The Čech Homology Theory in the Category of Šostak Fuzzy Topological Spaces

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Abstract

In this study, by using the covering of Šostak fuzzy topological spaces, an inverse system of simplicial complexes is constituted. Built upon these inverse systems of simplicial complexes, Čech homology groups are defined. It is proved that Čech homology groups are a functor. Later, axioms of homology theory are checked for this homology groups. To prove homotopic invariant of homology groups, we provide new homotopy relation in the category of Šostak fuzzy topological spaces.

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1. Introduction

After the introduction of fuzzy sets by Zadeh [18], Chang [1] was the first to introduce the concept of a fuzzy topology. Later, A.Šostak, K.C. Chattopadhyay, R.N.Hazra, T.K.Mondal, S.K.Samanta introduced Šostak fuzzy topological spaces by using a concept of gradation of openness on fuzzy sets. Methods
of algebraic topology aren’t used on fuzzy topology a lot, reason being absence of proper homotopy relation. Wang-jin, L., and Chong-you, Z. [17] defined singular homology groups of fuzzy topological spaces as related to this subject. They proved that these groups are invariant under the fuzzy homeomorphism. Salleh, A.R. [13] gave a fuzzy homotopy relation by showing that singular homology groups are homotopic invariant. There are many different approaches to define fuzzy homotopy concept of fuzzy topological spaces. In [14,15] Abdul Razah Salleh and Abu Osman define fuzzy topology of the unit interval. A fuzzy set in this topology is fuzzy open if and only if support of this fuzzy set belongs to usual topology of unit interval. On the basis of this, fuzzy path and their production is being introduced and proved that the set of fuzzy path (fuzzy loop) creates groupoid (group). In [11] F.Klawonn, definition of homotopy groups of fuzzy topological spaces elucidates the definition of homotopy groups of ordinary topological spaces on the basis of Huber’s [9] categorical approach. For this, Kan’s [10] approach of homotopy groups of topological spaces definition is being used. This approach is based on embeddings of complete semi-simplicial complexes into the topological space. In [4] G.Cuvalcioglu and M.Citil introduce another definition of fuzzy homotopy for the fuzzy set. In this case new topology in the fuzzy set is being introduced.

In this study, Čech homology (cohomology) groups are defined in the category of Šostak fuzzy topological spaces. Firstly, excision axiom is proved for these groups. In order to prove a homotopy axiom, Šostak fuzzy unit interval is provided. By using this idea, a new homotopy relation is defined in the category of Šostak fuzzy topological spaces. Finally, it is proved that Čech homology groups are homotopic invariant according to this homotopy relation.

2. Preliminaries

Definition 2.1. [8] For any fuzzy set $A \in F(X)$ and any $\lambda \in [0,1]$, the $\lambda$–cut and strong $\lambda$–cut of $A$ are respectively defined as follows:

$$A_\lambda = \{x \in X : A(x) \geq \lambda\}, \quad A_{(\lambda)} = \{x \in X : A(x) > \lambda\},$$

where $A(x) = \mu_A(x)$ since $A(x)$ is more convenient than $\mu_A(x)$.

Definition 2.2. [8] For any $A \in F(X), B \in F(Y)$ and $R \in F(X \times Y)$, if $R \subset A \times B$, then $R$ is called a fuzzy relation from $A$ to $B$.

Definition 2.3. [8] For $A \in F(X)$ and $B \in F(Y)$, a fuzzy relation $f \subset A \times B$ is called a fuzzy mapping from $A$ to $B$ if $f_\lambda$ is a mapping from $A_\lambda$ to $B_\lambda$ for any $\lambda \in [0,1]$. When $f$ is a fuzzy mapping from $A$ to $B$, we
denote it by \( f : A \rightarrow B \). \( f : A \rightarrow B \) is called a fuzzy injection if \( f_\lambda \) is an injection from \( A_\lambda \) to \( B_\lambda \) for any \( \lambda \in [0,1] \); it is called a fuzzy surjection if \( (\forall \lambda \in [0,1]) \ (B_\lambda) \subset f_\lambda (A_\lambda) \subset B_\lambda \); and it is called a fuzzy bijection if \( f \) is not only an injection but also a surjection.

**Definition 2.4.** [4] Let \( (X, \tau) \) be a topological space, \( A \in F(X) \) and \( \tau^* \subset F(X) \). Let [\( \tau^* = \{G : \nu \rightarrow I \mid \nu \in \tau\}, \ \tau^*_X = \{G_\lambda \mid G \in \tau^*\}, \ \lambda \in I\]. The pair \((A, \tau^*)\) is called fuzzy topological space if and only if for every \( \lambda \in I \), \((A_\lambda, \tau^*_X)\) is a topological space.

**Definition 2.5.** [4] Let \( A \in F(X) \), \( B \in F(Y) \), \((A, \tau_1)\) and \((B, \tau_2)\) be two fuzzy topological spaces and \( f : A \rightarrow B \) be a fuzzy function. \( f \) is fuzzy continuous if and only if for every \( \lambda \in I \), \( f_\lambda : A_\lambda \rightarrow B_\lambda \) is continuous.

**Definition 2.6.** [4] Let \((A, \tau_1)\) and \((B, \tau_2)\) be fuzzy topological spaces and \( f, g : A \rightarrow B \) be fuzzy continuous functions. \( f \) is fuzzy homotopic to \( g \) if there exists a fuzzy continuous function \( F : A \times I \rightarrow B \) such that for every \( \lambda \in I \),

\[
F_\lambda (x, 0) = f_\lambda (x), \quad F_\lambda (x, 1) = g_\lambda (x).
\]

The map \( F \) is called a fuzzy homotopy between \( f \) and \( g \), and we write \( f \sim g \).

**Definition 2.7.** [13] Let \( I^\Gamma \) be set of all monotonic decreasing maps \( \lambda : \mathbb{R} \rightarrow L \) (here \( \mathbb{R} \) is the set of real numbers and \( L \) is completely distributive lattice) satisfying:

1. \( \lambda (t) = 1 \) for \( t < 0 \),
2. \( \lambda (t) = 0 \) for \( t > 1 \).

For \( \lambda, \mu \in \mathbb{I}^\Gamma \), we define that \( \lambda \equiv \mu \) iff \( \lambda (t-) = \mu (t-) \) and \( \lambda (t+) = \mu (t+) \) for all \( t \in \mathbb{R} \), where \( \lambda (t-) = \inf_{s \leq t} \lambda (s) \) and \( \lambda (t+) = \sup_{s \geq t} \lambda (s) \). Then \( \equiv \) is an equivalence relation on \( \mathbb{I}^\Gamma \), \([\lambda]\) denotes the equivalence class of \( \lambda \in \mathbb{I}^\Gamma \), and the quotient set \( \mathbb{I}^\Gamma / \equiv \) is called the \( L \)-fuzzy unit interval which in symbols is written \( I(L) \).

We define an \( L \)-fuzzy topology \( \tau \) on \( I(L) \) by taking as a subbase \( \{L_t, R_t : t \in \mathbb{R}\} \), where we define \( L_t ([\lambda]) = (\lambda (t-))^\prime \) and \( R_t ([\lambda]) = (\lambda (t+))^\prime \). The topology \( \tau \) is called the standart topology on \( I(L) \), and the base of \( \tau \) is \( \{L_s \wedge R_t : s, t \in \mathbb{R}\} \).

**Definition 2.8.** [13] Let \( f, g : (X, \tau) \rightarrow (Y, \sigma) \) be \( L \)-fuzzy continuous maps. We say that \( f \) is \( L \)-fuzzy homotopic to \( g \) if there exists an \( L \)-fuzzy continuous map \( F : (X, \tau) \times (I(L), \tau) \rightarrow (Y, \sigma) \) such that \( F (a_\alpha, [\lambda_0]) = f (a_\alpha) \) and \( F (a_\alpha, [\lambda_1]) = g (a_\alpha) \) for every \( L \)-fuzzy point \( a_\alpha \) in \((X, \tau)\), where for \( i = 0, 1 \),
The map \( F \) is called an \( L \)-fuzzy homotopy between \( f \) and \( g \), and written \( F : f \cong_L g \) and simply \( f \cong_L g \) when no confusion arises.

**Definition 2.9.** [15] Let \((X,T)\) be a (usual) topological space. The collection \( T = \{ G \mid G \text{ is a fuzzy set on } X \text{ and } G_0 \in T \} \) is a fuzzy topology on \( X \), called the fuzzy topology on \( X \) introduced by \( T \). The pair \((X,F_T)\) is called a fuzzy topological space introduced by \((X,T)\).

Thus if \( \xi_I \) is an Euclidean subspace topology on \( I \) then \((I,\xi_I)\) denotes the fuzzy topological space introduced by the (usual) topological space \((I,\xi_I)\).

**Definition 2.10.** [15] Two fuzzy paths \( \alpha(A)\), \( \beta(B) \) in \( P(X,\tau) \), \((a_\lambda,b_\eta)\) are said to be fuzzy homotopic rel end points if there exists a fuzzy continuous function \( H : (I,\xi_I) \times (I,\xi_I) \to (X,\tau) \) such that

\[
(H(t,0))_{A(t)} = \alpha(t_{A(t)}), \quad (H(t,1))_{A(t)} = \beta(t_{B(t)}), \quad t \in I
\]

\[
(H(0,s))_{A(0)} = \alpha_\lambda = (H(0,s))_{B(0)}, \quad (H(1,s))_{A(1)} = \beta_\eta = (H(1,s))_{B(1)}, \quad s \in I.
\]

We can denote the function \( H \) as a fuzzy homotopy rel end points and write \( H : \alpha(A) \cong \beta(B) \). We abbreviate fuzzy homotopic rel end points to fuzzy homotopic, since in the case of fuzzy paths we have no need of other homotopies.

**Definition 2.11.** [14] Let \((X,\tau)\) and \((Y,\sigma)\) be fuzzy topological spaces. Two fuzzy continuous functions \( f, g : (X,\tau) \to (Y,\sigma) \) are said to be fuzzy homotopic if there exists a fuzzy continuous function \( H : (X,\tau) \times (I,\xi_I) \to (Y,\sigma) \) such that \( H(p,0) = f(p) \) and \( H(p,1) = g(p) \) for all fuzzy point \( p \) in \((X,\tau)\).

**Definition 2.12.** [17] Let \((X,\tau)\) be an \( L-fts \), and \( Z \) the (usual) additive group of integers. For \( n \geq 0 \), \( Q_n[(X,\tau),Z] \) denotes the free abelian group generated by the set \( S_n(X,\tau) \) of all \( L \)-fuzzy singular \( n \)-cubes in \((X,\tau)\), i.e. every \( \xi_n \in Q_n[(X,\tau),Z] \) has a unique representation as finite linear combination of \( L \)-fuzzy singular \( n \)-cubes,
We should not distinguish between an \( L \)-fuzzy singular \( n \)-cube \( \xi^{(n)} \) and \( 1 \xi^{(n)} \in Q_n[(X, \tau), Z] \); furthermore, \( 0 \) denotes \( \sum_{\alpha} 0\xi^{(n)}_{\alpha} \in Q_n[(X, \tau), Z] \).

For \( n \geq 1 \), let \( D_n[(X, \tau), Z] \) denote the subgroup of \( Q_n[(X, \tau), Z] \) generated by all degenerate \( L \)-fuzzy singular \( n \)-cubes, and let \( D_0[(X, \tau), Z] = \{0\} \).

Now we can define

\[
C_n[(X, \tau), Z] = \frac{Q_n[(X, \tau), Z]}{D_n[(X, \tau), Z]}, \quad n \geq 0,
\]

and called it a group of singular \( n \)-chains in \((X, \tau)\) (with integral coefficients).

**Definition 2.13.** [17] Let \((X, \tau)\) be an \( L \)-fts, \( n \geq 1 \). We define a homomorphism \( \partial_n : Q_n[(X, \tau), Z] \to Q_{n-1}[(X, \tau), Z] \) such that for \( \xi^{(n)} \in S_n(X, \tau) \),

\[
\partial_n\xi^{(n)} = \sum_{i=1}^{n} (-1)^{i} \left[ d_1^{i}\xi^{(n)}_i - d_0^{i}\xi^{(n)}_i \right].
\]

**Proposition 2.14.** [17] Let \((X, \tau)\) be an \( L \)-fts. We define

\[
\partial_{n-1}\partial_n = 0 (nullhomomorphism).
\]

**Definition 2.15.** [17] Let \((X, \tau)\) be an \( L \)-fts. We define

\[
Z_n[(X, \tau), Z] = \ker(\partial_n^*) = \{c_n \in C_n[(X, \tau), Z] \mid \partial_n^*c_n = 0\}, \quad n \geq 1
\]

and call them a group of singular \( n \)-cycles and singular \( n \)-boundaries in \((X, \tau)\), respectively.

Particularly, let \( Z_0[(X, \tau), Z] = C_0[(X, \tau), Z] \). Note that from proposition 2.14,

\[
B_n[(X, \tau), Z] \subseteq Z_n[(X, \tau), Z], \quad n \geq 0.
\]
Hence we define

\[ H_n[(X, \tau), Z] = \frac{Z_n[(X, \tau), Z]}{B_n[(X, \tau), Z]}, \quad n \geq 0, \]

and call it the singular n-homology of \( L - fts \) \( (X, \tau) \) (with integral coefficients).

**Definition 2.16** [12] The gradation of openness on \( X \) is a mapping \( \tau : I^X \rightarrow I \), satisfying the following conditions:

(i) \( \tau(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \tau(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = 1 \),

(ii) \( \tau(\lambda_1 \cap \lambda_2) \geq \tau(\lambda_1) \land \tau(\lambda_2) \),

(iii) \( \tau(\bigcup_{i \in \Delta} \lambda_i) \geq \bigwedge_{i \in \Delta} \tau(\lambda_i) \).

The pair \( (X, \tau) \) is called a fuzzy topological space or Čech fuzzy topological space.

### 3. Čech Homology Theory

Let \( \mathcal{SFTS} \) be a category of Čech fuzzy topological space. For each Čech fuzzy topological space \( (X, \tau) \), \( (X, \tau^r) \) is a Chang fuzzy topological space, \( \forall r \in [0, 1] \). We show the set of all open coverings of \( X \) as \( Cov_r(X) \). \( Cov_r(X) \) is a directed set according to the following relation:

If \( \alpha = \{U_i\}_{i \in I} \), \( \beta = \{V_j\}_{j \in J} \) are two open covering, then the covering \( \beta \) is called a refinement of \( \alpha \), denoted by \( \alpha < \beta \), if there exists a mapping \( p : J \rightarrow I \) such that \( V_j \prec U_{p(j)} \).

Let \( \alpha = \{U_i\}_{i \in I} \) be a open covering of Chang fuzzy topological space \( (X, \tau^r) \) and \( nerv(\alpha) = \left\{ (i_1, \ldots, i_n) : \bigwedge_{j=1}^n U_{i_j} \neq 0 \right\} \) be the simplicial complex consisting of all simplexes whose vertices are elements of \( I \). The map \( p : J \rightarrow I \) extends uniquely to a simplicial mapping \( p : nerv(\beta) \rightarrow nerv(\alpha) \). If \( \alpha < \beta \) are two coverings of \( (X, \tau^r) \), then any two mappings \( p, p' : nerv(\beta) \rightarrow nerv(\alpha) \) are contiguous simplicial maps [5]. Hence the simplicial mapping \( p_\alpha^\beta : nerv(\beta) \rightarrow nerv(\alpha) \) is uniquely defined in contiguous class of simplicial maps.

Taking these into consideration, we can easily demonstrate that

\[ nerv(X, \tau^r) = \left\{ (nerv(\alpha))_{\alpha \in Cov_r(X)}, \left\{ p_\alpha^\beta : nerv(\beta) \rightarrow nerv(\alpha) \right\}_{\alpha < \beta} \right\} \quad (1) \]
Čech homology theory

is an inverse system of simplicial complexes.

Let $G$ be an arbitrary group. If we apply the homology (cohomology) functor $H_q (H^q)$ for the inverse system (1), then we obtain the following inverse (direct) system of groups

$$[H_q (\text{nerv} (X, \tau^r); G) = \left( \{H_q (\text{nerv} \alpha, G)\}_{\alpha \in \text{Cov}_r (X)}, \{H_q (p^\beta_\alpha)\}_{\alpha < \beta} \right)]$$

$$[H^q (\text{nerv} (X, \tau^r); G) = \left( \{H^q (\text{nerv} \alpha, G)\}_{\alpha \in \text{Cov}_r (X)}, \{H^q (p^\beta_\alpha)\}_{\alpha < \beta} \right)].$$

(2)

**Definition 3.1.** The group $H_q ((X, \tau^r); G) = \varprojlim H_q (\text{nerv} ; G)$

$$[H^q ((X, \tau^r); G) = \varprojlim H^q (\text{nerv} ; G)]$$

is said to be $r$-level homology (cohomology) group of Šostak fuzzy topological space $(X, \tau)$.

Since $\tau^r \subset \tau^s$ for $r \geq s$, the inverse system (1) is a subsystem of the following system

$$\text{nerv} (X, \tau^s) = \left( \{\text{nerv} \alpha\}_{\alpha \in \text{Cov}_s (X)}, \{p^\beta_\alpha : \text{nerv} \beta \to \text{nerv} \alpha\}_{\alpha < \beta} \right).$$

Hence the family $\{\text{nerv} (X, \tau^r)\}_{r \in [0, 1]}$ is a direct system of inverse systems. Therefore, the family $\{H_q (X, \tau^r)\}_{r \in [0, 1]}$ is a direct (inverse) system.

**Definition 3.2.** The group

$$H_q ((X, \tau); G) = \lim_{\longrightarrow} H_q ((X, \tau^r); G) = \lim_{\longleftarrow} \lim_{\longrightarrow} H_q (\text{nerv} ; G)$$

$$\left[ H^q ((X, \tau); G) = \lim_{\longrightarrow} H^q ((X, \tau^r); G) = \lim_{\longleftarrow} \lim_{\longrightarrow} H^q (\text{nerv} ; G) \right]$$

is said to be homology (cohomology) group of Šostak fuzzy topological space $(X, \tau)$.

Let $f : (X, \tau) \to (Y, \tau')$ be a gp-map of Šostak fuzzy topological spaces. Then the map $f$ determines fuzzy continuous map $f^r : (X, \tau^r) \to (Y, \tau'^r)$ of Chang fuzzy topological spaces, for $\forall r \in [0, 1]$. For every open covering $\alpha = \{V_j\}_{j \in J}$ of the space $(Y, \tau'^r)$, the family $f^{-1} (\alpha) = \{f^{-1} (V_j)\}_{j \in J'}$ is a fuzzy open covering of the space $(X, \tau^r)$ and $J' \subset J$. If $(f^r)^{-1} (V_{j_1}) \wedge ... \wedge
(f^r)^{-1} (V_{j_k}) \neq 0, \text{ then } V_{i_1} \land \ldots \land V_{j_k} \neq 0. \text{ Because } (f^r)^{-1} (V_{j_1} \land \ldots \land (f^r)^{-1} (V_{j_k}) = (f^r)^{-1} (V_{j_1} \land \ldots \land V_{j_k}) \neq 0, \text{ i.e. the simplicial complex } nerv(f^r)^{-1} (\alpha) \text{ is a subcomplex of the simplicial complex } nerv\alpha. 

Let \( i_{\alpha,f^r} : nerv(f^r)^{-1} (\alpha) \rightarrow nerv\alpha \) be an embedding map. Then the family 

\[ f = \{(f^r)^{-1} : Cov(Y, \tau^{r''}) \rightarrow Cov(X, \tau^r)\}, \quad \{i_{\alpha,f} : nerv(f^r)^{-1} (\alpha) \rightarrow nerv\alpha\}_\alpha \]

is a morphism from the inverse system \( nerv(X, \tau^r) \) to the inverse system \( nerv(Y, \tau^{r''}) \). By using the morphism \( f \), we define the following homomorphism of \( r \)-level homology groups 

\[ f_* = \lim_{\rightarrow} H_q(f) : H_q((X, \tau^r); G) \rightarrow H_q((Y, \tau^{r''}); G). \]

The family of homomorphisms \( \{f_*\}_r \) determines the following homomorphism of homology groups

\[ f_* = \lim_{\rightarrow} f_*^r : H_q((X, \tau); G) \rightarrow H_q((Y, \tau'); G). \]

**Theorem 3.3.** The corresponding

\[ (X, \tau) \mapsto H_q((X, \tau); G) \quad [(X, \tau) \mapsto H^q((X, \tau); G)] \]

is a covariant (contravariant) functor from the category \( \tilde{SFTS} \) to the category of groups.

Let \((X, \tau)\) be \( \tilde{S} \)-Sostak fuzzy topological space, \( A \in X \) a fuzzy set. If \( \tau (A) = 1 \), we call \((X, A)\) as a pair of \( \tilde{S} \)-Sostak fuzzy topological spaces. The morphism \( f : (X, A) \rightarrow (Y, B) \) is a gp-map \( f : X \rightarrow Y \) such that \( f (A) \leq B \). The pairs of all \( \tilde{S} \)-Sostak fuzzy topological spaces and morphisms of their constitute a category. We denote this category as \( \tilde{SFTS}^2 \).

Let \((X, A)\) be a pair of \( \tilde{S} \)-Sostak fuzzy topological spaces. For \( \forall r \in [0,1] \), let’s take an open covering \( \alpha = \{U_i\}_{i \in I} \) of the space \((X, \tau^r)\). Since \( \tau (A) = 1 \), \( A \in \tau^r \) and the family \( \alpha \land A = \{U_i \land A\}_{i \in I} \) is an open covering of fuzzy set \( A \) over the space \((X, \tau^r)\). Then the pair \((\alpha, \alpha \land A)\) is an open covering of Chang fuzzy topological space \((X, \tau^r), (A, \tau_A^r)\). We denote family of this coverings as \( Cov_r(X, A) \). If the covering \( \beta \) is a refinement of the covering \( \alpha \), then the covering \( \beta \land A \) is also a refinement of the covering \( \alpha \land A \). So, the simplicial map \( p^\alpha_\beta : nerv\beta \rightarrow nerv\alpha \) induces the following map of pairs of simplicial complexes

\[ p^\alpha_\beta : (nerv\beta, nerv(\beta \land A)) \rightarrow (nerv\alpha, nerv(\alpha \land A)). \]

Hence for \((X, A) \in \tilde{SFTS}^2 \) and \( \forall r \in [0,1] \),

\[ \left(\{nerv(\alpha \land A)\}_{\alpha \in Cov_r(X)}\right) \cup \left\{p^\alpha_\beta\right\}_{\alpha < \beta} \]
is an inverse system of simplicial complexes.

**Definition 3.4.** The group \( H_q(((X, A), \tau); G) = \lim_{\rightarrow} H_q((\text{nerv}_\alpha, \text{nerv} (\alpha \land A)); G) \)

is said to be \( r \)-level homology group of the pair \((X, A)\). The group \( H_q(X, A; G) = \lim_{\rightarrow} H_q((X, A), \tau^r; G) \)

is said to be homology group. It is obvious that

\[
H_q : \tilde{\mathbf{FTS}}^2 \to \text{Group}
\]

is a functor.

For the pair \((X, A)\), let \( i : A \to X, j : X \to (X, A) \) be embedding maps. For every \( \alpha \in \text{Cov}_r(X, A) \), the mappings \( i, j \) induces simplicial mappings

\[
i_{\alpha, r} : \text{nerv}(\alpha \land A) \to \text{nerv}_\alpha, \quad j_{\alpha, r} : \text{nerv}_\alpha \to (\text{nerv}_\alpha, \text{nerv}(\alpha \land A)).
\]

Hence for every \( \alpha \in \text{Cov}_r(X, A) \), we obtain the following exact sequence of homology groups

\[
\ldots \leftarrow H_q(\text{nerv}_\alpha, \text{nerv}(\alpha \land A)) \leftarrow H_q(\text{nerv}_\alpha) \leftarrow H_q(\text{nerv}(\alpha \land A)) \leftarrow \ldots \quad (3)
\]

We denote this sequence as \( H(\text{nerv}_r, \alpha) \). If we take limit, then \( \lim_{\rightarrow} \lim_{\rightarrow} H(\text{nerv}_r, \alpha) \)

is

\[
\ldots \leftarrow H_q(X; G) \leftarrow H_q(A; G) \leftarrow H_{q+1}(X, A; G) \leftarrow H_{q+1}(X; G) \leftarrow \ldots
\]

Consequently, we obtain a sequence of homology groups of Šostak fuzzy topological spaces \((X, A)\). Since the sequence \((3)\) is not exact, this sequence is not generally exact.

**Theorem 3.5. (Excision Axiom)** Let \((X, A)\) be a pair of Šostak fuzzy topological spaces and \( U \) a fuzzy set such that \( \tau(U) = 1 \). Then embedding map \( f : (U', A \land U') \to (X, A) \) induces isomorphism of homology groups

\[
f* : H_q(U', A \land U') \to H_q(X, A).
\]

**Proof.** Let \( D \) be a subset of \( \text{Cov}((X, A), \tau^r) \) consisting of all coverings \( \alpha \) indexed by \( (V_\alpha, V^A_\alpha) \) such that

(a) If \( \alpha_v \cap U \neq 0 \), then \( v \in V^A_\alpha \) and \( \alpha_v \in A \).
The conclusion of the theorem is a consequence of the following three propositions:

(b) $D$ is cofinal subset in $\text{Cov}_r (X, A)$.

(c) $f^{-1} (D)$ is cofinal subset in $\text{Cov}_r (U', A \land U')$.

(d) If $\alpha \in D$ and $\beta = f^{-1} (\alpha)$, then

$$f_{\alpha*} : H_q (nerv_\beta (U'), nerv_\beta (A \land U')) \rightarrow H_q (nerv_\alpha (X), nerv_\alpha (A))$$

is an isomorphism.

In order to prove (b), let’s consider any covering $\alpha$ of $(X, A)$ with indexing pair $(V_\alpha, V_\alpha^A)$. Let $V'$ be a set disjoint with $V_\alpha$ and in a 1-1 correspondence with $V_\alpha^A$. For each $v \in V_\alpha^A$, the corresponding element of $V'$ will be denoted by $v'$. Consider the covering $\gamma$ of $((X, A), \tau')$ indexed by $(V_\alpha \cup V', V_\alpha^A \cup V')$ and defined as follows:

$$\gamma_v = \alpha_v \land (\overline{U})' \quad \text{for} \quad v \in V_\alpha,$$

$$\gamma'_v = \alpha_v \cap \text{Int} A \quad \text{for} \quad v' \in V'$$

Since $\overline{U} < \text{Int} A$, it follows that $\gamma$ is a covering of $((X, A), \tau')$. Clearly, $\alpha < \gamma$ and $\gamma \in D$, i.e. $D$ is a cofinal subset in $\text{Cov}_r (X, A)$.

In order to prove (c), let’s consider any covering $\beta$ of $(U', A \land U')$ indexed by $(V_\beta^*, V_\beta^A)$. Define $\alpha \in \text{Cov} (X, A)$ indexed by the same pair $(V_\beta^*, V_\beta^A)$ as follows:

$$\alpha_v = \beta_v \lor U, \quad v \in V_\beta^*.$$  

Then $\beta = f^{-1} (\alpha)$. Choose $\gamma \in D$ so that $\alpha < \gamma$. Then $\beta = f^{-1} (\alpha) < f^{-1} (\gamma)$, so that $f^{-1} (D)$ is a cofinal subset in $\text{Cov}_r (U', A \land U')$.

In order to prove (d), it suffices to prove that

$$nerv_\alpha (X) = nerv_\beta (U') \cup nerv_\alpha (A)$$

$$nerv_\beta (A \land U') = nerv_\beta (U') \cap nerv_\alpha (A).$$

Since $(nerv_\beta (U'), nerv_\beta (A \land U'))$ is a subcomplex of $(nerv_\alpha (X), nerv_\alpha (A))$, we have the inclusions
Thus it remains to prove
\[ \nerv_\beta (U') \cup \nerv_\alpha (A) \supset \nerv_\alpha (X), \quad \nerv_\beta (A \wedge U') \supset \nerv_\beta (U') \cap \nerv_\alpha (A) \]

Let \( s \) be a simplex of \( \nerv_\alpha (X) \) which is not in \( \nerv_\beta (U') \). Then \( \Car_\alpha (s) \neq 0 \) [5] and \( \Car_\alpha (s) \wedge U' = \Car_\beta (s) = 0 \). Consequently \( 0 \neq \Car_\alpha (s) < U \). This implies that for every vertex \( v \) of \( s \), we have \( \alpha_v \wedge U \neq 0 \), and therefore since \( \alpha \in D \) that \( v \in V_\alpha^A \). Since \( U < A \), it follows that \( \Car_\alpha (s) \wedge A \neq 0 \) so that \( s \) is a simplex of \( \nerv_\alpha (A) \). This proves the first part of (4).

Let \( s \) be a simplex of \( \nerv_\beta (U') \cap \nerv_\alpha (A) \). It follows that the vertices of \( s \) are in \( V_\alpha^A \) and that
\[ \Car_\alpha (s) \wedge U' = \Car_\beta (s) \neq 0. \]

If \( \Car_\alpha (s) < U' \), then
\[ \Car_\beta (s) \wedge (A \wedge U') = \Car_\alpha (s) \wedge U' \wedge A = \Car_\alpha (s) \wedge A \neq 0 \]
and \( s \) is in \( \nerv_\beta (A \wedge U') \). If \( \Car_\alpha (s) \wedge U \neq 0 \), then, since \( \alpha \) is in \( D \), it follows that \( \alpha_v < A \) for every vertex \( v \) of \( s \). Thus \( \Car_\alpha (s) < A \) and
\[ \Car_\beta (s) \wedge (A \wedge U') = \Car_\alpha (s) \wedge U' \wedge A = \Car_\alpha (s) \wedge U' \neq 0, \]
so that \( s \) is in \( \nerv_\beta (A \wedge U') \). For \( r \)-level homology groups and \( \forall r \in [0,1] \), this concludes the proof of theorem 3.5.

Since \( r \)-level homology groups are isomorphic, homology groups of their are isomorphic.

To prove homotopy axiom, let’s define a unit interval in \( ČFTS \). Let \( (I, \tau_e) \) be a unit interval in Euclidean topology. We define Chang fuzzy topology \( \left( I, T \right) \) as \( T = \{ G : I \rightarrow I \mid \sup pG \in \tau_e \} \). We define \( Čsostak \) fuzzy topology \( \left( I, \tau \right) \) by the formula
\[ \tau(A) = \begin{cases} 1, & \text{if } A \in T \\ 0, & \text{if } A \notin T \end{cases} \]

and we say \( \left( \Gamma I, \tau \right) \) as Šostak unit interval. Then for \( \forall r \in [0, 1], \left( \Gamma I, \tau^r \right) = \left( \Gamma I, T \right). \)

**Definition 3.6.** Let \( f, g : (X, \tau) \to (Y, \tau') \) be gp-maps of Šostak fuzzy topological spaces. If \( f^r, g^r : (X, \tau^r) \to (Y, \tau'^r) \) are fuzzy homotopic for \( \forall r \in [0, 1] \), then the gp-maps \( f, g \) are said to be Šostak fuzzy homotopic maps.

Since fuzzy homotopy relation is an equivalence relation, Šostak fuzzy homotopy relation is an equivalence relation too.

**Theorem 3.7. (Homotopy Axiom)** If \( f, g : (X, \tau) \to (Y, \tau') \) are Šostak fuzzy homotopic maps, then \( f_{eq} = g_{eq} = H_q(X, \tau; G) \to H_q(Y, \tau'; G). \)

It is enough to prove the theorem for \( r \)-level homology groups and \( \forall r \in [0, 1] \). Now let us give necessary lemmas.

Let \( \left( \Gamma I, T \right) \) be fuzzy unit interval.

**Lemma 3.8.** Every fuzzy open connected set in fuzzy unit interval \( \left( \Gamma I, T_e \right) \) is a fuzzy set such that its support is open interval \((a, b)\). Fuzzy sets supports of which are open interval in fuzzy unit interval \( \left( \Gamma I, T_e \right) \) constitute a base of topology \( T_e. \)

**Proof.** For every open connected fuzzy set \( G \), the set \( G_0 \subset I \) is a connected open set, i.e. \( G_0 = (a, b) \).

If \( G \in T \), then \( G_0 \in \tau_e \). In that case \( G_0 = \bigcup_{i \in I} (a_i, b_i) \) can be written. Now, for every \( i \in I \) fuzzy set \( G^i \) is defined by

\[ G^i(t) = \begin{cases} G(t), & t \in (a_i, b_i) \\ 0, & t \notin (a_i, b_i) \end{cases} \]

It is obvious that the sets \( G^i \) are open sets and \( G = \bigvee_{i \in I} G^i. \)

**Lemma 3.9.** For every finite open connected covering of unit interval \( \left( \Gamma I, T \right) \), the complex \( nerva \) is acyclic.
Proof. We shall first reduce the general case to the case when no comparison done between the set of covering \( \alpha = \{ G^1, \ldots, G^n \} \) in unit interval \( \left( \Gamma, I, T \right) \). We put fuzzy sets \( G^i \) in an ascending order of end points of intervals \( (G^i)_0 \). Therefore \( G^i \wedge G^{i+1} \neq \emptyset \) and \( G^i \wedge G^j = \emptyset, j \neq i-1, i+1 \) and \( 0 \in (G^1)_0 \) is obtained. Now, if we define simplicial mappings \( p_i : \text{nerv} \alpha \to \text{nerv} \alpha, i = 1, n \) as

\[
p_i(\nu_j) = \begin{cases} 
\nu_j, & j \leq i \\
\nu_i, & j > i \end{cases}
\]

then simplicial mappings \( p_i, p_{i+1} \) are contiguous. Thus mappings \( p_1, p_n \) are contiguous. Since \( p_1 \) is a constant map and \( p_n \) is an identity map, it follows that \( H_q(\text{nerv} \alpha) = H_q(*) = 0 \).

We note that open connected coverings taken from proof of Lemma 3.9 constitute a confinal subset of the covering of \( \left( \Gamma, I, T \right) \). We call these covers as a regular covering.

Let \( \alpha = \{ U_i \}_{i \in I} \in \text{Cov}_r(X) \) be a covering. Suppose to each \( i \in I \) there corresponds a regular covering \( \beta^i = \{ V^i_j \}_{j \in N^i} \) of unit interval \( \left( \Gamma, T \right) \). If \( W \) is defined by

\[
W = \{(i, j) : i \in I, j \in N^i \},
\]

then the family \( \gamma = \{ \gamma_{i, j} = U_i \times V^i_j \}_{(i, j) \in W} \) is an open covering of the fuzzy space \( X \times I \). These coverings are called a stacked covering.

Lemma 3.10. Stacked coverings form a cofinal subset of \( \text{Cov}(X \times I) \).

Proof. Let \( \delta = \{ A_k \}_{k \in K} \) be any covering of the space \( X \times I \). For each \( (x, t) \) choose open sets \( U \) \( (x, t) < X, V \) \( (x, t) < I \) such that \( U \times V < A_k \). For each fixed fuzzy point \( x \), the family \( \{ V(x, t) \} \) constitute an open covering of the unit interval \( \left( \Gamma, T \right) \). As indicated above, this family has a finite regular refinement \( \beta^{x, i} \). For each fuzzy open set \( \beta^{x, i} \), there is a fuzzy open set \( U(x, i) \) such that \( U(x, i) \times \beta^{x, i} < A_k \). If \( U(x, i) = \wedge U(x, i) \), then the family \( \{ U(x, i) \times \beta^{x, i} \} \) is stacked covering of fuzzy topological space \( X \times I \) and refinement of covering \( \delta \).

Lemma 3.11. Let family \( \gamma \) be a stacked covering of fuzzy topological space \( X \times I \) over base \( \alpha \). If the \( \text{nerv} \alpha \) is a simplex, then the \( \text{nerv} \gamma \) is acyclic.
Proof: Let $\alpha = \{ U_i \}_{i \in I}$, $\gamma = \{ U_i \times V_j \}_{(i, j) \in W}$. We define a covering $\delta$ of the space $\left( \Gamma, I, T \right)$ as follows:

$$\delta = \{ \delta_{i,j} = V_j \}_{(i, j) \in W}.$$

If $s$ is any simplex with vertices $(i_0, j_0), \ldots, (i_n, j_n) \in W$, then

$$\bigwedge_{k=0}^{n} (U_{i_k} \times V_{j_k}^{i_k}) = \left( \bigwedge_{k=0}^{n} U_{i_k} \right) \times \left( \bigwedge_{k=0}^{n} V_{j_k}^{i_k} \right) = \left( \bigwedge_{k=0}^{n} U_{i_k} \right) \wedge \left( \bigwedge_{k=0}^{n} \delta_{i_k, j_k} \right).$$

Since $\text{nerv}_\alpha$ is a simplex, then $\bigwedge_{k=0}^{n} \delta_{i_k, j_k} \neq 0$. It follows that $\bigwedge_{k=0}^{n} (U_{i_k} \times V_{j_k}^{i_k}) = 0 \Leftrightarrow \bigwedge_{k=0}^{n} \delta_{i_k, j_k} = 0$. Thus $\text{nerv}_\gamma = \text{nerv}_\delta$ and by Lemma 3.9, $\text{nerv}_\delta$ is an acyclic.

If family $\gamma$ is a stacked covering of fuzzy topological space $X \times I$ over base $\alpha$, then for the simplicial maps

$$l, u : \text{nerv}_\alpha \to \text{nerv}_\gamma$$

which is defined by

$$l(i) = (i, 0), \quad u(i) = (i, n^i),$$

$$l_q^* = u_q^* : H_q(\text{nerv}_\alpha) \to H_q(\text{nerv}_\gamma)$$

are obtained [3].

Proof of Theorem 3.7. We define fuzzy maps $f_0, g_0 : X \to X \times I$ for every fuzzy point $p \in X$ as $f_0(p) = (p, 0)$, $g_0(p) = (p, 1)$. Therefore if $H : X \times I \to X$ is a fuzzy homotopy between mappings $f$ and $g$, then $f = H \circ f_0$, $g = H \circ g_0$. Hence to prove the theorem, it is sufficient to show that $f_{0q} = g_{0q}$. Since stacked coverings is a confinal subset, we can use the stacked coverings to define homology groups of the fuzzy topological space $X \times I$. Let family $\gamma$ be a stacked covering of fuzzy topological space $X \times I$ over base $\alpha$. Consider the coverings $\gamma_0 = f_0^{-1}(\gamma)$, $\gamma_1 = g_0^{-1}(\gamma)$ and simplicial mappings

$$i_{f_0, \gamma} : \text{nerv}_{f_0}^{-1}(\gamma) \to \text{nerv}_\gamma, \quad i_{g_0, \gamma} : \text{nerv}_{g_0}^{-1}(\gamma) \to \text{nerv}_\gamma.$$
If maps $u': \text{nerv}_\alpha \to \text{nerv}_\gamma$, $\pi: \text{nerv}_\gamma \to \text{nerv}f_{0}^{-1}(\gamma)$ are defined by

$$u'(i) = (i, n'i), \quad \pi(i, j) = (i, 0)$$

then for mappings in (4)

$$u = i_{g_0, \gamma} \circ u', \quad l = i_{f_0, \gamma} \pi u'$$

are obtained. Further it can be observed that $(i_{g_0, \gamma})_{*q} = (i_{f_0, \gamma})_{*q}$ and consequently $f_{0*q} = g_{0*q}$.

References


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