The Localization of Strictly \( \pi \)-Regular Semigroups

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Abstract

In this paper, we prove that the localization of strictly \( \pi \)-regular semigroups on the \( \text{Reg}S \) exists and is unique, and the localization is the maximum homomorphism image of group. The minimum group congruence of strictly \( \pi \)-regular semigroups is also given.

Keywords: strictly \( \pi \)-regular semigroups; localization; the minimum group congruence

1 Introduction and preliminaries

Localization is an important tool for the exchange algebra. In 1985, the localization theory was extended to the semigroups, later in the regular semigroups and non-regular semigroups (such as: strong \( \pi \)-inverse semigroups, C-rpp Semigroups Etc.) proved that the localization on the idempotent semilattice exists. Let \( S \) is a semigroup, \( A \) is its subsemigroup, if there exist a monoid \( T \) and morphism \( f : S \rightarrow T \) such that for every \( a \in A \), \( f(a) \) in \( T \) is reversible, and \( (T, f) \) is an extensive pair in the following significance: if there exist a monoid \( Q \) and morphism \( g : S \rightarrow Q \), for every \( a \in A \), \( g(a) \) in \( Q \) is reversible, then there exists a unique morphism \( h : T \rightarrow Q \) such that \( hf = g \), then the semigroup \( T \) is called the localization of \( S \) on \( A \).[2]

Let \( S \) be a semigroup, the set of all the idempotents of \( S \) is denoted by \( E_S \), the set containing all regular elements of \( S \) is denoted \( \text{Reg}S \). A semigroup \( S \) is called \( \pi \)-regular if for every element \( a \) of \( S \) there exists \( m \in \mathbb{Z}^+ \) (the set of positive integers) such that \( a^m \) is regular. Let us denote by \( r(a) \) the least positive integer \( m \) such that \( a^m \) is regular of \( S \) and call it the regular index of an element \( a \). A \( \pi \)-regular semigroup \( S \) will be called strictly \( \pi \)-regular, if \( \text{Reg}S \) is completely regular and a ideal of \( S \).[6]
This paper proves that the localization of strictly $\pi$-regular semigroups on the \(\text{Reg}\mathcal{S}\) exists, and up to isomorphism it is unique, thereby characterizes the minimum group congruence of strictly $\pi$-regular semigroups.

## 2 The localization of strictly $\pi$-regular semigroups

Let \(S\) be a semigroup, we define the relation

\[ R = \{(ea^r(a)f(a^r(a))', (ea^r(a)f(a^r(a))')|e, f \in E_S, a \in S, (a^r(a))' \in V(a^r(a))\}, \]

the minimum congruence containing \(R\) is denoted by \(\rho^\ast\), and \(R^c = \{(xsy, xty): x, y \in S^1, (s, t) \in R\}\).

Let \(S\) be a strictly $\pi$-regular semigroup, we define the relation \(\sim\) on \(S \times \text{Reg}\mathcal{S}\)

\[(a, x) \sim (b, y) \iff (\exists g \in E_S)(gxaxg)\rho^\ast = (gybyg)\rho^\ast.\]

The relation effect of these three lemmas is as follows:

**Lemma 1.** \(\sim\) is an equivalence on \(S\).

**Proof.** It is clear that \(\sim\) is reflexive and antisymmetric. To show that it is transitive. First we show that if \((a, x) \sim (b, y)\), then for any \(s \in V(xax)\) and \(t \in V(yby)\) there exists \(f \in E_S\) such that

\[(fsf)\rho^\ast = (ftf)\rho^\ast.\]

In fact, suppose \((a, x) \sim (b, y)\), then there exists \(k \in E_S\), then \((kxaxk)\rho^\ast = (kybyk)\rho^\ast\). Since \(\text{Reg}\mathcal{S}\) is ideal, then \(xax, yby \in \text{Reg}\mathcal{S}\); hence let \(s \in V(xax), t \in V(yby)\), and let \(f_1 = ktkybyk, f_2 = kxaxksk\), obviously, \(f_1, f_2 \in \text{Reg}\mathcal{S}\). From the definition of \(\rho^\ast\), we have

\[f_1 \rho^\ast = f_1^2 \rho^\ast, \quad f_2 \rho^\ast = f_2^2 \rho^\ast, \quad kf_1 = f_1 k = f_1, \quad kf_2 = f_2 k = f_2,\]

and

\[(f_1sf_2)\rho^\ast = (f_1f_1sf_2)\rho^\ast = (f_1ktkybyksf_2)\rho^\ast = (f_1ktxaxksksf_2)\rho^\ast = (f_1ktxaxksfsf_2)\rho^\ast = (f_1tf_2)\rho^\ast.\]

Let \(h \in V((f_2f_1^2)), f = f_2f_1hf_2f_1 \in E_S\), then

\[(fsf)\rho^\ast = (f_2f_1hf_2f_1sf_2f_1hf_2f_1)\rho^\ast = (f_2f_1hf_2f_1tf_2f_1hf_2f_1f_1)\rho^\ast = (ftf)\rho^\ast.\]

Now suppose \((a, x) \sim (b, y), (b, y) \sim (c, z)\), then there exist \(u, v \in E_S\) such that

\[(uxaxu)\rho^\ast = (uybyu)\rho^\ast, \quad (vybyv)\rho^\ast = (vzczv)\rho^\ast.\]
Let \( s_1 \in V(yby), \ t_1 \in V(zcz) \), then there exists \( f \in E_S \) such that \( (fs_1f)x\rho^* = (ft_1f)x\rho^* \). Let \( w_1 \in V((uf)^2), w_2 \in V((fu)^2) \), then \( e_1 = fuv_2fu, e_2 = uv_1uf \).

Let \( h_1 = e_1zczh_2e_1, h_2 = e_2s_1e_1ybye_2, \) from the definition of \( \rho^* \), we have

\[
\begin{align*}
    h_1\rho^* &= h_1^2\rho^*, & h_2\rho^* &= h_2^2\rho^*, & h_1e_1 &= e_1h_1 = h_1, & h_2e_2 &= e_2h_2 = h_2,
\end{align*}
\]
and

\[
(\rho^*xaxh_2)\rho^* = (h_1h_1xaxh_2)\rho^* = (h_1e_1zczh_2e_1xaxh_2)\rho^* = (h_1zcz(ufv)w_f)(uxaxu)(fwufh)\rho^* = (h_1zczxh_2(ufv)w_f)(uxaxu)(fwufh)\rho^* = (h_1zczxh_2)^2\rho^*.
\]

Let \( h \in V((h_2h_1)^2) \), \( g_1 = h_2h_1h_1h_2 \), then \( g_1 \in E_S \), thus \( (g_1(xax)(g_1)\rho^* = (g_1(zcz)g_1)^2\rho^*, \) that is, \( (a, x) \sim (c, z) \).

Therefore, “\( \sim \)” is an equivalence on \( S \).

**Lemma 2.** For every \( (a, x) \in S \times \text{Reg}S \) there exist \( b \in S \) and \( h \in E_S \) such that \( (a, x) \sim (b, h) \), and \( (a, e) \sim (a, f) \) for any \( a \in S \) and \( e, f \in E_S \).

**Proof.** Suppose \( (a, x) \in (S \times \text{Reg}S)/\sim, p \in V(x^2), t \in V((xp)^2) \), and let \( f = xp^2xtxp^2x \), easy to know \( f, xp \in E_S \), and because

\[
px(xax)xp = px^2px^2ax^2p = px(xp)(xax)(xp)xp,
\]

hence \( (f(xax)f)x\rho^* = (f(xp)(xax)(xp)f)x\rho^* \). Vividly, \( (a, x) \sim (xax, xpx) \).

Let \( b = xax \) and let \( h = xp \), we have that \( (a, x) \sim (b, h) \).

Suppose \( a \in S \) and let \( e, f \in E_S \). We first show that \( (ea, e) \sim (ea, f) \).

From the definition of \( \rho^* \), we obtain that \( (efe)x\rho^*, (ff)x\rho^*, (e^2\rho^* \in E(S/\rho^*), \) so that there exist \( h_1, u, v \in E_S \) such that

\[
\begin{align*}
    h_1\rho^* &= (efe)x\rho^*, & u\rho^* &= (fe^2)x\rho^* & v\rho^* &= (efe)x\rho^* ,
\end{align*}
\]
and

\[
(ue(ea)u(ea)u(ea)u(ea)h_1(ea)u)\rho^* = (ue(ea)u(ea)u(ea)h_1(ea)u)\rho^* = (ue(ea)u(ea)u(ea)u(ea)u(ea)h_1(ea)u)\rho^* = (ue(ea)u(ea)u(ea)u(ea)u(ea)h_1(ea)u)\rho^*.
\]

Let \( x_1 = u(ea)v(ea)u, y_1 = u(ea)h_1(ea)u, t_1 \in V((y_1x_1)^2) \), and let \( w_1 = y_1x_1t_1y_1x_1 \in E_S \), then \( (w_1(ea)e)w_1)x\rho^* = (w_1(ea)f)w_1x\rho^* \), that is, \( (ea, e) \sim (ea, f) \).
Next we show that \((ea, f) \sim (af, e)\). Let \(x_2 = u(ea)u(ea)'u\) and let \(y_2 = u(ea)'u(ea)u\). Since
\[
x_2\rho^x = x_2^2 \rho^x, \quad y_2\rho^x = y_2^2 \rho^x,
\]
then
\[
(x_2(f(ea)f)y_2)\rho^x = (u(ea)u(ea)'u(f(ea)f)u(ea)'u(ea))\rho^x = (u(ea)u(ea)'ufe)\rho^x u(e(f)e)u(ea)'u(ea)u\rho^x = (u(ea)u(ea)'u(e(f)e)u(ea)'u(ea)u)\rho^x = (x_2(e(af)e)y_2)\rho^x.
\]
Hence we may let \(t_2 \in V((y_2x_2)^2)\) and let \(w_2 = y_2x_2t_2y_2x_2\). Obviously, \(w_2 \in E_S\), and we have \((w_2(f(ea)f)w_2)\rho^x = (w_2(e(af)e)w_2)\rho^x\), that is, \((ea, f) \sim (af, e)\).

Thus \((a, e) \sim (ea, e) \sim (ea, f) \sim (af, e) \sim (af, f) \sim (a, f)\), we have shown that \((a, e) \sim (a, f)\).

In the following, the equivalence class containing \((a, e)\) is denoted by \(a/e\). By Lemma 2 we know that \((S \times \text{Reg}S)/\sim = \{a/e | a \in S, e \in E_S\}\).

**Lemma 3.** \((S \times \text{Reg}S)/\sim = \{a/e | a \in S, e \in E_S\}\) is monoid where the multiplication is given by \(a/e \cdot b/e = ab/e\), and the unit element is \(e/e, e \in E_S\).

**Proof.** Next we are ready to consider that the multiplication is well-defined.

If \(a/e = a_1/e, b/e = b_1/e\), then there exist \(x, y \in E_S\) such that
\[
(xeaex)\rho^x = (xea_1e_1x)\rho^x, \quad (yebey)\rho^x = (yeb_1e_1y)\rho^x.
\]
Let \(s = ea, t = ea_1, u = be, v = b_1e\), we have
\[
(exesexc)\rho^x = (exetcx)\rho^x, \quad (yeyeueye)\rho^x = (yeyeve)\rho^x.
\]
Suppose that \(k_1 \in V((ex)^2), k_2 \in V((ey)^2)\) and let \(l_1 = exek_1ex, l_2 = eyek_2ey\), clearly, \(l_1, l_2 \in E_S\), and we deduce that
\[
(l_1sl_1)\rho^{x} = (l_1tl_1)\rho^{x}, \quad (l_2ul_2)\rho^{x} = (l_2vl_2)\rho^{x}.
\]
To show that \(ab/e = a_1b_1/e\), we must first show that \(ab/e = a_1b_1/e\). Let \(s' \in V(ese), t' \in V(ete)\). Since
\[
(l_1sel_1)\rho^{x} = (l_1sl_1)\rho^{x} = (l_1tl_1)\rho^{x} = (l_1etel_1)\rho^{x},
\]
then there exists \(f \in E_S\) such that \((fs'f)\rho^x = (ft'f)\rho^x\). Let \(u' \in V(u), p \in V((fu)u)^2, q \in V((uu'f)^2)\), and let \(f_1 = (fupfu), f_2 = (uu'fquu'f)\). Apparently, \(f_1, f_2 \in E_S\). Let \(g_1 = f_1f_2f'_1f_1, g_2 = f_2sf_1sf_2\), then by the definition of \(\rho^x\), we can know
\[
g_1\rho^x = g_1^2 \rho^x, \quad g_2\rho^x = g_2^2 \rho^x, \quad g_1f_1 = f_1g_1 = g_1, \quad g_2f_2 = f_2g_2 = g_2.
\]
thus
\[
(g_1 suu' g_2)\rho^\sharp = (g_1 g_1 suu' g_2)\rho^\sharp = (g_1 f_1 t f_2 t' f_1 suu' g_2)\rho^\sharp \\
= (g_1 t f_2 f t' f f_1 s f_2 g_2)\rho^\sharp = (g_1 t f_2 s f f_1 s f_2 g_2)\rho^\sharp \\
= (g_1 tuu' f_2 s f f_1 s f_2 g_2)\rho^\sharp = (g_1 tuu' g_2)\rho^\sharp \\
= (g_1 tuu' g_2)\rho^\sharp.
\]

Let \( g' \in V((u' g_2 g_1)^2) \), and let \( w = u' g_2 g_1 g' u' g_2 g_1 \), then \( w \in E_S \), and \( (wsuw)\rho^\sharp = (w(tu)w)\rho^\sharp \). Since \( s = ea \), \( t = ea_1 \), \( u = be \), hence \( (w(eabe)w)\rho^\sharp = (w(ea_1 be)w)\rho^\sharp \), that is, \( ab/e = a_1 b/e \).

Similarly, \( a_1 b/e = a_1 b_1/e \). So \( ab/e = a_1 b_1/e \). Thus the multiplication is well-defined and satisfies associative law. It is clear that \( (ea, e) \sim (ae, e) \), thus we obtain that \( e/e \) is the identity element of \( (S \times \text{Reg}S)/\sim \).

Therefore, \( (S \times \text{Reg}S)/\sim \) is monoid where the multiplication is \( a/e \cdot b/e = ab/e \).

Then we have the following result:

**Theorem 4.** Let \( S \) be a strictly \( \pi \)-regular semigroup, then the localization of \( S \) on the \( \text{Reg}S \) exists and is unique. Moreover the localization is the maximum homomorphism image of groups of \( S \).

**Proof.** By Lemma 3, \( (S \times \text{Reg}S)/\sim \) is monoid. We define the natural map \( \varphi^\sharp \) from \( S \) onto \( (S \times \text{Reg}S)/\sim \) by \( a\varphi^\sharp = a/e \). For every \( a/e, b/e \in (S \times \text{Reg}S)/\sim \), we have \( \varphi^\sharp(a) \cdot \varphi^\sharp(b) = a/e \cdot b/e = ab/e = \varphi^\sharp(ab) \), then \( \varphi^\sharp \) is morphism and onto. For every \( x \in \text{Reg}S \), let \( x' \in V(x) \), then \( x/e \cdot x'/e = xx'/e = e/e \), thus \( x'/e \) is an inverse element of \( x/e \). We come to conclusion \( \varphi^\sharp(x) = x/e \) is reversible in \( (S \times \text{Reg}S)/\sim \) for any \( x \in \text{Reg}S \), and \( \varphi^\sharp(e) \) is identity element of \( (S \times \text{Reg}S)/\sim \).

If \( S' \) is monoid and there exists a morphism \( \psi : S \rightarrow S' \) such that \( \psi(x) \) is reversible in \( S' \) for every \( x \in \text{Reg}S \), then \( \psi(e) \) is identity element of \( S' \) for every \( e \in E_S \).

Let
\[
\xi : (S \times \text{Reg}S)/\sim \rightarrow S'
\]
\[
a/e \mapsto \psi(a).
\]
Suppose \( a/e = b/e \), now we show that \( \psi(a) = \psi(b) \). According to \( a/e = b/e \), we know there exists \( x \in E_S \) such that \( (xeaex)\rho^\sharp = (xebex)\rho^\sharp \). If \( xeaex = xebex \), clearly, \( \psi(a) = \psi(b) \), for \( \psi(x) \), \( \psi(e) \) is reversible in \( S' \). If \( xeaex \neq xebex \), then \( (xeaex, xebex) \in \rho^\sharp \), there exist \( z_1, z_2, \ldots, z_{n-1} \in S \) such that
\[
(xeaex, z_1), \ (z_1, z_2), \ \cdots, \ (z_{n-1}, xebex) \in R^c \bigcup (R^{-1})^c \bigcup 1_S.
\]
If \( (z_i, z_{i+1}) \in 1_S \), clearly, \( \psi(z_i) = \psi(z_{i+1}) \). Otherwise \( (z_i, z_{i+1}) \in R^c \bigcup (R^{-1})^c \). Suppose \( (z_i, z_{i+1}) \in R^c \), then there exist \( s, t \in S^1 \), \( d \in S \), \( (d^{r(d)})' \in V(d^{r(d)}) \),
and $f \in E_S$ such that
\[(z_i, z_{i+1}) = (sed^{r(d)} f (d^{r(d)})' t, s(ed^{r(d)} f (d^{r(d)})' t))^2 t).
\]

According to for all $x \in \text{Reg} S$, $\psi(x)$ is reversible in $S'$, we can know that
\[\psi(z_i) = \psi(sed^{r(d)} f (d^{r(d)})' t) = \psi(st), \quad \psi(z_{i+1}) = \psi(s(ed^{r(d)} f (d^{r(d)})' t)^2 t) = \psi(st).
\]

Then we get $\psi(z_i) = \psi(z_{i+1})$. Similarly, if $(z_i, z_{i+1}) \in (R^{-1})^c$, we can show $\psi(z_i) = \psi(z_{i+1})$. Therefore, $\psi(a) = \psi(b)$. So $\xi$ is homomorphism of monoids.

Let $a \in S$, then $\xi \varphi^z(a) = \xi(\varphi^z(a)) = \xi(a/e) = \psi(a)$, so $\xi \varphi^z = \psi$. If there exists homomorphism of monoid $\xi' : (S \times \text{Reg} S)/\sim \to S'$ such that $\xi' \varphi^z = \psi$. We have $\xi'(a/e) = \xi'(\varphi^z(a)) = \xi' \varphi^z(a) = \psi(a) = \xi(a/e)$ for any $a/e \in (S \times \text{Reg} S)/\sim$, hence $\xi' = \xi$. Thus $(S \times \text{Reg} S)/\sim$ is the localization of $S$ on the $\text{Reg} S$.

Let us show that the localization is unique. Suppose morphism $\varphi' : S \to C$ where the semigroup $C$ is the localization of $S$ on the $\text{Reg} S$. According to the definition of the localization, we deduce that there exists the unique morphism $\chi : (S \times \text{Reg} S)/\sim \to C$ such that $\chi \varphi^z = \varphi'$. Since $C$ is the localization of $S$, we know that there exists the unique morphism $\phi : C \to (S \times \text{Reg} S)/\sim$ such that $\phi \varphi' = \varphi^z$, thus $\phi \chi \varphi^z = \varphi^z$, and $\chi \phi \varphi' = \varphi'$, we have $\phi \chi$ is the identity morphism of $(S \times \text{Reg} S)/\sim$, $\chi \phi$ is the identity morphism of $C$, so $\phi$ and $\chi$ are mutually inverse. That is, $C \cong (S \times \text{Reg} S)/\sim$. Therefore, up to isomorphism the localization is unique.

Suppose $(a^{r(a)})' \in V(a^{r(a)})$), then we have $(a^{r(a)})' a^{r(a)} - 1/e \cdot a/e = e/e$, so $(a^{r(a)})' a^{r(a)} - 1/e$ is an inverse element of $a/e$, hence $(S \times \text{Reg} S)/\sim$ is a group. Suppose morphism $\phi : S \to G$ where $G$ is another homomorphism image of group of $S$, by the definition of the localization, we can know that there exists the unique morphism $\phi' : (S \times \text{Reg} S)/\sim \to G$ such that $\phi' \varphi^z = \phi$. So $(S \times \text{Reg} S)/\sim$ is the maximum homomorphism image of group.

We now immediately deduce

**Corollary 5.** Let $S$ be a strictly $\pi$-regular semigroup, then

$$\beta = \{(a, b) \in S \times S : \text{there exists } e \in E_S \text{ such that } (eae) \varphi^z(ebe)\}$$

is the minimum group congruence of $S$.

**Proof.** By the theorem 4, we obtain the localization $(S \times \text{Reg} S)/\sim$ on $\text{Reg} S$ is the maximum homomorphism image of group, and it is clear that $\beta = \ker \varphi^z$. 
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References


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