An Approximation Model of Continuous Functions

S. Jahedi, M. J. Mehdipour and R. Rafizadeh

Department of Mathematics
Shiraz University of Technology
Shiraz 71555-313, Iran
jahedi@sutech.ac.ir
mehdipour@sutech.ac.ir
math0011@gmail.com

Abstract

Fuzzy transform is a powerful tool for approximation of continuous functions. The paper aims at constructing approximation models on the basis of a generalization of fuzzy partition. We will prove the best approximation properties in the corresponding approximation spaces.

Mathematics Subject Classification: 41A30, 41A45

Keywords: Fuzzy partition, fuzzy transform, approximation theory

1 Introduction

In classical mathematics, various kinds of transforms (Laplace, Fourier,...) are used in methods for construction of approximation models.

Fuzzy transform is a powerful tool for approximation of continuous functions. This method has been developed by I.Perfilieva [2]. By the help of fuzzy transform, ordinary and partial differential equation can be approximately solved [3]. Fuzzy transform, also, can be applied to data compression [4].

This paper generalizes the technique of [1] by generalizing fuzzy partition to weighted partition. In this method we do not limit ourselves to spatial partition by normal fuzzy subsets. We use an arbitrary continuous function $\phi$ from $[a, b]$ (the universe set) into $(0, 1]$ as a weight and then we construct the weighted fuzzy partition. We introduce weighted fuzzy transform, its inverse, and prove the convergence of approximated function to the original function.
2 Main Results

First of all we introduce the concept of weighted fuzzy partition and weighted fuzzy transform. Let us introduce weighted fuzzy partition of \([a, b]\).

**Definition 2.1** Let \(\phi\) be a continuous function from \([a, b]\) into \((0, 1]\) and \(x_0 = x_1 < ... < x_n = x_{n+1}\) be a partition of \([a, b]\) with \(x_1 = a, \ x_n = b\). We say that \(B_1, ..., B_n\) is a weighted fuzzy partition of \([a, b]\) if the following statements hold.

(i) \(B_k \in C([a, b])\), for all \(k = 1, ..., n\).
(ii) \(B_k(x_k) = \phi(x_k)\), for all \(k = 1, ..., n\).
(iii) \(B_k(x) = 0\) whenever \(x \notin (x_{k-1}, x_{k+1})\), for all \(k = 1, ..., n\).
(iv) \(\sum_{k=1}^{n} B_k(x) = \phi(x)\) for all \(x \in [a, b]\).

We say that it is uniform if the nodes \(x_1, ..., x_n\), \(n \geq 3\), are equidistant. This means that \(x_k = a + \Delta(k - 1), k = 1, ..., n\), where \(\Delta = \frac{b-a}{n-1}\).

Let us to introduce the weighted fuzzy transform.

**Definition 2.2** Let \(\phi\) be a continuous function from \([a, b]\) into \((0, 1]\) and \(B_1, ..., B_n\) be a weighted fuzzy partition of \([a, b]\). If \(f \in C([a, b])\) then the \(n\)-tuple of real numbers \([F_1, ..., F_n]\) given by
\[
F_k = \frac{\int_{a}^{b} f(x)B_k(x)dx}{\int_{a}^{b} B_k(x)dx}, \quad k = 1, ..., n
\]
will be called weighted fuzzy transform of \(f\) with respect to \(B_1, ..., B_n\).

From linearity of integration, we obtain that weighted fuzzy transform is a linear mapping from \(C([a, b])\) into \(\mathbb{R}^n\) so that
\[
F[\alpha f + \beta g] = \alpha F[f] + \beta F[g]
\]
for \(\alpha, \beta \in \mathbb{R}^n\) and function \(f, g \in C([a, b])\).

We claim that the components of weighted fuzzy transform of a function from \(C([a, b])\) are weighted mean values of its function, where the weights are given by \(B_k\) for \(k = 1, ..., n\). We establish this claim in the following theorem.

**Theorem 2.3** Let \(B_1, ..., B_n\) be a weighted fuzzy partition of \([a, b]\) and \(f \in C([a, b])\). Then \(F_k = \min G_k(y)\) where
\[
G_k(y) = \int_{a}^{b} (f(x) - y)^2 B_k(x)\ dx.
\]
defined between values of \(f(a)\) and \(f(b)\).
Proof. Since \((f(x) - y)^2B_k(x)\) is continuously differentiable with respect to \(y\), it follows that
\[
G'(y) = -2 \int_a^b (f(x) - y)B_k(x) \, dx.
\]
So if \(G'(y) = 0\), then
\[
y = \frac{\int_a^b f(x)B_k(x) \, dx}{\int_a^b B_k(x) \, dx}.
\]
Now we only need to note that \(F_k = y = \min G_k(y)\). □

**Proposition 2.4** Let \(A_1, ..., A_n\) be a basic function of F-transform on \([a, b]\) and \(\phi\) be a continuous map from \([a, b]\) into \((0, 1]\). Then \(A_1, ..., A_n\) is a weighted fuzzy partition if and only if \(\phi = 1\).

**Proof.** Let \(x \in [a, b]\). Since \(A_1, ..., A_n\) is a basic function, \(\sum_{k=1}^n A_k(x) = 1\). Now, if \(A_1, ..., A_n\) is a weighted fuzzy partition, then \(\sum_{k=1}^n A_k(x) = \phi(x)\) and so \(\phi(x) = 1\). The converse is clear. □

Following proposition shows that how the basic function in fuzzy transform which has been defined in [1] can be embed into a larger space.

**Proposition 2.5** Let \(\phi\) be a continuous function from \([a, b]\) into \((0, 1]\). Then the set of all basic functions can be embeded into the set of all weighted fuzzy partition.

**Proof.** Let \(A_1, ..., A_n\) be basic functions and suppose that \(a = x_1 < ... < x_n = b\) are nodes of the partition of \([a, b]\). Define
\[
B_1(x) = \begin{cases} 
\phi(x_1)A_1(x) & x \in [x_1, x_2] \\
0 & \text{otherwise}
\end{cases}
\]
\[
B_k(x) = \begin{cases} 
\phi(x) - \phi(x_{k-1})A_{k-1}(x) & x \in [x_{k-1}, x_k] \\
\phi(x_k)A_k(x) & x \in [x_k, x_{k+1}] \\
0 & \text{otherwise}
\end{cases}
\]
for \(k = 2, ..., n - 1\), and
\[
B_n(x) = \begin{cases} 
\phi(x) - \phi(x_{n-1})A_{n-1}(x) & x \in [x_{n-1}, x_n] \\
0 & \text{otherwise}
\end{cases}
\]
Clearly \(B_1, ..., B_n\) is a weighted fuzzy partition of \([a, b]\). □
Recall that the modulus of continuity of $f$,
\[
\omega(\Delta, f) = \max_{|\delta| \leq \Delta} \max_{x \in [a, b - \delta]} |f(x + \delta) - f(x)|.
\]

Now we investigate the relation between components of weighted fuzzy transform and smoothness of the given function $f$. On the other hand we want to estimate the components of weighted fuzzy transform of a function with respect to its smoothness.

**Theorem 2.6** Let $B_1, \ldots, B_n$, $n \geq 3$, be a weighted fuzzy partition of $[a, b]$. Then the following statements hold.

(i) $|f(t) - F_1| < \omega(\max_{2 \leq k \leq n}\{|x_k - x_{k-1}|\}, f)$ for all $t \in [x_1, x_2]$.

(ii) $|f(t) - F_k| < \omega(2\max_{2 \leq k \leq n}\{|x_k - x_{k-1}|\}, f)$ for all $t \in [x_{k-1}, x_{k+1}]$ and $k = 2, \ldots, n-1$.

(iii) $|f(t) - F_n| < \omega(\max_{2 \leq k \leq n}\{|x_k - x_{k-1}|\}, f)$ for all $t \in [x_{n-1}, x_n]$.

where $\omega(\max_{2 \leq k \leq n}\{|x_k - x_{k-1}|\}, f)$ is the modulus of continuity of function $f$ with respect to $\max_{2 \leq k \leq n}\{|x_k - x_{k-1}|\}$.

**Proof.** First let us prove the theorem for fix $k$ in the range $1 \leq k \leq n-1$, and let $t \in [x_k, x_{k+1}]$.

\[
|f(t) - F_k| = \left| \int_{x_{k-1}}^{x_{k+1}} \frac{B_k(x)}{B_k(x)} f(t) - \frac{f(x)B_k(x)}{B_k(x)} \, dx \right| \leq \int_{x_{k-1}}^{x_{k+1}} |f(t) - f(x)| \frac{B_k(x)}{B_k(x)} \, dx.
\]

Since $t \in [x_{k-1}, x_{k+1}]$, we have $|t-x| \leq 2\max_{2 \leq k \leq n}\{|x_k - x_{k-1}|\}$. With $\delta = t-x$

\[
|f(t) - F_k| \leq \max_{|\delta| \leq 2\max_{2 \leq k \leq n}\{|x_k - x_{k-1}|\}} \max_{x \in [a, b - \delta]} |f(x + \delta) - f(x)| = \omega(2\max_{2 \leq k \leq n}\{|x_k - x_{k-1}|\}, f).
\]

Similarly, (i) and (iii) will be proved. □

Now we introduce the inverse weighted fuzzy transform.

**Definition 2.7** The function

\[
f_{F,n}(x) = \sum_{k=1}^{n} F_k \frac{B_k(x)}{\phi(x)}
\]

is called the inverse weighted fuzzy transform of $f \in C([a, b])$ with respect to weighted fuzzy partition $B_1, \ldots, B_n$. 
Theorem 2.8 Let \( f \in C([a, b]) \) and \( \phi \) be a continuous function on \([a, b]\) into \((0, 1]\). Then for any positive \( \varepsilon \) there exist \( n_\varepsilon \) and a weighted fuzzy partition \( B_1, ..., B_{n_\varepsilon} \) such that \( |f(x) - f_{F,n}(x)| < \varepsilon \) for all \( x \in [a, b] \).

Proof. Fix \( \varepsilon \geq 0 \). By the fact that \( f \) is uniformly continuous on \([a,b]\) we can find \( \delta(\varepsilon) \geq 0 \) and \( a = x_1 < ... < x_n = b \) such that \( |f(x) - f(y)| < \varepsilon \) whenever \( x, y \in [x_{k-1}, x_{k+1}] \), and \( 2 \leq k \leq n - 1 \).

Choose the weighted fuzzy partition \( B_1, ..., B_n \) on \([a, b]\). Put \( n = n_\varepsilon \), then for \( 1 \leq k \leq n_\varepsilon - 1 \) and \( t \in [x_k, x_{k+1}] \),

\[
|f(t) - F_k| = |f(t) - \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)B_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} B_k(x)dx}|
\leq \frac{\int_{x_{k-1}}^{x_{k+1}} |f(t) - f(x)|B_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} B_k(x)dx}
< \varepsilon.
\]

Similar computation shows that \( |f(t) - F_{k+1}| < \varepsilon \). By the above argument we have

\[
|f(t) - \sum_{i=1}^{n_\varepsilon} F_i \frac{B_i(t)}{\phi(t)}| = |f(t)\sum_{i=1}^{n_\varepsilon} \frac{B_i(t)}{\phi(t)} - \sum_{i=1}^{n_\varepsilon} F_i \frac{B_i(t)}{\phi(t)}|
\leq \sum_{i=1}^{n_\varepsilon} |f(t) - F_i| \frac{B_i(t)}{\phi(t)}
< \varepsilon \sum_{i=1}^{n_\varepsilon} \frac{B_i(t)}{\phi(t)} = \varepsilon.
\]

So the proof is complete. \( \square \)

References


Received: July, 2010