On Fully Pseudo \((m, n)\)-Stable Modules

Muna J. MohammedAli

Department of Mathematics, College of Science for Women
University of Baghdad, Baghdad, Iraq
a73n79a80@yahoo.com

Abstract

Let \(R\) be a commutative ring with non-zero identity element. For two fixed positive integers \(m\) and \(n\). A right \(R\)-module \(M\) is called fully pseudo \((m, n)\)-stable, if \(\theta(N) \subseteq N\) for each \(n\)-generated submodule \(N\) of \(M^m\) and \(R\)-monomorphism \(\theta : N \rightarrow M^m\). In this paper we give some characterization theorems and properties of fully pseudo \((m, n)\)-stable modules which generalize the results of fully pseudo stable modules.

Mathematics Subject Classification(2000): 13C99, 13C11

Keywords: fully \((m, n)\)-stable module, fully pseudo \((m, n)\)-stable module, \((m, n)\)-multiplication module.

1 Introduction

Throughout, \(R\) is a commutative ring with non-zero identity and all modules are unitary. We use the notation \(R^{m \times n}\) for the set of all \(m \times n\) matrices over \(R\). For \(A \in R^{m \times n}\), \(A^T\) will denote the transpose of \(A\). In general, for an \(R\)-module \(N\), we write \(N^{m \times n}\) for the set of all formal \(m \times n\) matrices whose entries are elements of \(N\). Let \(M\) be a right \(R\)-module and \(N\) be a left \(R\)-module. For \(x \in M^{l \times m}\), \(s \in R^{m \times n}\) and \(y \in N^{n \times k}\), under the usual multiplication of matrices, \(xs\) (resp. \(sy\)) is a well defined element in \(M^{l \times m}\) (resp. \(N^{n \times k}\)). If \(X \in M^{l \times m}\), \(S \in R^{m \times n}\) and \(Y \in N^{n \times k}\), define

\[
\ell_{M^{l \times m}}(S) = \{u \in M^{l \times m} : us = 0, \forall s \in S\}
\]

\[
r_{N^{n \times k}}(S) = \{v \in N^{n \times k} : sv = 0, \forall s \in S\}
\]

\[
\ell_{R^{m \times n}}(Y) = \{s \in R^{m \times n} : sy = 0, \forall y \in Y\}
\]

\[
r_{R^{m \times n}}(X) = \{s \in R^{m \times n} : xs = 0, \forall x \in X\}
\]
We will write $N^n = N^{1 \times n}$, $N_n = N^{n \times 1}$. Fully pseudo stable module have been discussed in [1], an $R$-module $M$ is called fully pseudo stable, if $\theta(N) \subseteq N$ for each submodule $N$ of $M$ and $R$-monomorphism $\theta$ from $N$ into $M$. It is an easy matter to see that $M$ is fully pseudo stable, if and only if $\theta(xR) \subseteq xR$ for each $x$ in $M$ and $R$-monomorphism $\theta : xR \to M$. In this paper, for two fixed positive integers $m$ and $n$, we introduce the concepts of fully pseudo $(m, n)$-stable modules, if and only if the distinct $n$-generated submodules of $M^m$ are not isomorphic.

2 Results

Definition 2.1 An $R$-module $M$ is called fully pseudo $(m, n)$-stable, if $\theta(N) \subseteq N$ for each $n$-generated submodule $N$ of $M^m$ and $R$-monomorphism $\theta : N \to M^m$. The ring $R$ is fully pseudo $(m, n)$-stable, if $R$ is fully pseudo $(m, n)$-stable as $R$-module.

An $R$-module $M$ is fully pseudo $(m, n)$-stable, if and only if for each $R$-monomorphism $\theta : N(= \sum_{i=1}^{n} \alpha_i R) \to M^m$ (where $\alpha_i \in M^m$) and each $w \in N$, there exists $t = (t_1, \ldots, t_n) \in R^n$ such that $\theta(w) = \sum_{i=1}^{n} \alpha_i t_i = (\alpha_1, \ldots, \alpha_n)t^T$, if $r = (r_1, \ldots, r_n) \in R^n$, then $\theta((\alpha_1, \ldots, \alpha_n)r^T) = (\alpha_1, \ldots, \alpha_n)t^T$.

It is clear that $M$ is fully pseudo $(1, 1)$-stable, if and only if $M$ is fully pseudo stable.

It is an easy matter to see that an $R$-module $M$ is fully pseudo $(m, n)$-stable, if and only if it is fully pseudo $(m, q)$-stable for all $1 \leq q \leq n$, if and only if it is fully pseudo $(p, n)$-stable for all $1 \leq p \leq m$, if and only if it is fully pseudo $(p, q)$-stable for all $1 \leq p \leq m$ and $1 \leq q \leq n$.

Recall that an $R$-module $M$ is called fully $(m, n)$-stable, if $\theta(N) \subseteq N$ for each $n$-generated submodule $N$ of $M^m$ and $R$-homomorphism $\theta : N \to M^m$. The ring $R$ is fully $(m, n)$-stable, if $R$ is fully $(m, n)$-stable as $R$-module. It is clear that every fully $(m, n)$-stable is fully pseudo $(m, n)$-stable. But the converse is not true.

Recall that an $R$-module $M$ is uniform if any non-zero submodules of $M$ has non-zero intersection.

Proposition 2.2 Every uniform fully pseudo $(m, n)$-stable $R$-module is fully $(m, n)$-stable
**proof.** let $M$ be fully pseudo $(m, n)$-stable module. For any $n$-generated submodule $N$ of $M^m$ and $R$-homomorphism $\theta : N \to M^m$. If $ker\theta = 0$, nothing to prove. Otherwise let $x \in ker \theta \cap ker(I_N + \theta)$, then $\theta(x) = 0$ and $(I_N + \theta)(x) = 0$. Now $x = x + \theta(x) = (I_N + \theta)(x) = 0$. Thus $ker \theta \cap ker(I_N + \theta) = 0$. But $M$ is uniform, hence $ker(I_N + \theta) = 0$, that is $(I_N + \theta) : N \to M^m$ is an $R$-monomorphism. Since $M$ is fully pseudo $(m, n)$-stable, then $(I_N + \theta)(N) \subseteq (N)$, hence $\theta(N) \subseteq (N)$.

**Corollary 2.3** [1, Proposition 2.2] Every uniform fully pseudo stable $R$-module is fully stable module.

**Theorem 2.4** An $R$-module $M$ is fully pseudo $(m, n)$-stable, if and only if distinct $n$-generated submodules of $M^m$ are not isomorphic.

**proof.** suppose that distinct $n$-generated submodules of $M^m$ are not isomorphic, and there exists an $n$-generated submodule $N$ of $M^m$ and $R$-monomorphism $\theta : N \to M^m$ such that $\theta(N) \not\subseteq N$, then $N$ and $\theta(N)$ are two distinct $n$-generated submodules of $M^m$. By assumption, then $\theta(N)$ is not isomorphic to $N$ which is an absurd. Conversely, suppose that $M$ is a fully pseudo $(m, n)$-stable and $M$ has two $n$-generated submodules $N_1$ and $N_2$ such that $N_1 \not\subseteq N_2$. No loss of generality if it is assumed that $N_1 \not\subseteq N_2$. There exists a non-zero element $x$ in $N_1$ not in $N_2$. Let $\theta : N_1 \to N_2$ be an isomorphism, consider the following two $R$-monomorphism $i_{N_1} \circ \theta : N_1 \to M^m$ and $i_{N_1} \circ \theta^{-1} : N_2 \to M^m$. Since $M$ is fully pseudo$(m, n)$-stable, then $(i_{N_2} \circ \theta)(N_1) \subseteq N_1$ and $(i_{N_1} \circ \theta^{-1})(N_2) \subseteq N_2$. Now $x = (i_{N_1} \circ \theta^{-1} \circ i_{N_2} \circ \theta)(x) \in N_2$ which is contradiction.

**Corollary 2.5** [1, Proposition 2.4] An $R$-module $M$ is fully pseudo-stable, if and only if distinct cyclic submodules of $M$ are not isomorphic.

**Corollary 2.6** Let $M$ be a uniform $R$-module. Then $M$ is fully $(m, n)$-stable, if and only if distinct $n$-generated submodules of $M^m$ are not isomorphic.

In [7], prove that, let $M$ be a right $R$-module and $I_R$ an $n$-generated submodule of $R^m_R$, then $\ell_{M^n}(I) \cong Hom_R(R^n/I, M)$.

**Theorem 2.7** Let $M$ be an $R$-module. Then the following statement are equivalent.
1. \( M \) is fully \((m, n)\)-stable.

2. distinct \( n \)-generated submodules of \( M^m \) are not isomorphic and \( \sum_{i=1}^{n} \alpha_i R \cong \text{Hom}_R(\sum_{i=1}^{n} \alpha_i R, M) \) for each \( n \)-elements subset \( \{\alpha_1, \ldots, \alpha_n\} \) of \( M^m \)

**proof.** Assume that \( M \) is fully \((m, n)\)-stable \( R \)-module, then the distinct \( n \)-generated submodules of \( M^m \) are not isomorphic. By \([2, \text{proposition (2.9)}]\), for each \( n \)-element subset \( \{\alpha_1, \ldots, \alpha_n\} \subset M^m \), we have

\[
\ell_{M^m}(\alpha_1 R + \ldots + \alpha_n R) \cong \text{Hom}_R(\sum_{i=1}^{n} \alpha_i R, M)
\]

Conversely, assume that distinct \( n \)-generated submodules of \( M^m \) are not isomorphic and \( \alpha_1 R + \ldots + \alpha_n R \cong \text{Hom}_R(\sum_{i=1}^{n} \alpha_i R, M) \). By \([6]\), we have

\[
\ell_{M^m}(r_{R^m}(\alpha_1 R + \ldots + \alpha_n R)) \cong \text{Hom}_R((\alpha_1 R + \ldots + \alpha_n R), M)
\]

Hence \( M \) is fully \((m, n)\)-stable.

**Corollary 2.8** \([1, \text{Theorem 2.8}]\) Let \( M \) be an \( R \)-module. Then the following statement are equivalent.

1. \( M \) is fully-stable.

2. distinct cyclic submodules of \( M \) are not isomorphic and \( xR \cong \text{Hom}_R(xR, M) \) for each \( x \in M \)

**Corollary 2.9** Let \( M \) be an \( R \)-module. Then the following statement are equivalent.

1. \( M \) is a fully \((m, n)\)-stable.

2. \( M \) is a fully pseudo \((m, n)\)-stable and \( \sum_{i=1}^{n} \alpha_i R \cong \text{Hom}_R(\sum_{i=1}^{n} \alpha_i R, M) \) for each \( n \)-elements subset \( \{\alpha_1, \ldots, \alpha_n\} \) of \( M^m \)

**Proposition 2.10** Let \( M \) be an \( R \)-module. Then the following statement are equivalent.

1. distinct \( n \)-generated submodules of \( M^m \) are not isomorphic.

2. \( r_{R^m}\{\alpha_1, \ldots, \alpha_n\} = r_{R^m}\{\beta_1, \ldots, \beta_n\} \) for each \( n \)-element two subsets \( \{\alpha_1, \ldots, \alpha_n\} \) and \( \{\beta_1, \ldots, \beta_n\} \) of \( M^n \) implies that \( \alpha_1 R + \ldots + \alpha_n R = \beta_1 R + \ldots + \beta_n R \)
proof. Assume that (1) is true. Define \( \theta : \alpha_1 R + \cdots + \alpha_n R \to \beta_1 R + \cdots + \beta_n R \) by \( \theta(\sum_{i=1}^n \alpha_i r_i) = \sum_{i=1}^n \beta_i r_i \) for each \( r_i \in R \) and \( i = 1, \ldots, n \). Because \( r_{R_n}\{\alpha_1, \ldots, \alpha_n\} = r_{R_n}\{\beta_1, \ldots, \beta_n\} \), then \( \theta \) is an isomorphism, then by (1), we have \( \alpha_1 R + \cdots + \alpha_n R = \beta_1 R + \cdots + \beta_n R \). Conversely, assume that (2) holds, and there exists two \( n \)-generated submodules \( \alpha_1 R + \cdots + \alpha_n R \neq \beta_1 R + \cdots + \beta_n R \) with \( \alpha_1 R + \cdots + \alpha_n R \cong \beta_1 R + \cdots + \beta_n R \), then without lose of generality there exists an element \( w \in \alpha_1 R + \cdots + \alpha_n R \) and \( w \notin \beta_1 R + \cdots + \beta_n R \). Let \( f : \alpha_1 R + \cdots + \alpha_n R \to \beta_1 R + \cdots + \beta_n R \) be isomorphism. Now \( w \neq f(w) \), otherwise \( w \in \beta_1 R + \cdots + \beta_n R \). We claim that \( r_{R_n}(w) = r_{R_n}(f(w)) \). For let \( \eta \in r_{R_n}(w) \), then \( w\eta = 0 \), hence \( f(w)\eta = 0 \), thus \( r_{R_n}(w) \subseteq r_{R_n}(f(w)) \). Let \( \zeta \in r_{R_n}(f(w)) \), then \( 0 = f(w)\zeta = f(w\zeta) \), \( w\zeta \in \ker(f) = 0 \), \( w\zeta = 0 \) or \( \zeta \notin r_{R_n}(w) \). Therefore \( r_{R_n}(w) = r_{R_n}(f(w)) \). But \( Rw \neq Rf(w) \) which is a contradiction.

Corollary 2.11 [1, Proposition 2.11]Let \( M \) be an \( R \)-module . Then the following statements are equivalent.

1. distinct cyclic submodules are not isomorphic.
2. \( r_R(x) = r_R(y) \) implies \( Rx = Ry \) for each \( x, y \in M \)

S. K. Jain and S. Singh in[4] introduced the concept of a pseudo-injective module . An \( R \)-module \( M \) is said to be pseudo-injective, if each \( R \)-monomorphism \( \theta : N \to M \) of any submodule \( N \) of \( M \) can be extended to an \( R \)-endomorphism of \( M \). An \( R \)-module \( M \) is said to be principally pseudo-injective, if each \( R \)-monomorphism from cyclic submodule \( N \) of \( M \) can be extended to an \( R \)-endomorphism of \( M \) [4].

Lemma 2.12 Every fully pseudo-stable module is principally pseudo-injective module.

proof. Is clear

Motivated by concept of principally pseudo-injective, we introduce the following definition.

Definition 2.13 An \( R \)-module \( M \) is called \((m, n)\)-pseudo injective, if each \( R \)-monomorphism from \( n \)-generated submodule of \( M^m \) to \( M \) can be extended to an \( R \)-homomorphism from \( M^m \) to \( M \).

It is clear that \( M \) is principally pseudo-injective, if and only if \( M \) is \((1, 1)\)-pseudo injective. An \( R \)-module \( M \) is called \( n \)-pseudo injective if it is \((1, n)\)-pseudo injective for all positive integers \( n \).
It is an easy matter to see that an R-module M is \((m, n)\)-pseudo injective, if and only if it is \((m, q)\)-pseudo injective for all \(1 \leq q \leq n\), if and only if it is \((p, n)\)-pseudo injective for all \(1 \leq p \leq m\), if and only if it is \((p, q)\)-pseudo injective for all \(1 \leq p \leq m\) and \(1 \leq q \leq n\).

In [9] prove the following proposition. Write \(A_m = \{ n \in M \mid r_R(n) = r_R(m) \}\) and \(B_m = \{ \alpha \in S \mid \kappa \alpha \cap mR = 0 \}\) for each \(m \in M\).

**Proposition 2.14** Let \(M\) be an R-module. The following statements are equivalent for each \(m \in M\):

1. \(M\) is principally pseudo injective
2. \(A_m = B_m\)
3. If \(A_m = A_n\) then \(B_m m = B_n n\).
4. For every R-monomorphism \(\alpha : 0 \to mR \to M\) and \(\beta : 0 \to mR \to M\), there exists \(\gamma \in \text{End}(M_R)\) such that \(\alpha = \gamma \beta\).

Let \(M\) be an R-module, \(\alpha_i\) be a non-zero element in \(M^n\), \(i = 1, \ldots, m\). We write \(A_{\alpha_i} = \{ \beta_i \in M^n \mid r_{R^n}(\beta_i) = r_{R^n}(\alpha_i) \}\) and \(B_{\alpha_i} = \{ c \in S_n \mid r_{R^n}(c) \cap \alpha_i R = 0 \}\).

**Theorem 2.15** Let \(M\) be an R-module. The following statements are equivalent:

1. \(M\) is \((m, n)\)-pseudo injective.
2. \(A_{\alpha_i} = B_{\alpha_i} \alpha_i\), for each \(\alpha_i \in M^n\).
3. \(A_{\alpha_i} = A_{\beta_i}\), then \(B_{\alpha_i} \alpha_i = B_{\beta_i} \beta_i\).
4. For every R-monomorphism \(\theta : \sum_{i=1}^n \alpha_i R \to M^m\) and \(\varphi : \sum_{i=1}^n \alpha_i R \to M\), there exists \(\gamma : M^m \to M\) such that \(\theta = \gamma \varphi\).

**proof.** (1)\(\Rightarrow\)(2) \(\theta : \alpha_1 R + \ldots + \alpha_n R \to M\) is well-defined by \(\theta(\sum_{i=1}^n \alpha_i r_i) = \sum_{i=1}^n \beta_i r_i\), for each \(r_i \in R\). \((m, n)\)-pseudo-injectivity of \(M\) implies there exists \(\gamma : M^m \to M\) such that \(\theta = \gamma i\). In particular, there is \(c = (c_1, \ldots, c_n) \in S_n\) with \(\beta_i = \sum_{k=1}^n c_k \alpha_i\), \(i = 1, \ldots, n\). If \(\sum_{i=1}^n \alpha_i r_i \in r_{R^n}(c) \cap \alpha_i R\), then

\[
\gamma(\sum_{i=1}^n \alpha_i r_i) = \theta(\sum_{i=1}^n \alpha_i r_i) = \sum_{i=1}^n \beta_i r_i = \sum_{i=1}^n (\sum_{k=1}^n c_k \alpha_i) r_i
\]
It is clear that pseudo-injective, right \( r \) frequently, \( \sum \) then \( r = \sum s_k \alpha_i \) where \( s = (s_1, \ldots, s_n) \in S_n \) and \( t \in r_{R^n(s)} \cap \alpha_i R = 0 \). It is clear that \( r_{R^n}(\{\alpha_1, \ldots, \alpha_n\}) \subseteq r_{R^n}(\{s\alpha_1, \ldots, s\alpha_n\}) \).

If \( t \in r_{R^n}(\{s_1\alpha_1, \ldots, s_n\alpha_n\}) \), then \( \sum_{k=1}^nt_k(\sum_{i=1}^ns_i\alpha_i) \) and hence \( \sum_{k=1}^nt_k\alpha_i \in r_{R^n(s)} \cap \alpha_i R \), so \( \sum_{i=1}^nt_i\alpha_i = 0 \). Thus \( r_{R^n}(\{\alpha_1, \ldots, \alpha_n\}) = r_{R^n}(\{\beta_1, \ldots, \beta_n\}) \).

The other equivalence in (2) follows by symmetry.

(2)\(\Leftrightarrow(2')\) and (3)\(\Leftrightarrow(3')\) are trivial.

(2) \(\Rightarrow\) (3) Let \( A_{\alpha_i} = A_{\beta_i} \). Then \( A_{\alpha_i} = B_{\alpha_i}, A_{\beta_i} = B_{\beta_i} \). So \( B_{\alpha_i} \subseteq B_{\beta_i} \).

(3) \(\Rightarrow\) (4) For each \( \alpha_i \in M^n, i = 1, \ldots, n \), let \( \theta : \sum_{i=1}^n\alpha_i R \to M^n \) and \( \varphi : \sum_{i=1}^n\alpha_i R \to M \) be \( R \)-monomorphisms. Then by (3), \( r_{R^n}(\varphi\alpha_i) = r_{R^n}(\theta\alpha_i) \). So \( A_{\alpha_i} = A_{\varphi\alpha_i}, B_{\alpha_i} = B_{\varphi\alpha_i} \). Because \( r_{R^n}(1_{M^n}) \cap \alpha_i R = 0, 1_{M^n} \in B_{\alpha_i} \).

Then \( \theta\alpha_i \in B_{\varphi\alpha_i} \). There exists \( \gamma \in B_{\varphi\alpha_i} \) such that \( \theta = \gamma\varphi \).

(4) \(\Rightarrow\) (1) Let \( \varphi = i\sum_{i=1}^n\alpha_i R \). It is clear.

**Theorem 2.16** Given an \( R \)-module \( M_R \). Then \( M_R \) is \((m, n)\)-pseudo injective, if and only if the right \( R^{m \times n} \)-module \( M^{m \times n} \) is principally pseudo-injective.

**proof.** \(\Rightarrow\) Let \( U, V \in M^{m \times n} \) with \( r_{R^{m \times n}}(U) = r_{R^{m \times n}}(V) \) and write

\[
V = \begin{pmatrix}
V_1 \\
\vdots \\
V_m
\end{pmatrix}
\]

Then for each \( i = 1, \ldots, m \), \( r_{R^{m \times n}}(U) = r_{R^{m \times n}}(V_i) \). Consequently, \( r_{R^n}(U) = r_{R^n}(V_i) \). Since \( M \) is \((m, n)\)-pseudo injective, by theorem (2.15), \( B_U = B_{V_i} \), put \( B_V = \begin{pmatrix}
B_{V_1} & V_1 \\
\vdots & \ddots \\
B_{V_m} & V_m
\end{pmatrix} \). So \( A_U = B_V \). Therefore the right \( R^{m \times n} \)-module \( M^{m \times n} \) is principally pseudo-injective by [9, proposition (2.1)].

\(\Leftarrow\) Suppose that \( \alpha_i, \beta_i \in M^n \) and \( r_{R^n}(\alpha_i) = r_{R^n}(\beta_i) \). Let \( U = \begin{pmatrix} \alpha_i \\ 0 \end{pmatrix} \) and \( V = \begin{pmatrix} \beta_i \\ 0 \end{pmatrix} \in M^{m \times n} \). Then \( r_{R^{m \times n}}(V) = r_{R^{m \times n}}(U) \). Since \( M_{R^{m \times n}}^{m \times n} \) is principally pseudo-injective, \( A_V = A_U \). Then \( M \) is \((m, n)\)-pseudo-injective by theorem(2.15).
Let $M$ be an $R$-module, $\alpha_i$ be a non-zero element in $M^n$, $i = 1, \ldots, m$ and $t$ in $R^n$. We write $W(\alpha_i) = \{r \in R_n \mid \ell_{M^n}(r) \cap \alpha_i R = 0\}$.

Theorem 2.17  The following are equivalent for an $R$-module $M$

1. $M$ is fully pseudo $(m,n)$-stable.

2. $r_{R_n}(\{\alpha_1, \ldots, \alpha_n\}) = r_{R_n}(\{\beta_1, \ldots, \beta_n\})$ if and only if $\beta_i \in \alpha_i W(\alpha_i)$ if and only if $\alpha_i \in \beta_i W(\beta_i)$ for each two $n$-element subsets $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$ of $M^n$.

2' $r_{R_n}(A) = r_{R_n}(B)$ if and only if $B \in AW(A)$ if and only if $A \in BW(B)$ for each $A, B \in M^{m \times n}$

3. For any $R$-monomorphisms $\theta, \varphi : \alpha_1 R + \ldots + \alpha_n R \rightarrow M^m$ where $\alpha_i \in M^n$, there is $t \in R^n$ such that $\theta = \varphi \cdot t$.

proof. (1)$\Rightarrow$(2) $\theta : \alpha_1 R + \ldots + \alpha_n R \rightarrow M^m$ is well-defined by $\theta(\sum_{i=1}^n \alpha_i r_i) = \sum_{i=1}^n \beta_i r_i$, for each $r_i \in R$. Full pseudo $(m,n)$-stability of $M$ implies $\theta(\sum_{i=1}^n \alpha_i R) \subseteq \sum_{i=1}^n \alpha_i R$. In particular, there is $t = (t_1, \ldots, t_n) \in R_n$ with $\beta_i = \sum_{i=1}^n \alpha_i t_k$, $i = 1, \ldots, n$. If $\sum_{i=1}^n \alpha_i r_i \in \ell_{M^n}(t) \cap \alpha_i R$, then $0 = \sum_{i=1}^n (\sum_{i=1}^n \alpha_i r_i) t_k = \sum_{i=1}^n (\sum_{i=1}^n \alpha_i t_k) r_i = \sum_{i=1}^n \beta_i r_i = \theta(\sum_{i=1}^n \alpha_i r_i)$, so $\sum_{i=1}^n \alpha_i r_i = 0$, thus $t \in W(\alpha_i)$ and hence $\beta_i \in \alpha_i W(\alpha_i)$. Conversely if $\beta_i \in \alpha_i W(\alpha_i)$, then $\beta_i = \sum_{k=1}^n \alpha_i s_k$ where $s = (s_1, \ldots, s_n) \in R_n$ and $\ell_{M^n}(s) \cap \alpha_i R = 0$. It is clear that $r_{R_n}(\{\alpha_1, \ldots, \alpha_n\}) \subseteq r_{R_n}(\{\alpha_1 s, \ldots, \alpha_n s\})$. If $t \in r_{R_n}(\{\alpha_1 s_1, \ldots, \alpha_n s_n\})$, then $\sum_{k=1}^n (\sum_{i=1}^n \alpha_i s_i) t_k$ and hence $\sum_{k=1}^n \alpha_i t_k \in \ell_{M^n}(s) \cap \alpha_i R$, so $\sum_{i=1}^n \alpha_i t_k = 0$. Thus $r_{R_n}(\{\alpha_1, \ldots, \alpha_n\}) = r_{R_n}(\{\beta_1, \ldots, \beta_n\})$.

The other equivalence in (2) follows by symmetry.

(2) $\Leftrightarrow$ (2') Is trivial

(2) $\Rightarrow$ (3) Let $\theta, \varphi : \alpha_1 R + \ldots + \alpha_n R \rightarrow M^m$ be two $R$-monomorphisms. Then

$$r_{R_n}(\{\alpha_1, \ldots, \alpha_n\}) = r_{R_n}(\{\beta_1, \ldots, \beta_n\}).$$

By the hypothesis, $\theta(\alpha_i) \in \varphi(\alpha_i) \varphi(\alpha_i)$. So there is $t \in W(\varphi(\alpha_i))$, such that $\theta(\alpha_i) = \sum_{i=1}^n \varphi(\alpha_i) t_i = \sum_{i=1}^n \varphi(\alpha_i) t_i$. Also by symmetry there is $s \in R_n$ such that $\varphi = \theta s$.

(3) $\Rightarrow$ (1) For each $\alpha_i \in M^n, i = 1, \ldots, m$, let $f : \alpha_1 R + \ldots + \alpha_n R \rightarrow M^n$ be an $R$-monomorphism. Then by (3), there is an element $t \in R^n$ such that $f = it$ where $i$ is the inclusion map of $\alpha_1 R + \ldots + \alpha_n R$ into $M^n$ and hence $f(\alpha_1 R + \ldots + \alpha_n R) \subseteq \alpha_1 R + \ldots + \alpha_n R$. 

Theorem 2.18 Given an $R$-module $M_R$. Then $M_R$ is fully pseudo $(m,n)$-stable, if and only if the right $R^{n\times n}$-module $M^{m\times n}$ is fully pseudo-stable.

**proof.** The proof is similar to the proof of theorem (2.16)

Theorem 2.19 Given an $R$-module $M_R$. If $M_R$ is fully pseudo-$(m,n)$-stable, then $M$ is $(m,n)$-pseudo injective.

The following proposition is the converse of theorem (2.19)

Proposition 2.20 Let $M$ be an $(m,n)$-multiplication $R$-module. If $M$ is $(m,n)$-pseudo injective, then $M$ is a fully pseudo $(m,n)$-stable module.

**proof.** The proof is essentially the same as that of [2, proposition (2.17)] by replacing the $R$-homomorphism $f : N \rightarrow M^m$ by $R$-monomorphism.

References


[7] Zhang, X., Chen J. and Zhang J., *On (m, n) injective modules and (m, n)-coherent rings*, *Algebra Colloquium* 12(1), (2005), 149-160.


Received: March, 2009