Rad(TM)$.

From (1.12) and (1.13), we get

$$\bar{g}(h^l(X, \bar{P}Y), \xi) = g(A^*_\xi X, \bar{P}Y), \tag{1.14}$$

$$g(h^*(X, \bar{P}Y), N) = g(A_N X, \bar{P}Y), \tag{1.15}$$

$$\bar{g}(h^l(X, \xi), \xi) = 0, \qquad A^*_\xi \xi = 0 \tag{1.16}$$

In general the induced connection $\nabla$ on $M$ is not a metric connection. Since $\bar{\nabla}$ is a metric connection, by using (1.7), we obtain

$$(\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y) \tag{1.17}$$

However, it is important to note that $\nabla^*$, $\nabla^{*t}$ are metric connections on $S(TM)$ and $Rad(TM)$, respectively.

A plane section $\prod$ in $T_x\bar{M}$ of a Kenmotsu manifold $\bar{M}$ is called a $\phi$-section if it is spanned by a unit vector $X$ orthogonal to $V$ and $\phi X$, where $X$ is a non-null vector field on $\bar{M}$. The sectional curvature $K(\prod)$ with respect to $\prod$ determined by $X$ is called a $\phi$-sectional curvature. If $\bar{M}$ has a $\phi$-sectional curvature $c$ which does not depend on the $\phi$-section at each point, then $c$ is constant in $\bar{M}$. Then, $\bar{M}$ is called an indefinite Kenmotsu space form and is denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by [5]

$$\begin{aligned}\bar{R}(X,Y)Z = {} & \frac{(c-3)}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ & + \frac{(c+1)}{4}\left\{\begin{array}{l}\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \bar{g}(X,Z)\eta(Y)V \\ -\bar{g}(Y,Z)\eta(X)V + \bar{g}(\phi Y,Z)\phi X + \bar{g}(\phi Z,X)\phi Y - 2\bar{g}(\phi X,Y)\phi Z\end{array}\right\}\end{aligned} \tag{1.18}$$

for any $X$, $Y$, and $Z$ vector fields on $\bar{M}$.

**Definition 1.1.** [9]. A lightlike submanifold ($M$, $g$) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is totally umbilical in $M$ if there is a smooth transversal vector field $\mathrm{H} \in \Gamma(\mathrm{tr}(TM))$ on $M$, called the transversal curvature vector field of $M$, such that for all $X, Y \in \Gamma(TM)$,

$$h(X, Y) = Hg(X, Y) \tag{1.19}$$

Using (1.7) and (1.19), it is easy to see that $M$ is totally umbilical if and only if on each coordinate neighborhood $\mathscr{U}$, there exist smooth vector fields $\mathscr{H}^l \in \Gamma(l\mathrm{tr}(TM))$ and $\mathscr{H}^s \in \Gamma(S(TM^\perp))$ such that

$$\left.\begin{array}{ll} h^l(X,Y)=\mathscr{H}^l\ g(X,Y), & D^l(X,W)=0 \\ h^s(X,Y)=\mathscr{H}^s g(X,Y), & \forall X,Y\in\Gamma(TM),\quad W\in\Gamma(S(TM^{\perp})) \end{array}\right\} \quad (1.20)$$

Similar to the concept of contact totally umbilical submanifold of Sasakian manifold introduced in the book of Yano and Kon [*cf.* [10], page 374], we define:

**Definition 1.2.** If the second fundamental form $h$ of a submanifold $M$, tangent to the structure vector field $V$, of an indefinite Kenmotsu manifold $\overline{M}$ **,** is of the form

$$h(X,Y)=\left[g(X,Y)-\eta(X)\eta(Y)\right]\alpha \quad (1.21)$$

for any $X,Y \in \Gamma(TM)$, where $\alpha$ is a vector field transversal to $M$, then $M$ is called contact totally umbilical and totally geodesic if $\alpha = 0$.

The above definition also holds for a lightlike submanifold $M$. For a contact totally umbilical submanifold $M$, we have

$$\left.\begin{array}{l} h^l(X,Y)=\left[g(X,Y)-\eta(X)\eta(Y)\right]\alpha_L, \\ h^s(X,Y)=\left[g(X,Y)-\eta(X)\eta(Y)\right]\alpha_S \end{array}\right\} \quad (1.22)$$

where $\alpha_S \in\Gamma(S(TM^{\perp}))$ and $\alpha_L \in\Gamma(ltr(TM))$.

A general notion of minimal lightlike submanifold $M$, introduced by Bejancu and Duggal [3], is as follows:

**Definition 1.3.** A lightlike submanifold ($M$, $g$, $S(TM)$) isometrically immersed in a semi-Riemannian manifold $(\overline{M},\overline{g})$ is minimal if

(i) $h^s = 0$ on Rad($TM$);

(ii) trace$h$ = 0, where trace is written with respect to g restricted to $S(TM)$.

**Definition 1.4.** [4]. A lightlike submanifold $M$ of a semi-Riemannian manifold is said to be an *irrotational submanifold* if $\overline{\nabla}_X\xi \in \Gamma(TM)$, for all $X \in \Gamma(TM)$ and $\xi \in\Gamma(\mathrm{Rad}TM)$.
From (1.7), we conclude that $M$ is an irrotational lightlike submanifold if and only if the following are satisfied:

$$h^s(X,\xi)=0, \qquad h^l(X,\xi)=0 \quad (1.23)$$

**Definition 1.5.** A lightlike submanifold $M$, of an indefinite Kenmotsu manifold

$\overline{M}$ , is screen real submanifold if Rad($TM$) and $S(TM)$ are, respectively, invariant and anti-invariant with respect to $\phi$.

The above definition is the lightlike version (*cf.* [7]) of totally real submanifolds of an almost Hermitian (or contact) manifold [10].

The following result is important for our work.

**Proposition 1.1** [6]. *The lightlike second fundamental forms of a lightlike submanifold M do not depend on* $S(TM)$, $S(TM^{\perp})$, and $ltr(TM)$.

## 2. Invariant submanifolds

Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$. For any vector field $X$ tangent to $M$, we write

$$\phi X = PX + FX \tag{2.1}$$

where $PX$ and $FX$ are the tangential and transversal parts of $\phi X$, respectively. It is known that if $M$ is tangent to the structure vector field $V$, then $V$ belongs to $S(TM)$. Using this fact, we say that $M$ is invariant in $\overline{M}$ if $M$ is tangent to the structure vector field $V$ and

$$\phi X = PX, \quad \text{that is}, \phi X \in \Gamma(TM), \forall X \in \Gamma(TM) \tag{2.2}$$

From (1.2) and (1.7), we have

$$h^l(X,V) = 0, \quad h^s(X,V) = 0, \ \nabla_X V = -X + \eta(X)V, \tag{2.3}$$

Moreover, for invariant submanifolds, using (1.2), (1.7) and (2.2), we get

$$h(X, \phi Y) = \phi h(X,Y) = h(\phi X, Y), \forall X, Y \in \Gamma(TM) \tag{2.4}$$

Now, we have

**Proposition 2.1.** *Let* $(M, g, S(TM), S(TM^{\perp}))$ *be a lightlike submanifold of an indefinite Kenmotsu manifold* $\overline{M}$. *If the second fundamental forms* $h^l$ *and* $h^s$ *are parallel, then M is totally geodesic.*

Proof: Let us assume that $h^l$ is parallel. Then we have

$$(\nabla^t_X h^l)(Y,V) = \nabla_X h^l(Y,V) - h^l(\nabla_X Y, V) - h^l(Y, \nabla_X V) = 0 \tag{2.5}$$

Thus using (2.3), we get $h^l(Y, X) = 0$. Similarly, we have $h^s(Y, X) = 0$, which completes the proof.

**Theorem 2.2.** *Let* $(M, g, S(TM), S(TM^{\perp}))$ *be an invariant lightlike submanifold of codimension two of an indefinite Kenmotsu manifold* $\overline{M}$. *Then,* Rad$TM$ *defines a totally geodesic foliation on M. Moreover,* $M = M_1 \times M_2$ *is a lightlike product manifold if and only if* $h^* = 0$, *where* $M_1$ *is a leaf of the radical distribution and* $M_2$ *is a semi-Riemannian manifold.*

Proof: Since rank (Rad$TM$) = 2, we can write $X, Y \in \Gamma(\text{Rad}TM)$ as a linear combination of $\xi$ and $\phi\xi$, that is, $X = A_1 \xi + B_1 \phi\xi$, $Y = A_2 \xi + B_2 \phi\xi$. Thus using (1.7), we find

$$g(\nabla_X Y, \overline{P}Z) = -A_2 A_1 \overline{g}(\xi, h^l(\xi, \overline{P}Z)) - A_2 B_1 \overline{g}(h^l(\phi\xi, \overline{P}Z), \xi)$$
$$- B_2 A_1 \overline{g}(\phi\xi, h^l(\xi, \overline{P}Z)) - B_2 B_1 \overline{g}(h^l(\phi\xi, \overline{P}Z), \phi\xi) \qquad (2.6)$$

Now, by using (1.16) and (2.4), we obtain $g(\nabla_X Y, \overline{P}Z) = 0$. This shows that Rad$TM$ defines a totally geodesic foliation. Then, the proof of the theorem follows from Theorem 2.6, page 162[6].

**Theorem 2.3.** *Let* $(M, g, S(TM), S(TM^{\perp}))$ *be an invariant lightlike submanifold of codimension two of an indefinite Kenmotsu manifold* $\overline{M}$. *Suppose* $(M', g)$ *is a nondegenerate submanifold of* $\overline{M}$ *such that* $M'$ *is a leaf of integrable* $S(TM)$. *Then M is totally geodesic, with an induced metric connection if* $M'$ *is being so immersed as a submanifold of* $\overline{M}$.

*Proof.* Since dim (Rad$TM$) = dim ($ltr(TM)$) = 2, $h^l(X,Y) = A_1 N + B_1 \phi N$, where $A_1$ and $B_1$ are functions on $M$. Thus $h^l(X, \xi) = 0$ if and only if $\overline{g}(h^l(X, \xi), \xi) = 0$ and $\overline{g}(h^l(X, \xi), \phi\xi) = 0$, for all $X \in \Gamma(TM)$, and $\xi \in \Gamma(\text{Rad}TM)$. From (1.16), we have $\overline{g}(h^l(X, \xi), \xi) = 0$. Using (2.4), we get $\overline{g}(h^l(X, \xi), \phi\xi) = -\overline{g}(h^l(\phi X, \xi), \xi) = 0$. Similarly, $h^l(X, \phi\xi) = 0$. On $M'$, we have

$$\overline{\nabla}_X Y = \nabla'_X Y + h'(X,Y), \qquad \forall X, Y \in \Gamma(TM'), \qquad (2.7)$$

where $\nabla'$ is a metric connection on $M'$ and $h'$ is the second fundamental form of $M'$. Thus $h'(X,Y) = h^*(X,Y) + h^l(X,Y)$, *for* all $X, Y \in \Gamma(TM')$. Also, $g(X,Y) = g'(X,Y)$, for all $X, Y \in \Gamma(TM)$, which completes the proof.

**Theorem 2.4.** *Let* $(M, g, S(TM), S(TM^{\perp}))$ *be a lightlike submanifold, tangent to structure vector field V, of an indefinite Kenmotsu manifold* $\overline{M}$. *If M is totally umbilical, then M is totally geodesic.*

Proof. Using (1.2), (1.7), (2.1), and considering the transversal parts, we get

$$h^s(X,V) + h^l(X,V) = 0, \qquad \forall X \in \Gamma(TM) \qquad (2.8)$$

Thus from (2.8), we have $h^l(X, V) = 0$ and $h^s(X, V) = 0$ which imply that $h^l(V,V) = 0$ and $h^s(V,V) = 0$. As $M$ is totally umbilical and $V$ is nonnull, using (1.20), we have $h^l = 0$ and $h^s = 0$. Hence $M$ is totally geodesic.

## 3. Contact CR-lightlike submanifolds

In this section, we state the following definition for a *contact CR-lightlike submanifold*:

**Definition 3.1.** Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold, tangent to structure vector field $V$, immersed in an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$.

We say that $M$ is a contact $CR$-lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(A) Rad$TM$ is a distribution on $M$ such that Rad$TM \cap \phi$(Rad$TM$) = {0};

(B) there exist vector bundles $D_0$ and $D'$ over $M$ such that

$$\left.\begin{aligned} &S(TM) = \{\phi(RadTM) \oplus D'\} \perp D_0 \perp \{V\}, \\ &\phi D_0 = D_0, \phi(D') = L_1 \perp ltr(TM) \end{aligned}\right\} \tag{3.1}$$

where $D_0$ is nondegenerate and $L_1$ is vector subbundle of $S(TM^{\perp})$.

Thus we have the following decomposition:

$$TM = D \oplus D' \perp \{V\}, \qquad D = RadTM \perp \phi(RadTM) \perp D_0 \tag{3.2}$$

A contact CR-lightlike submanifold is proper if $D_o \neq \{0\}$ and $L_1 \neq \{0\}$.

It is easy to see that any contact CR-lightlike three-dimensional submanifold is 1-lightlike.

Hereafter, $(R_q^{2m+1}, \phi_0, V, \eta, g)$ will denote the manifold $R_q^{2m+1}$ with its usual Kenmotsu structure given by

$$\left.\begin{aligned} &\eta = dz, \qquad V = \partial z, \\ &\bar{g} = \eta \otimes \eta + e^{2z}(-\sum_{i=1}^{q/2} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i), \\ &\phi_0(\sum_{i=1}^{m}(X_i \partial x^i + Y_i \partial y_i) + Z\partial z) = \sum_{i=1}^{m}(Y_i \partial x^i - X_i \partial y_i) \end{aligned}\right\} \tag{3.3}$$

where $(x^i, y^i, z)$ are the Cartesian coordinates.

**Example 3.2.** Let $\overline{M} = (R_2^9, \bar{g})$ be a semi-Euclidean space, where $g$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\} \tag{3.4}$$

Consider a submanifold $M$ of $R_2^9$, defined by

$$x^1 = y^4, \quad x^2 = \sqrt{1-(y^2)^2}, \qquad y^2 \neq \pm 1 \tag{3.5}$$

Then a local frame of $TM$ is given by

$$\left.\begin{aligned} &Z_1 = e^{-z}(\partial x_1 + \partial y_4), \qquad Z_2 = e^{-z}(\partial x_4 - \partial y_1), \\ &Z_3 = e^{-z}\partial x_3, \qquad Z_4 = e^{-z}\partial y_3, \qquad Z_5 = e^{-z}(-\frac{y^2}{x^2}\partial x_2 + \partial y_2), \\ &Z_6 = e^{-z}(\partial x_4 + \partial y_1), \qquad Z_7 = V = \partial z \end{aligned}\right\} \tag{3.6}$$

Hence, Rad$TM$ = span$\{Z_1\}$, $\phi_0$ Rad$TM$ = span$\{Z_2\}$, and Rad$TM \cap \phi_0$ Rad$TM$ =

$\{0\}$. Hence, (A) holds. Next, $\phi_0(Z_3) = -Z_4$ implies that $D_0 = \{Z_3, Z_4\}$ is invariant with respect to $\phi_0$. By direct calculations, we get

$$S(TM^{\perp}) = span\left\{W = e^{-z}(\partial x_2 + \frac{y^2}{x^2}\partial y_2)\right\} \quad such\ that \quad \phi_0(W) = -Z_5 \tag{3.7}$$

and $l$tr($TM$) is spanned by $N = = \frac{e^{-z}}{2}(-\partial x_1 + \partial y_4)$ such that $\phi_0(N) = \frac{1}{2} Z_6$. Hence, $M$ is a contact CR-lightlike submanifold.

**Proposition 3.3.** There exist no isotropic or totally lightlike contact CR-lightlike submanifolds in an indefinite Kenmotsu manifold.

Proof. If $M$ is isotropic or totally lightlike, then $S(TM) = \{0\}$. Hence, conditions (A) and (B) of Definition 3.1 are not satisfied.

**Proposition 3.4.** Let $M$ be a contact $CR$-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then,

(i) *D is integrable if and only if*
$B\,h^S(X,\phi Y) = B\,h^S(Y,\phi X), \ \ \forall X,Y \in \Gamma(D)$

(ii) and $D' \oplus D$ is integrable.

Proof. Suppose $D$ is integrable, then $g([X,Y],V) = 0$, $g([X,Y],W) = 0$, for $X,Y \in \Gamma(D)$ and $W \in \Gamma(D')$. Also from (1.7), we derive $g([X,Y],V) = \bar{g}(\bar{\nabla}_X Y,V) - \bar{g}(\bar{\nabla}_Y X,V)$. Then, using (1.2) and the fact that $\bar{\nabla}$ is a metric connection, we have

$g([X,Y],V) = -g(Y, -X + \eta(X)\ V) + g(-Y + \eta(Y)\ V,X)$. Hence, $g([X,Y],V) = 0$ implies that $D$ is integrable.

Similarly, we have

$$g([X,Y],W) = \bar{g}(\ h^S(X,\phi Y) - h^S(Y,\phi X),\phi W\ ),$$

Hence, $g([X,Y],W)=0$ if and only if $B\,h^S(X,\phi Y) = B\,h^S(Y,\phi X)$ , $\forall X,Y \in \Gamma(D)$.

It is easy to see that $D' \oplus D$ is also integrable.

Denote the orthogonal complement subbundle to the vector subbundle $L_1$ in $S(TM^{\perp})$ by $L_1^{\perp}$. For a contact CR-lightlike submanifold $M$, we put

$$\phi X = fX + \omega X, \qquad \forall X \in \Gamma(TM), \tag{3.8}$$

where $fX \in \Gamma(D)$ *a*nd $\omega X \in \Gamma(L_1 \perp l\,tr(TM))$. Similarly, we have

$$\phi W = BW + CW, \qquad \forall W \in \Gamma(S(TM^{\perp})), \tag{3.9}$$

where $BW \in \Gamma(\phi L_1)$ and $CW \in \Gamma(L_1^{\perp})$.

**Proposition 3.5.** *Let M be a contact CR-lightlike submanifold of an indefinite Kenmotsu manifold* $\overline{M}$ *. Then,* $D\perp\{V\}$ *is integrable if and only if*

$$h(X, \phi Y) = h(\phi X, Y) \tag{3.10}$$

Proof. From (1.2), (1.7), (3.8), (3.9), and using the transversal parts, we obtain $\omega(\nabla_X Y) = -Ch^s(X,Y) + h(X, \phi Y)$, for all $X, Y \in \Gamma(D\perp\{V\})$.

Consequently, $\omega[X,Y] = h(X,\phi Y) - h(\phi X, Y)$, for all $X, Y \in \Gamma(D\perp\{V\})$. Therefore, $D\perp\{V\}$ *is integrable if and only if* $h(X,\phi Y) = h(\phi X, Y)$.

**Proposition 3.6.** *Let M be a contact CR-lightlike submanifold of an indefinite Kenmotsu manifold* $\overline{M}$ *. Then* $D\perp\{V\}$ *is a totally geodesic foliation if and only if*

$$h^l(X,\phi Y) = 0, \textit{ and } \; h^s(X,Y) \text{ has no component in } L_1. \tag{3.11}$$

Proof. In view of conditions (A) and (B) of Definition 3.1, $D\perp\{V\}$ defines a totally geodesic foliation if and only if $g(\nabla_X Y, \phi\xi) = g(\nabla_X Y, W) = 0$ for $X, Y \in \Gamma(D\perp\{V\})$ *and* $W \in \Gamma(\phi L_1)$. Then, from (1.7), we have

$$g(\nabla_X Y, \phi\xi) = -\bar{g}(\phi \bar{\nabla}_X Y, \xi).$$

Using (1.2) and (1.7), we get

$$g(\nabla_X Y, \phi\xi) = -\bar{g}(h^l(X,\phi Y), \xi) \tag{3.12}$$

In a similar way, we obtain

$$g(\nabla_X \phi Y, W) = -\bar{g}(h^s(X,Y), \phi W) \tag{3.13}$$

Thus, from (3.12) and (3.13) and as $D\perp\{V\}$ is invariant with respect to $\phi$, we obtain (3.11), which completes the proof.

**Proposition 3.7.** *Let M be a contact CR-lightlike submanifold of an indefinite Kenmotsu manifold* $\overline{M}$ *. Then,* $D\check{y}$ *does not define a totally geodesic foliation.*

Proof. In view of conditions (A) and (B) of Definition 3.1, $D\check{y}$ defines a totally geodesic foliation if and only if

$$\bar{g}(\nabla_Z W, N) = g(\nabla_Z W, \phi N) = g(\nabla_Z W, X) = g(\nabla_Z W, V) = 0, \tag{3.14}$$

*where* $Z, W \in \Gamma(D')$, $N \in \Gamma(l\mathrm{tr}(TM))$, and $X \in \Gamma(D_0)$.

From (1.2) and (1.7), we find

$$g(\nabla_Z W, V) = -g(W, Z) \tag{3.15}$$

which is not zero and contradicts the equation (3.14), proving our assertion.

Now we prove a Theorem for irrotational lightlike submanifolds of indefinite Kenmotsu manifold:

**Theorem 3.8.** *Let M be an irrotational contact CR-lightlike submanifold of an indefinite Kenmotsu manifold* $\overline{M}$. *Then M does not define contact CR-lightlike product.*

Proof: By classical definition of product manifolds, a submanifold $M$ is a contact $CR$-lightlike product if $D \perp \{V\}$ and $D\check{y}$ defines totally geodesic foliations in $M$. From Proposition 3.7., $D\check{y}$ does not define a totally geodesic foliation, which completes the proof.

We have the following:

**Lemma 3.9.** *Let M be a totally contact umbilical proper contact CR-lightlike submanifold of an indefinite Kenmotsu manifold* $\overline{M}$. *Then* $\alpha_L = 0$.

Proof. Let $M$ be a totally contact umbilical proper contact CR-lightlike submanifold. Then, by direct calculations, using (1.2), (1.7), (1.9), and taking the tangential parts, we have

$$A_{\phi z}Z + f\nabla_Z Z + \phi h^l(Z,Z) + Bh^s(Z,Z) = 0 \tag{3.16}$$

for $Z \in \Gamma(\phi L_1)$. Hence, we obtain $\overline{g}(A_{\phi z}Z, \phi\xi) + \overline{g}(h^l(Z,Z),\xi) = 0$.

Using (1.10), we find

$$g(h^s(Z,\phi\xi),\phi Z) + \overline{g}(h^l(Z,Z),\xi) = 0.$$

Thus from (1.22), we get

$$-g(Z,Z)\overline{g}(\alpha_L,\xi) = 0$$

Since $\phi L_1$ is nondegenerate, we get $\alpha_L = 0$, which completes the proof.

**Theorem 3.10.** *Let M be a totally contact umbilical proper contact CR-lightlike submanifold of an indefinite Kenmotsu manifold* $\overline{M}$. *Then either M is totally geodesic or* $\dim(\phi L_1) = 1$.

Proof. Assume *M proper* is totally contact umbilical. From (1.2), (1.7), (3.9), and Lemma 3.9., we get

$$\omega(\nabla_X X) + Ch^s(X,X) = 0 \text{ for } X \in \Gamma(D_o).$$

Hence,

$$h^s(X,X) \in \Gamma(L_1) \tag{3.17}$$

Now from (3.16) and (1.10), we have

$$\overline{g}(h^s(Z,Z),\phi W) = \overline{g}(h^s(Z,W),\phi Z) \text{ for } Z,W \in \Gamma(\phi L_1)$$

Since $M$ is totally contact umbilical, we obtain

$$g(Z,Z)\bar{g}(\alpha_s,\phi W) = g(Z,W)\bar{g}(\alpha_s,\phi Z) \tag{3.18}$$

Interchanging $Z$ and $W$ and subtracting the two, we obtain

$$\bar{g}(\alpha_s,\phi Z) = \frac{g(Z,W)^2}{g(Z,Z)g(W,W)}\bar{g}(\alpha_s,\phi Z) \tag{3.19}$$

Considering (3.17), (3.19) has solutions if either (a) dim($L_1$) = 1, or (b) $\alpha_s = 0$. Thus the proof follows from Lemma 3.9.

It is known that CR-submanifolds of Riemannian manifold are generalization of both invariant and totally real submanifolds $[1]$. Therefore, it is important to know whether contact CR-lightlike submanifolds admit invariant submanifolds and, also, are there any real submanifolds. From Definition 1.4., we have

**Proposition 3.11.** *Contact CR-lightlike submanifolds are nontrivial.*

Proof. Suppose $M$ is an invariant lightlike submanifold of an indefinite Kenmotsu manifold. Then we can easily see that the radical distribution is invariant. Thus condition (A) of Definition 3.1 is not satisfied. Similarly, one can prove that the screen real lightlike case is not possible.

## 4. Contact SCR-lightlike submanifolds

We know from Proposition 3.11 that contact CR-lightlike submanifolds exclude the invariant and the screen real subclasses, and therefore, do not serve the central purpose of introducing a CR-structure. To include these two subclasses, we introduce a new class, called *contact screen Cauchy-Riemann* (*SCR*)*-lightlike submanifold,* as follows:

**Definition 4.1.** Let ($M$, $g$, $S(TM)$, $S(TM^{\perp})$) be a lightlike submanifold tangent to the structure vector field $V$ of an indefinite Kenmotsu manifold $\overline{M}$. Then $M$ is a contact SCR-lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(1) There exist real nonnull distributions $D$ and $D^{\perp}$ such that

$$S(TM) = D \oplus D^{\perp} \perp \{V\}, \qquad \phi(D^{\perp}) \subset S(TM^{\perp}), \qquad D \cap D^{\perp} = \{0\}, \tag{4.1}$$

where $D^{\perp}$ is orthogonally complementary to $D \perp \{V\}$ in $S(TM)$.

(2) The distributions $D$ and Rad($TM$) are invariant with respect to $\phi$.

It follows that $l$tr($TM$) is also invariant with respect to $\phi$. Hence we have

$$TM = \bar{D} \oplus D^{\perp} \perp \{V\}, \qquad \bar{D} = D \perp Rad(TM) \tag{4.2}$$

Denote the orthogonal complement of $\phi(D^{\perp})$ in $S(TM^{\perp})$ by $\mu$. We say that $M$ is a

proper contact SCR-lightlike submanifold of $\overline{M}$ if $D \neq \{0\}$ and $D^{\perp} \neq \{0\}$. For a contact SCR-lightlike submanifold, we have:

(i) Condition (2) implies that dim(Rad$TM$) = $r = 2p \geq 2$;

(ii) For proper $M$, (2) implies that dim($D$) = $2s \geq 2$, dim($D^{\perp}$) $\geq 1$. Thus, dim($M$) $\geq$ 5, dim($\overline{M}$) $\geq 9$.

For any $X \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$, we put

$$\phi X = P'X + F'X, \qquad \phi W = B'W + C'W, \tag{4.3}$$

where $P'X \in \Gamma(\overline{D})$, $F'X \in \Gamma(\phi D^{\perp})$, $B'W \in \Gamma(D^{\perp})$, $C'V \in \Gamma(\mu)$.

*We have the following:*

**Example 4.2.** Let $M$ be a submanifold of $\overline{M} = (R_2^9, \overline{g})$ defined by

$$x^1 = x^2, \qquad y^1 = y^2, \qquad x^4 = \sqrt{1-(y^4)^2}, \qquad y^4 \neq \pm 1 \tag{4.4}$$

It is easy to see that a local frame of $TM$ is given by,

$$\left.\begin{array}{ll} Z_1 = e^{-z}(\partial x_1 + \partial x_2), & Z_2 = e^{-z}(\partial y_1 + \partial y_2), \\ Z_3 = e^{-z}\partial x_3, & Z_4 = e^{-z}\partial y_3, \\ Z_5 = e^{-z}(-y^4\partial x_4 + x^4\partial y_4), & V = \partial z, \end{array}\right\} \tag{4.5}$$

Then, Rad$TM$ = Span $\{Z_1, Z_2\}$ and $\phi_0(Z_1) = -Z_2$. Thus, Rad$TM$ is invariant with respect to $\phi_0$. Also, $\phi_0(Z_3) = -Z_4$ implies that $D$ = Span$\{Z_3, Z_4\}$. By direct calculations, we get that $S(TM^{\perp})$ = span$\{W = e^{-z}(x^4\partial x_4 + y^4\partial y_4)\}$ such that $\phi_0(W) = -Z_5$, and the lightlike transversal bundle $l$tr($TM$) is spanned by

$$N_1 = \frac{e^{-z}}{2}(-\partial x_1 + \partial x_2), \qquad N_2 = \frac{e^{-z}}{2}(-\partial y_1 + \partial y_2) \tag{4.6}$$

It follows that $\phi_0(N_2) = N_1$. Thus, $l$tr($TM$) is also invariant. Hence, $M$ is a contact SCR lightlike submanifold.

The following results can be easily proved by using Definition 4.1:

a) A contact $SCR$-lightlike submanifold of $\overline{M}$ is invariant (resp., screen real) and only if $D^{\perp} = \{0\}$ (resp., $D = \{0\}$).

b) Any contact $SCR$-coisotropic or isotropic and or totally lightlike submanifold of $\overline{M}$ is an invariant lightlike submanifold. Consequently, there exist no proper contact $SCR$ or screen real coisotropic or isotropic or totally lightlike submanifold of $\overline{M}$.

We now prove:

**Theorem 4.3.** *Let M be a contact SCR-lightlike submanifold of an indefinite Kenmotsu manifold* $\overline{M}$ *. Then the induced connection* $\nabla$ *is a metric connection if and only if the following two conditions hold:*

1. $h^s(X, \xi)$ *has no components in* $\phi(D^{\perp})$
2. $A_x^* X$ *has no components in D, for all* $X \in \Gamma(TM)$, $\xi \in \Gamma(\mathrm{Rad}TM)$.

Proof. Equation (1.2) implies that $\overline{\nabla}_X \phi\xi = \phi\overline{\nabla}_X \xi$ and from (1.7), (1.13), (4.2), we get

$$\nabla_X \phi\xi = B'h^s(X,\xi) + \phi\nabla_X^{*t}\xi - P'A_\xi^* X \tag{4.7}$$

We know that the induced connection is a metric connection if and only if Rad$TM$ is parallel with respect to $\nabla$. Suppose that Rad$TM$ is parallel. Then from (4.7), we have $B\check{y}h^s(X,\xi) = 0$ and $P\check{y}A_x^* X = 0$. Hence $h^s(X,\xi)$ has no components in $\phi(D^{\perp})$ and $A_x^* X$ has no components in $D$. Conversely, assume that (1) and (2) are satisfied. Then from (4.7), we get $\nabla_X \phi\xi \in \Gamma(\mathrm{Rad}TM)$. Thus, Rad$TM$ is parallel and $\nabla$ is a metric connection, which completes the proof.

**Proposition 4.4.** *There exists a Levi-Civita connection on an irrotational screen real lightlike submanifold of an indefinite Kenmotsu manifold.*

Proof. From (1.7), we have

$$\overline{g}(h^l(X,Y),\xi) = \overline{g}(\overline{\nabla}_X Y,\xi) = \overline{g}(\phi\overline{\nabla}_X Y,\phi\xi) = \overline{g}(-(\overline{\nabla}_X\phi)Y + \overline{\nabla}_X\phi Y,\phi\xi) \tag{4.8}$$

for all $X,Y \in \Gamma(TM)$.

From (1.2), we obtain

$$\overline{g}(h^l(X,Y),\xi) = \overline{g}(\overline{C}_X \phi Y, \phi\xi)$$

Since $\overline{C}$ is a metric connection, we have

$$\overline{g}(h^l(X,Y),\xi) = -\overline{g}(\phi Y, \overline{C}_X \phi\xi).$$

Using (1.7), (1.13) and (1.14) in the above equation, we obtain

$$\overline{g}(h^l(X,Y),\xi) = -\overline{g}(\phi Y, h^s(X,\phi\xi)).$$

$M$ being irrotational implies that $\overline{g}(h^l(X,Y), \xi) = 0$, that is, $h^l = 0$. Then the proof follows from (1.17).

From (1.2), (1.7), and (4.3), we have the following:

$$(\nabla_X P')Y = A_{F'Y}X + B'h^s(X,Y) - g(\phi X,Y)V + \eta(Y)P'X, \tag{4.9}$$

$$(\nabla_X F')Y = C'h^s(X,Y) - h^s(X,P'Y) + \eta(Y)F'X, \tag{4.10}$$

$$\phi h^l(X,Y) = h^l(X,P'Y) + D^l(X,F'Y), \qquad \forall X,Y \in \Gamma(TM) \tag{4.11}$$

The following results are similar to those proved in Propositions 3.5 and 3.6.

**Proposition 4.5.** *Let $M$ be a contact SCR-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then*

(1) *the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi X}Y = A_{\phi Y}X, \ \forall X, Y \in \Gamma(D^{\perp}); \tag{4.12}$$

(2) *the distribution $\overline{D}$ and $\overline{D} \perp \{V\}$ is integrable if and only if*

$$h^s(X, P'Y) = h^s(P'X, Y), \qquad \forall X, Y \in \Gamma(\overline{D}); \tag{4.13}$$

**Theorem 4.6.** *Let $M$ be a contact SCR-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then $\overline{D} \perp \{V\}$ defines a totally geodesic foliation in $M$ if and only if $h^s(X, \phi Y)$ has no components in $\phi(D^{\perp})$, for $X, Y \in \Gamma(\overline{D} \perp \{V\})$.*

Proof. From (1.7), we have

$$g(\nabla_X Y, Z) = \overline{g}(\overline{C}_X Y, Z), \text{ for } X, Y \in \Gamma(\overline{D} \perp \{V\}) \text{ and } Z \in \Gamma(D^{\perp}).$$

Using (1.2), we get

$$g(\nabla_X Y, Z) = \overline{g}(\overline{C}_X \phi Y, \phi Z).$$

Hence,

$$g(\nabla_X Y, Z) = g(h^s(X, \phi Y), \phi Z), \text{ which proves our assertion.}$$

**Theorem 4.7.** *Let $M$ be a contact SCR-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then $D^{\perp}$ and $\overline{D}$ do not define a totally geodesic foliation on $M$.*

Proof: By definition of contact *SCR*-lightlike submanifold, $D^{\perp}$ defines a totally geodesic foliation in $M$ if and only if

$$g(\nabla_X Y, Z) = g(\nabla_X Y, V) = \overline{g}(\nabla_X Y, N) = 0, \tag{4.14}$$

for $X, Y \in \Gamma(D^{\perp})$, $Z \in \Gamma(\overline{D})$, and $N \in \Gamma(l\mathrm{tr}(TM))$.

From (1.7) and (1.2), we obtain

$$g(\nabla_X Y, V) = g(Y, X) \tag{4.15}$$

for $X, Y \in \Gamma(D^{\perp})$. Since $\Gamma(D^{\perp})$ is non degenerate, we can choose non null vectors $X, Y \in \Gamma(D^{\perp})$ such that $g(Y, X) \neq 0$, which imply that $D^{\perp}$ does not define a totally geodesic foliation on $M$. Similarly we can show that $\overline{D}$ does not define a totally geodesic foliation on $M$.

**Lemma 4.8.** *Let M be a contact SCR-lightlike submanifold of an indefinite Kenmotsu manifold* $\overline{M}$ *. Then*

$$h^l(X,V)=0,\ h^s(X,V)=0,\ \nabla_X V=-X+\eta(X)V, \qquad \forall X\in\Gamma(TM), \tag{4.16}$$

$$\nabla_X V=X, \qquad \forall X\in\Gamma(\overline{D}), \tag{4.17}$$

$$\nabla_X V=X, \qquad \forall X\in\Gamma(D^{\perp}), \tag{4.18}$$

Proof. Using (1.2) and (1.7), we get

$$\nabla_X V+h^l(X,V)+h^s(X,V)=-X+\eta(X)V, \qquad \textit{for } X\in\Gamma(TM).$$

Then, (4.16)~(4.18) follow directly.

**Theorem 4.9.** *Any totally contact umbilical proper contact SCR-lightlike submanifold M of* $\overline{M}$ *admits a metric connection.*

*Proof.* From (4.11), we obtain $h^l(X, \phi Y) = h^l(\phi Y, X)$, for all $X, Y \in \Gamma(\overline{D})$. Using this and (1.22), we get $g(X,\phi Y)\alpha_L = g(\phi X,Y)\alpha_L$ . Thus, $g(X,\phi Y)\alpha_L = 0$, and since $\overline{D}$ is nondegenerate, it follows that $\alpha_L = 0$. Thus, $h^l(X, Y) = 0$ for $X, Y \in \Gamma(\overline{D})$. Again, from (4.11), we have $\phi h^l(X,Y)=D^l(X,F'Y)$, $\forall X,Y\in\Gamma(D^{\perp})$ and using (1.22), we find that $\phi h^l(X, Y) = 0$.Thus, $h^l(X, Y) = 0$ for $X, Y \in \Gamma(D^{\perp})$. From Lemma 4.8, if $X \in \Gamma(TM)$ and $Y = V$, then from (4.16), we get that $h^l(X, V) = 0$. Thus $h^l = 0$ on $M$. Finally, our assertion follows from (1.15).

**Theorem 4.10.** *Let M be a totally contact umbilical contact SCR-lightlike submanifold of* $\overline{M}$ *. If* dim $(D^{\perp}) > 1$*, then M is totally geodesic.*

*Proof.* The proof is similar to the proof of Theorem 3.10.

**Theorem 4.11.** *There exist no totally contact umbilical proper contact SCR-lightlike submanifolds in* $\overline{M}$ $(c)$ *with c* № -1.

Proof. Suppose $M$ is totally umbilical proper *SCR*-lightlike submanifold of $\overline{M}$ . From equation (1.18) and Gauss equation (3.9) (*cf.* [6], page171), we get

$$-\frac{1}{2}(1+c)g(X,X)g(Z,Z)=\bar{g}((\nabla_X h^s)(\phi X,Z),\phi Z)-\bar{g}((\nabla_{\phi X}h^s)(X,Z),\phi Z), \quad (4.19)$$

$\forall X\in\Gamma(D), Z\in\Gamma(D^{\perp})$, where

$$(\nabla_X h^s)(\phi X,Z)=\nabla^s_X h^s(\phi X,Z)-h^s(\nabla_X\phi X,Z)-h^s(\phi X,\nabla_X Z)$$

Since $M$ is totally contact umbilical, we have $h^s(\phi X,Z)=0$, and from (1.22), we get

$$\left.\begin{aligned}-h^s(\nabla_X\phi X,Z)&=-g(\nabla_X\phi X,Z)\alpha_s,\\ -h^s(\phi X,\nabla_X Z)&=-g(\phi X,\nabla_X Z)\alpha_s\end{aligned}\right\} \quad (4.20)$$

Thus from (4.20), we obtain

$$(\nabla_X h^s)(\phi X,Z)=-g(\nabla_X\phi X,Z)\alpha_s-g(\phi X,\nabla_X Z)\alpha_s \quad (4.21)$$

On the other hand, since $\bar{g}(\phi X,Z)=0$, taking the covariant derivative with respect to $X$, we obtain $g(\nabla_X\phi X,Z)=-g(\phi X,\nabla_X Z)$. Hence, we get

$$(\nabla_X h^s)(\phi X,Z)=0 \quad (4.22)$$

In a similar way, we have

$$(\nabla_{\phi X}h^s)(X,Z)=0 \quad (4.23)$$

Thus from (4.19), (4.22) and (4.23), we obtain

$$-\frac{1}{2}(1+c)g(X,X)g(Z,Z)=0 \quad (4.24)$$

This implies $(c+1)\,g(X,X)g(Z,Z)=0$. Since $D$ and $D^{\perp}$ are nondegenerate, we can choose non-null vector fields $X$ and $Z$, so $c=-1$, which proves the result.

## 5. Minimal lightlike submanifolds

In this section, we study minimal lightlike submanifold of an indefinite Kenmotsu manifold. As in the semi-Riemannian case, any lightlike totally geodesic submanifold $M$ is minimal. Thus, from Theorem 2.4, any totally umbilical lightlike submanifold of indefinite Kenmotsu manifold, with structure vector field tangent to submanifold, is minimal. Furthermore, from Theorems 3.10 and 4.10 of this paper, it follows that totally contact umbilical contact CR-lightlike submanifold with ($\dim(\varphi L_1) > 1$) and totally contact umbilical contact SCR-lightlike submanifolds with ($\dim(D^{\perp} > 1)$) are minimal.

**Example 5.1.** Let $\overline{M}=(R^{11}_4,\bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,-,+,+,+,-,-,+,+,+,+)$ with respect to the canonical basis

$$(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial z) \tag{5.1}$$

Suppose $M$ is a submanifold of $R_4^{11}$ given by

$$\left.\begin{array}{ll} x^1 = u^1, & y^1 = -u^5 \\ x^2 = \cosh u^2 \cosh u^3, & y^2 = \cosh u^2 \sinh u^3 \\ x^3 = \sinh u^2 \cosh u^3, & y^3 = \sinh u^2 \sinh u^3 \\ x^4 = u^4, & y^4 = -u^6 \\ x^5 = u^1 \cos\theta + u^5 \sin\theta, & y^5 = u^1 \sin\theta - u^5 \cos\theta \\ & z = u^7 \end{array}\right\} \tag{5.2}$$

Then it is easy to see that a local frame of $TM$ is given by

$$\left.\begin{array}{l} Z_1 = e^{-z}(\partial x_1 + \cos\theta\, \partial x_5 + \sin\theta\, \partial y_5) \\ Z_2 = e^{-z}(\sin\theta \partial x_5 - \partial y_1 - \cos\theta\, \partial y_5) \\ Z_3 = e^{-z}(\sinh u^2 \cosh u^3 \partial x_2 + \cosh u^2 \cosh u^3 \partial x_3 \\ \qquad + \sinh u^2 \sinh u^3 \partial y_2 + \cosh u^2 \sinh u^3 \partial y_3) \\ Z_4 = e^{-z}(\cosh u^2 \sinh u^3 \partial x_2 + \sinh u^2 \sinh u^3 \partial x_3 \\ \qquad + \cosh u^2 \cosh u^3 \partial y_2 + \sinh u^2 \cosh u^3 \partial y_3) \\ Z_5 = e^{-z}\partial x_4, \qquad Z_6 = -e^{-z}\partial y^4, \qquad Z_7 = \partial z. \end{array}\right\} \tag{5.3}$$

We see that $M$ is a 2-lightlike submanifold with $RadTM = span\{Z_1, Z_2\}$ and $\phi_0 Z_1 = Z_2$. Thus $RadTM$ is invariant with respect to $\phi_0$. Since $\phi_0(Z_5) = Z_6, D = \{Z_5, Z_6\}$ is also invariant.

Moreover, since $\phi_0 Z_3$ and $\phi_0 Z_4$ are perpendicular to $TM$ and they are nonnull , we can choose

$$S(TM^{\perp}) = Span\{\, \phi_0 Z_3 \,,\, \phi_0 Z_4 \,\}$$

Furthere more, the lightlike transversal bundle $ltr(TM)$ spanned by

$$\left.\begin{array}{l} N_1 = \dfrac{e^{-z}}{2}(-\partial x_1 + \cos\theta\, \partial x_5 + \sin\theta\, \partial y_5) \\ N_2 = \dfrac{e^{-z}}{2}(\sin\theta\, \partial x_5 + \partial y_1 - \cos\theta\, \partial y_5) \end{array}\right\} \tag{5.4}$$

is also invariant. Thus we conclude that $M$ is a contact $SCR-$lightlike submanifold of $R_4^{11}$. Then a quasiorthonormal basis of $\bar{M}$ along $M$ is given by

$$\left.\begin{array}{ll}
\xi_1 = Z_1, & \xi_2 = Z_2 \\
e_1 = \dfrac{1}{\sqrt{\cosh^2 u^3 + \sinh^2 u^3}} Z_3, & e_2 = \dfrac{1}{\sqrt{\cosh^2 u^3 + \sinh^2 u^3}} Z_4 \\
e_3 = Z_5, \quad e_4 = Z_6, & Z = Z_7 \\
W_1 = \dfrac{1}{\sqrt{\cosh^2 u^3 + \sinh^2 u^3}} \phi_0 Z_3, & W_2 = \dfrac{1}{\sqrt{\cosh^2 u^3 + \sinh^2 u^3}} \phi_0 Z_4 \\
N_1, \quad N_2. &
\end{array}\right\} \tag{5.5}$$

where $\varepsilon_1 = g(e_1, e_1) = 1$, $\varepsilon_2 = g(e_2, e_2) = -1$, and $g$ is the degenerate metric on $M$. By direct calculation and using Gauss formula (1.7), we get

$$\left.\begin{array}{l}
h^s(X, \xi_1) = h^s(X, \xi_2) = h^s(X, e_3) = h^s(X, e_4) = 0, \quad h^l = 0, \forall X \in \Gamma(TM) \\
h^s(e_1, e_1) = \dfrac{e^{-z}}{(\cosh^2 u^3 + \sinh^2 u^3)^{3/2}} W_2, \; h^s(e_2, e_2) = \dfrac{e^{-z}}{(\cosh^2 u^3 + \sinh^2 u^3)^{3/2}} W_2
\end{array}\right\} \tag{5.6}$$

Therefore,

$$trace\, h_{g|_{S(TM)}} = \varepsilon_1 h^s(e_1, e_1) + \varepsilon_2 h^s(e_2, e_2) = h^s(e_1, e_1) - h^s(e_2, e_2) = 0 \tag{5.7}$$

Thus $M$ is a minimal contact $SCR-$lightlike submanifold of $R_4^{11}$ .

Now we prove characterisation results for minimal lightlike submanifold of all the cases discussed in the paper.

**Theorem 5.2.** *Let M be a contact SCR-lightlike submanifold of an indefinite Kenmotsu manifold* $\bar{M}$ *. Then M is minimal if and only if*

$$trace\, A_{W_j|S(TM)} = 0, \quad A^*_{\xi_k} = 0 \; on \; D^\perp, \; D^l(X, W) = 0 \tag{5.8}$$

for $X \in \Gamma(RadTM)$ and $W \in \Gamma(S(TM^\perp))$.

Proof. Since $\bar{\nabla}_V V = 0$, from (1.7), we get $h^l(V, V) = h^s(V, V) = 0$.

Now, take an orthonormal frame $\{e_1, ....., e_{m-r}\}$ such that $\{e_1, ....., e_{2a}\}$ are tangent to $D$

and $\{e_{2a+1},.....,e_{m-2r}\}$ are tangent to $D^{\perp}$.

First from $[3]$, we know that $h^l = 0$ on $Rad(TM)$. Now from (4.11), for $Y, Z \in \Gamma(D)$, we have

$$h^l(\phi Y, Z) = \phi h^l(Y, Z) \tag{5.9}$$

Hence, we obtain $h^l(\phi Z, \phi Y) = -h^l(Y, Z)$ .

Thus

$$\sum_{i=1}^{2a} h^l(e_i, e_i) = 0.$$

Since $trace\, h\big|_{S(TM)} = \sum_{i=1}^{m-2r} \varepsilon_i (h^l(e_i, e_i) + h^s(e_i, e_i))$, then $M$ is minimal if and only if

$$\sum_{i=1}^{2a} \varepsilon_i h^s(e_i, e_i) + \sum_{2a+1}^{m-2r} \varepsilon_i (h^l(e_i, e_i) + h^s(e_i, e_i)) = 0 \tag{5.10}$$

On the other hand, we have

$$\left.\begin{aligned} trace\, h\big|_{S(TM)} = & \sum_{i=1}^{2a} \frac{1}{n-2r} \sum_{j=1}^{n-2r} \varepsilon_i \bar{g}(h^s(e_i, e_i), W_j) W_j \\ & + \frac{1}{n-2r} \sum_{j=1}^{n-2r} \sum_{i=2a+1}^{m-2r} \varepsilon_i \bar{g}(h^s(e_i, e_i), W_j) W_j \\ & + \sum_{k=1}^{2r} \frac{1}{2r} \sum_{i=2a+1}^{m-2r} \varepsilon_i \bar{g}(h^l(e_i, e_i), \xi_k) N_k. \end{aligned}\right\} \tag{5.11}$$

Using (1.10) and (1.14), we get

$$\left.\begin{aligned} trace\, h\big|_{S(TM)} = & \sum_{i=1}^{2a} \frac{1}{n-2r} \sum_{j=1}^{n-2r} \varepsilon_i \bar{g}(A_{W_j} e_i, e_i) W_j \\ & + \frac{1}{n-2r} \sum_{j=1}^{n-2r} \sum_{i=2a+1}^{m-2r} \varepsilon_i \bar{g}(A_{W_j} e_i, e_i) W_j \\ & + \sum_{k=1}^{2r} \frac{1}{2r} \sum_{i=2a+1}^{m-2r} \varepsilon_i \bar{g}(A^*_{\xi_k} e_i, e_i) N_k. \end{aligned}\right\} \tag{5.12}$$

On the other hand, from (1.10), we obtain

$$\bar{g}(h^s(X, Y), W) = -\bar{g}(Y, D^l(X, W)), \tag{5.13}$$

$\forall X, Y \in \Gamma(RadTM)$, and $\forall W \in \Gamma(S(TM^{\perp}))$.

Thus our assertion follows from (5.12) and (5.13).

**Theorem 5.3.** *Let M be an irrotational screen real lightlike submanifold of an indefinite Kenmotsu manifold* $\bar{M}$ *. Then M is minimal if and only if*

$$trace\, A_{W_a} = 0 \quad on\ S(TM) \tag{5.14}$$

Proof. Proposition 4.4, implies that $h^l = 0$ . Thus $M$ is minimal if and only if

$h^s = 0$ on $RadTM$ and $trace\, h^s\big|_{S(TM)} = 0$. Thus the proof follows from Theorem 5.2.

**Theorem 5.4.** *Let $M$ be an invariant lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then $M$ is minimal in $\bar{M}$ if and only if $D^l(X,W)=0$ for $X \in \Gamma(RadTM)$ and $W \in \Gamma(S(TM^{\perp}))$.*

Proof. If $M$ is invariant, then $\phi Rad(TM) = Rad(TM)$ and $\phi S(TM) = S(TM)$.
Hence, $\phi(ltr(TM)) = ltr(TM)$ and $\phi(S(TM^{\perp})) = S(TM^{\perp})$.
Then using (2.4), we obtain

$$h(\phi X, Y) = \phi h(X,Y) \tag{5.15}$$

for $X, Y \in \Gamma(TM)$, *and consequently* $h(\phi X, \phi Y) = -h(X,Y)$.
Thus

$$trace\, h\big|_{S(TM)} = \sum_{i=1}^{m-2r} \varepsilon_i \left\{ h(e_i,e_i) + h(\phi e_i, \phi e_i) \right\} = \sum_{i=1}^{m-2r} \varepsilon_i \left\{ h(e_i,e_i) - h(e_i,e_i) \right\} = 0 \tag{5.16}$$

From (1.10), we get

$\bar{g}(h^s(X,Y),W) = -\bar{g}(D^l(X,W),Y)$ for $X, Y \in \Gamma(RadTM)$ $W \in \Gamma(S(TM^{\perp}))$.
The proof follows from Definition 1.3., and the fact that $h^l = 0$ on $RadTM$.

**Theorem 5.5.** *Let M be an irrotational contact CR- lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then M is minimal in $\bar{M}$ if and only if*

(1) $A^*_{\xi}\phi\xi$ *and* $A_N \phi N$ *have no components in* $D'$,

(2) $D^s(\phi N, N)$ *has no components in* $L_1^{\perp}$,

(3) $trace\, A_{W_a|D_0 \perp \phi L_1} = 0$,

*for* $N \in \Gamma(ltr(TM))$ *and* $\xi \in \Gamma(RadTM)$, *where* $D' = \phi(ltr(TM)) \perp \phi(L_1)$.

Proof. Suppose $M$ is irrotational. From (1.2) and (1.7), we have

$$\bar{g}(h^l(\phi\xi,\phi\xi),\xi_1) = -\bar{g}(\bar{\nabla}_{\phi\xi}\xi, \phi\xi_1).$$

Then using (1.7) and (1.13), we obtain

$$\bar{g}(h^l(\phi\xi,\phi\xi),\xi_1) = g(A^*_{\xi}\phi\xi, \phi\xi_1), \qquad \forall \xi, \xi_1 \in \Gamma(RadTM). \tag{5.17}$$

In a similar way, from (1.2), (1.7), (1.13), (3.8) and (3.9), we get

$$\bar{g}(h^s(\phi\xi,\phi\xi),W) = g(A^*_{\xi}\phi\xi, BW), \quad \forall \xi \in \Gamma(RadTM), W \in \Gamma(S(TM^{\perp})). \tag{5.18}$$

Now using (1.2), (1.7), (1.9), (3.8) and (3.9), we derive

$$h(\phi N, \phi N) = -\omega A_N \phi N + CD^s(\phi N, N), \qquad \forall\ N \in \Gamma(ltr(TM)). \tag{5.19}$$

Then the proof follows from (5.17)~(5.19) and Theorem 5.2.

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