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On the Total Domination Subdivision Numbers of Grid Graphs

Nasrin Soltankhah

Department of Mathematics, Alzahra University
Vanak Square 19834, Tehran, Iran
soltan@alzahra.ac.ir, soltankhah.n@gmail.com

Abstract

A set S of vertices in a graph $G(V, E)$ is called a total dominating set if every vertex $v \in V$ is adjacent to an element of S. The total domination number of a graph G denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G. Total domination subdivision number denoted by sd_{γ_t} is the minimum number of edges that must be subdivided to increase the total domination number.

Here we investigate the problem of total domination subdivision numbers of grid graphs $P_{m,n}$ and determine the total domination subdivision numbers of grid graphs $P_{m,n}$ for $m = 2, 3$ and 4, and $n \geq 2$. Also Haynes et al. [4] showed that $1 \leq sd_{\gamma_t}(P_{m,n}) \leq 4$ for any grid graph $P_{m,n}$. We improve this bound and prove that $sd_{\gamma_t}(P_{m,n}) \leq 3$.

Mathematics Subject Classification: 05C69

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1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A set S of vertices in a graph $G(V, E)$ is called a *total dominating set* (**TDS**) if every vertex $v \in V$ is adjacent to an element of S. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G, and a total dominating set of minimum cardinality is called a γ_t-set of G or a $\gamma_t(G)$-set. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [1] and is by now well studied in graph theory.

The *total domination subdivision number* $sd_{\gamma_t}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the total domination number.

(An edge $uv \in E(G)$ is *subdivided* if the edge uv is deleted, but a new vertex x is added, along with two new edges ux and vx. The vertex x is called a *subdivision vertex*). Since the total domination number of the graph K_2 does not change when its only edge is subdivided, we assume that the graph is of order $n \geq 3$.

The Cartesian product of two graphs G and H is the graph denoted by $G\Box H$, with $V(G\Box H) = V(G)\times V(H)$ (where $\times$ denotes the Cartesian product of sets) and
$((u, u'), (v, v')) \in E(G\Box H)$ if and only if $u = v$ and $(u', v') \in E(H)$ or $u' = v'$ and $(u, v) \in E(G)$. If each G and H is a path P_m and P_n (respectively), then we will call $P_m\Box P_n$ a grid graph. For notational convenience, we denote $P_m\Box P_n$ simply by $P_{m,n}$.

The concept of total domination subdivision number $sd_{\gamma_t}(G)$ was introduced by Haynes et al in [4] and studied in [2, 5, 6] and elsewhere. Constant upper bounds on the total domination subdivision number for several families of graphs were determined in[4, 5]. For grid graphs $P_{m,n}$, in [4] was shown that $1 \leq sd_{\gamma_t}(P_{m,n}) \leq 4$. In this paper we improve this bound and prove that $sd_{\gamma_t}(P_{m,n}) \leq 3$. Also the determination of total domination subdivision number of grid graphs was raised as an open problem in Section 4 of [4]. We study this problem and determine the total domination subdivision numbers of grid graphs $P_{m,n}$ for $m = 2, 3$ and 4, and $n \geq 2$.

First we state the following known results about total domination number of grid graphs, and are useful in the sequel.

The total domination number of a cycle or a path is easy to compute.

THEOREM A *For $n \geq 3$,*

$$\gamma_t(C_n) = \gamma_t(P_n) = \begin{cases} \frac{n}{2}, & n \equiv 0 \pmod 4 \\ \lfloor \frac{n}{2} \rfloor + 1, & \text{otherwise.} \end{cases}$$

An immediate consequence of Theorem A now follows:

THEOREM B *[4] For a path or cycle on $n \geq 3$ vertices,*

$$sd_{\gamma_t}(P_n) = sd_{\gamma_t}(C_n) = \begin{cases} 1, & n \equiv\ 0, 1 \pmod 4 \\ 3, & n \equiv\ 2 \pmod 4 \\ 2, & n \equiv\ 3 \pmod 4). \end{cases}$$

THEOREM C *[3] For $n \geq 2$, $\gamma_t(P_{2,n}) = 2\lfloor \frac{n+2}{3} \rfloor$.*

The last theorem implies the following recurrence relation.

$$\gamma_t(P_{2,n}) = \gamma_t(P_{2,n-3}) + 2, \quad \text{for } n \geq 5. \tag{1}$$

THEOREM D *[3] For $n \geq 3$, $\gamma_t(P_{3,n}) = n$.*

The last theorem implies the following recurrence relation.

$$\gamma_t(P_{3,n}) = \gamma_t(P_{3,n-1}) + 1, \quad \text{for } n \geq 4. \tag{2}$$

THEOREM E *[8] Let $n \geq 4$. Then*

$$\gamma_t(P_{4,n}) = \begin{cases} 6\lfloor \frac{n}{5} \rfloor + 2, & n \equiv 0, 1 \pmod 5 \\ 6\lfloor \frac{n}{5} \rfloor + 4, & n \equiv 2 \pmod 5 \\ 6\lceil \frac{n}{5} \rceil, & n \equiv 3, 4 \pmod 5). \end{cases}$$

The last theorem implies the following recurrence relation:

$$\gamma_t(P_{4,n}) = \gamma_t(P_{4,n-5}) + 6, \quad \text{for } n \geq 9. \tag{3}$$

2 Necessary definitions and lemmas

In this section we state some definitions and lemmas which will used in our discussion.
Notation 1. Let $1, \cdots, m$ and $1, \cdots, n$ be the vertices of P_m and P_n, respectively. Then the vertices of $P_{m,n}$ are denoted by $x_{i,j}$, where $i = 1, \cdots, m$ and $j = 1, \cdots, n$.

DEFINITION F*or a fixed i, $1 \leq i \leq m$, the set of vertices of the i-th copy of P_n in $P_{m,n}$, $x_{i1}, \cdots x_{in}$, is called a row (i-th row) of $P_{m,n}$ and denote it by $_i(P_n)$. For a fixed j, $1 \leq j \leq n$ the set of vertices of the j-th copy of P_m in $P_{m,n}$, $x_{1j}, \cdots x_{mj}$, is called a column (j-th column) of $P_{m,n}$ and denote it by $(P_m)j$.*

Notation 2. We denote a graph obtained by subdividing i edges of $P_{m,n}$, by notation ${P^i}_{m,n}$. And also we denote the set of such graphs by ${\mathcal{P}^i}_{m,n}$.

LEMMA 2.1 *(i) For any graph $P^1_{2,n} \in {\mathcal{P}^1}_{2,n}$ obtained from $P_{2,n}$ by subdividing an arbitrary edge, where $n > 5$,*

$$\gamma_t(P^1_{2,n}) = \gamma_t(P^1_{2,n-3}) + 2. \tag{4}$$

(ii) For a specific graph ${P^{2^*}}_{2,n} \in {\mathcal{P}^2}_{2,n}$ obtained from $P_{2,n}$ by subdividing the edges $x_{1,1}x_{2,1}$, $x_{1,2}x_{2,2}$, where $n > 5$,

$$\gamma_t({P^{2^*}}_{2,n}) = \gamma_t({P^{2^*}}_{2,n-3}) + 2. \tag{5}$$

PROOF. (i) By symmetry on $P_{2,n}$ we can assume the arbitrary subdivided edge belongs to the first $n-3$ columns. Obviously the subgraph induced by the vertices of $n-3$ first columns is a $P^1_{2,n-3}$.

Let S be a total dominating set on $P^1_{2,n}$. Last three columns of $P^1_{2,n}$ is a block B, $B \cong P_{2,3}$. Clearly S contains at least two vertices of the columns

$(P_2)_{n-1},\ (P_2)_n$ (of the set $\{x_{1,n-1},\ x_{2,n-1},\ x_{1,n},\ x_{2,n}\}$). It means that S must contain at least two vertices of B to totally dominate B. This results that the size of S is at least two more than each $\gamma_t(P^1_{2,n-3})$-set. Since S is an arbitrary total dominating set, then we have $\gamma_t(P^1_{2,n}) \geq \gamma_t(P^1_{2,n-3})+2,$ for $n > 5$. On the other hand consider a $\gamma_t(P^1_{2,n-3})$-set S totally dominate $P^1_{2,n-3}$. And consider two vertices $x_{1,n-1}, x_{2,n-1}$ which totally dominate B (last three columns). It follows that $S \cup \{x_{1,n-1}, x_{2,n-1}\}$ with cardinality $\gamma_t(P^1_{2,n-3})+2$ is a TDS of $P^1_{2,n}$. This implies that $\gamma_t(P^1_{2,n}) \leq \gamma_t(P^1_{2,n-3})+2,$ for $n > 5$. Hence the equation (4) holds. The proof of (ii) is very similar to (i). ■

LEMMA 2.2 *(i) For any graph $P^1_{4,n} \in \mathcal{P}^1{}_{4,n}$ obtained from $P_{4,n}$ by subdividing an arbitrary edge, where $n > 9$,*

$$\gamma_t(P^1_{4,n}) \geq \gamma_t(P^1_{4,n-5}) + 6. \tag{6}$$

(ii) For a specific graph $P^{2^*}{}_{4,n} \in \mathcal{P}^2{}_{4,n}$ obtained from $P_{4,n}$ by subdividing the edges $x_{1,1}x_{1,2},\ \ x_{2,1}x_{2,2}$, where $n > 9$,

$$\gamma_t(P^{2^*}{}_{4,n}) \geq \gamma_t(P^{2^*}{}_{4,n-5}) + 6. \tag{7}$$

PROOF. (i) By symmetry on $P_{4,n}$ we can assume the arbitrary subdivided edge belongs to the first $n-5$ columns. Obviously the subgraph induced by the vertices of $n-5$ first columns is a $P^1_{4,n-5}$.

Let S be a total dominating set on $P^1_{4,n}$. Last five columns of $P^1_{4,n}$ is a block B, $B \cong P_{4,5}$. At most the first column of B can be totally dominated from the adjacent graph (block) $P^1_{4,n-5}$. (It remains a $P_{4,4}$.) To totally dominate $P_{4,4}$ we need at least 6 vertices, (by Theorem E, $\gamma_t(P_{4,4}) = 6$). It means that S must contain at least six vertices of B. This results that the size of S is at least six more than each $\gamma_t(P^1_{4,n-5})$-set. Since S is an arbitrary total dominating set, we have $\gamma_t(P^1_{4,n}) \geq \gamma_t(P^1_{4,n-5}) + 6,$ for $n > 9$. Hence the inequality (6) holds. The proof of (ii) is very similar to (i). ■

For the definitions and notations not defined here we refer the reader to text [7].

3 Main results

In this section we first determine the value of total domination subdivision number in $P_{2,n}$, $P_{3,n}$ and $P_{4,n}$. Then we prove that $sd_{\gamma_t}(P_{m,n}) \leq 3$ for any $n, m \geq 2$.

THEOREM 3.1 *For $n \geq 2$,*

$$sd_{\gamma_t}(P_{2,n}) = \begin{cases} 1, & n \equiv\ 0, 2 \pmod 3 \\ 2, & n \equiv\ 1 \pmod 3). \end{cases}$$

PROOF. We prove the statement in three cases.

Case 1: for $n \equiv 2 \pmod 3$, $sd_{\gamma_t}(P_{2,n}) = 1$.
Let $P^1_{2,n}$ be obtained from $P_{2,n}$ by adding new vertex x subdividing the edge $x_{1,1}x_{2,1}$. By induction on n , we show that $\gamma_t(P^1_{2,n}) > \gamma_t(P_{2,n})$ ($sd_{\gamma_t}(P_{2,n}) = 1$). For $n = 2$, by Theorem B, we have $sd_{\gamma_t}(P_{2,2}) = 1$. For $n = 5$, we show that $sd_{\gamma_t}(P_{2,5}) = 1$. By Theorem C, $\gamma_t(P_{2,5}) = 4$. Let S be a total dominating set on $P^1_{2,5}$. Last two columns of $P^1_{2,5}$ is a block B, $B \cong P_{2,2}$. To totally dominate $P_{2,2}$ we need at least 2 vertices, since by Theorem C $\gamma_t(P_{2,2}) = 2$, then $|S \cap B| \geq 2$. The first two columns of $P^1_{2,5}$ is a block B', $B' \cong P^1_{2,2} \cong C_5$. To totally dominate C_5 we need at least 3 vertices. At most the third column $(P_2)_3$ of $P^1_{2,5}$ can be totally dominated from the adjacent columns $(P_2)_2$ and $(P_2)_4$ (from the adjacent blocks B and B'). Therefore $|S| \geq 5$ and we can see that to totally dominate $P^1_{2,5}$ we need at least 5 vertices. This implies $sd_{\gamma_t}(P_{2,5}) = 1$.

Assume $n > 5$ and that the statement is true for $n-3$. Then by induction assumption, Lemma 2.1, and relation 1 the statement is true for n as follows:

$$\gamma_t(P^1_{2,n}) = \gamma_t(P^1_{2,n-3}) + 2 > \gamma_t(P_{2,n-3}) + 2 = \gamma_t(P_{2,n}).$$

Case 2: for $n \equiv 0 \pmod 3$, $sd_{\gamma_t}(P_{2,n}) = 1$.

For the smallest n, $n = 3$, we show that $sd_{\gamma_t}(P_{2,3}) = 1$. By Theorem C, $\gamma_t(P_{2,3}) = 2$. Let $P^1_{2,3}$ be obtained from $P_{2,3}$ by adding a new vertex x subdividing of the edge $x_{1,1}x_{2,1}$ and S be a $\gamma_t(P^1_{2,3})$- set. Clearly S contains at least two vertices of the set $\{x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}\}$. But none of these vertices dominates x. Thus $|S| \geq 3$ and $sd_{\gamma_t}(P_{2,3}) = 1$. The rest of proof of this case is very similar to the previous case and is therefore omitted.

Case 3: for $n \equiv 1 \pmod 3$, $sd_{\gamma_t}(P_{2,n}) = 2$.

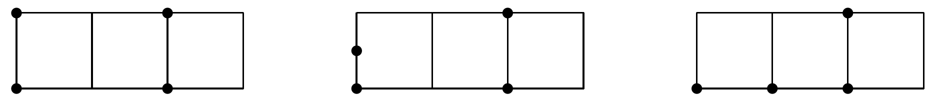
Let $P^1_{2,n}$ be obtained from graph $P_{2,n}$ by adding a new vertex x subdividing an arbitrary edge e. By induction on n , we show that $\gamma_t(P^1_{2,n}) = \gamma_t(P_{2,n})$ for $n \equiv 1 \pmod 3$. For the base case n, $n = 4$, we show that $\gamma_t(P^1_{2,4}) = \gamma_t(P_{2,4})$ (by Theorem C, $\gamma_t(P_{2,4}) = 4$).

For this purpose by symmetry on $P_{2,4}$ it is enough to examine subdividing of the edge e, for $e = x_{1,1}x_{2,1},\ x_{1,1}x_{1,2},\ x_{1,2}x_{2,2}$, or $x_{1,2}x_{1,3}$.
If $e = x_{1,1}x_{1,2}$ or $x_{1,2}x_{1,3}$. Take $S_1 = \{x_{1,1}, x_{2,1}, x_{1,3}, x_{2,3}\}$, (see Figure 1(a)).
If $e = x_{1,1}x_{2,1}$. Take $S_2 = S_1 - \{x_{2,1}\} \cup \{x\}$, (see Figure 1(b)).
If $e = x_{1,2}x_{2,2}$. Take $S_3 = \{x_{1,1}, x_{1,2}, x_{1,3}, x_{2,3}\}$, (see Figure 1(c)).

In all above cases, S_i $(1 \leq i \leq 3)$ is a TDS of size 4 for $P^1_{2,4}$. Assume $n \geq 7$ and that the statement is true for $n-3$. Then by induction assumption, Lemma 2.1, and relation 1 the statement is true for n as follows:

$$\gamma_t(P^1_{2,n}) = \gamma_t(P^1_{2,n-3}) + 2 = \gamma_t(P_{2,n-3}) + 2 = \gamma_t(P_{2,n}).$$

Consequently $sd_{\gamma_t}(P_{2,n}) \geq 2$.



(a) (b) (c)

Figure 1:

We show next that $sd_{\gamma_t}(P_{2,n}) \leq 2$. Let ${P^{2*}}_{2,n}$ be obtained from $P_{2,n}$ by adding new vertices x and y subdividing the edges $x_{1,1}x_{2,1}$ and $x_{1,2}x_{2,2}$, respectively.

By induction on n, we show that $\gamma_t({P^{2*}}_{2,n}) > \gamma_t(P_{2,n})$ ($sd_{\gamma_t}(P_{2,n}) \leq 2$). For the smallest n, $n = 4$, we show that $sd_{\gamma_t}(P_{2,4}) \leq 2$. Let S be a $\gamma_t({P^{2*}}_{2,4})$- set such that $|S \cap \{x_{1,4}, x_{2,4}\}|$ is minimal. It is easy to see that $S \cap \{x_{1,4}, x_{2,4}\} = \emptyset$ and so $x_{1,3}, x_{2,3} \in S$. Now at least three vertices other than $x_{1,3}$ and $x_{2,3}$ are needed to dominate $x, y, x_{1,1}$, and $x_{2,1}$. Thus $|S| \geq 5$ and so $sd_{\gamma_t}(P_{2,4}) \leq 2$. Assume $n \geq 7$ and that the statement is true for $n - 3$. Then by induction assumption, Lemma 2.1, and relation 1 the statement is true for n as follows:

$$\gamma_t({P^{2*}}_{2,n}) = \gamma_t({P^{2*}}_{2,n-3}) + 2 > \gamma_t(P_{2,n-3}) + 2 = \gamma_t(P_{2,n}).$$

Therefore $sd_{\gamma_t}(P_{2,n}) = 2$, for $n \equiv 1 \pmod 3$. ■

THEOREM 3.2 *For $n \geq 3$, $sd_{\gamma_t}(P_{3,n}) = 1$.*

PROOF. Let $n \geq 3$, we will prove that subdividing the edge $x_{1,1}x_{2,1}$ increases the total domination number. Let $P^1_{3,n}$ be obtained from $P_{3,n}$ by adding a new vertex x subdividing of the edge $x_{1,1}x_{2,1}$. Last $n-1$ columns of $P^1_{3,n}$ is a block B, $B \cong P_{3,n-1}$. We know that to totally dominate $P_{3,n-1}$, we need at least $n-1$ vertices and the choice of these vertices is unique. And this is if the vertices of the middle row are totally dominating vertices, see [3]. In the first column to totally dominate x we need at least two other vertices. This results that $\gamma_t(P^1_{3,n}) = n + 1$. This implies that $sd_{\gamma_t}(P_{3,n}) = 1$, for $n \geq 3$. ■

Now we determine the total domination subdivision numbers of $P_{4,n}$.

THEOREM 3.3 *For $n \geq 4$,*

$$sd_{\gamma_t}(P_{4,n}) = \begin{cases} 1, & n \equiv 1, 2, 4 \pmod 5 \\ 2, & n \equiv 0, 3 \pmod 5). \end{cases}$$

PROOF. We prove the theorem in each following cases.

Case 1: for $n \equiv 4 \pmod 5$, $sd_{\gamma_t}(P_{4,n}) = 1$.

Let $P^1_{4,n}$ be obtained from $P_{4,n}$ by adding a new vertex x subdividing the edge $x_{1,1}x_{2,1}$. By induction on n, we show that $\gamma_t(P^1_{4,n}) > \gamma_t(P_{4,n})$ ($sd_{\gamma_t}(P_{4,n}) =$ 1). For $n = 4$, we show that $sd_{\gamma_t}(P_{4,4}) = 1$. By Theorem E, $\gamma_t(P_{4,4}) = 6$. Let

S be a total dominating set on $P^1_{4,4}$. Last two columns of $P^1_{4,4}$ is a block B, $B \cong P_{4,2}$. To totally dominate $P_{4,2}$ we need at least 4 vertices, since by Theorem C $\gamma_t(P_{4,2}) = 4$, then $|S \cap B| \geq 4$. The first column of $P^1_{4,4}$ is a block B', $B' \cong P_5$. To totally dominate P_5 we need at least 3 vertices then $|S \cap B'| \geq 3$. At most the second column $(P_4)_2$ of $P^1_{4,4}$ can be totally dominated from the adjacent columns $(P_4)_1$ and $(P_4)_3$ (from the adjacent blocks B and B'). Therefore $|S| \geq 7$ and we can see that to totally dominate $P^1_{4,4}$ we need at least 7 vertices. This implies $sd_{\gamma_t}(P_{4,4}) = 1$. For $n = 9$, by a similar argument, we show that $sd_{\gamma_t}(P_{4,9}) = 1$. By Theorem E, $\gamma_t(P_{4,9}) = 12$. Let S be a total dominating set on $P^1_{4,9}$. Last four columns of $P^1_{4,9}$ is a block B, $B \cong P_{4,4}$. To totally dominate $P_{4,4}$ we need at least 6 vertices, since by Theorem E, $\gamma_t(P_{4,4}) = 6$, then $|S \cap B| \geq 6$. The first four columns of $P^1_{4,9}$ is a block B', $B' \cong P^1_{4,4}$. To totally dominate $P^1_{4,4}$, by above result, we need at least 7 vertices. At most the fifth column $(P_4)_5$ of $P^1_{4,9}$ can be totally dominated from the adjacent columns $(P_4)_4$ and $(P_4)_6$ (from the adjacent blocks B and B'). Therefore $|S| \geq 13$ and we can see that to totally dominate $P^1_{4,9}$ we need at least 13 vertices. This implies $sd_{\gamma_t}(P_{4,9}) = 1$.

Assume $n > 9$ and that the statement is true for $n - 5$ ($sd_{\gamma_t}(P_{4,n-5}) = 1$). Then by induction assumption, Lemma 2.2, and relation 3, the statement is true for n as follows:

$$\gamma_t(P^1_{4,n}) \geq \gamma_t(P^1_{4,n-5}) + 6 > \gamma_t(P_{4,n-5}) + 6 = \gamma_t(P_{4,n}).$$

Case 2: for $n \equiv 3 \pmod 5$, $sd_{\gamma_t}(P_{4,n}) = 2$.

Let $P^1_{4,n}$ be obtained from graph $P_{4,n}$ by adding a new vertex x subdividing an arbitrary edge e. By induction on n, we show that $\gamma_t(P^1_{4,n}) = \gamma_t(P_{4,n})$ for $n \equiv 3 \pmod 5$. For the base case n, $n = 8$, we show that $\gamma_t(P^1_{4,8}) = \gamma_t(P_{4,8})$ ($sd_{\gamma_t}(P_{4,8}) \geq 2$). By Theorem E, $\gamma_t(P_{4,8}) = 12$.

For this purpose by symmetry on $P_{4,8}$ it is enough to examine subdividing edge e in following cases.

(i) If $e = x_{1,1}x_{2,1}, x_{1,1}x_{1,2}, x_{1,2}x_{2,2}, x_{2,1}x_{2,2}, x_{1,3}x_{2,3}, x_{2,3}x_{3,3}, x_{3,3}x_{3,2}$, or $x_{3,3}x_{3,4}$. Take $S_1 = \{x_{2,1}, x_{3,1}, x_{1,3}, x_{1,2}, x_{3,3}, x_{4,3}, x_{2,5}, x_{3,5}, x_{4,7}, x_{4,8}, x_{1,7}, x_{1,8}\}$, (see figure 2(a)).

(ii) If $e = x_{1,2}x_{1,3}$. Take $S_2 = S_1 - \{x_{1,2}\} \cup \{x\}$, (see figure 2(b)).

(iii) If $e = x_{1,4}x_{1,5}, x_{1,4}x_{2,4}$, or $x_{2,4}x_{2,5}$. Take
$S_3 = \{x_{2,1}, x_{3,1}, x_{1,3}, x_{1,4}, x_{3,3}, x_{4,3}, x_{2,5}, x_{3,5}, x_{4,7}, x_{4,8}, x_{1,7}, x_{1,8}\}$, (see figure 2(c)).

(iv) If $e = x_{1,3}x_{1,4}$. Take $S_4 = S_3 - \{x_{1,4}\} \cup \{x\}$, (see figure 2(d)).

(v) If $e = x_{2,1}x_{3,1}$, or $x_{2,2}x_{3,2}$. Take
$S_5 = \{x_{2,1}, x_{3,1}, x_{2,2}, x_{3,2}, x_{4,4}, x_{4,5}, x_{4,6}, x_{1,4}, x_{1,5}, x_{1,6}, x_{2,8}, x_{3,8}\}$, (see figure 2(e)).

(vi) If $e = x_{2,4}x_{3,4}$. Take

$S_6 = \{x_{1,1}, x_{1,2}, x_{4,1}, x_{4,2}, x_{2,4}, x_{2,5}, x_{3,4}, x_{3,5}, x_{4,7}, x_{4,8}, x_{1,7}, x_{1,8}\}$, (see figure 2(f)).
In all above cases S_i, $(1 \le i \le 6)$ is a TDS of size 12 for $P^1_{4,8}$.

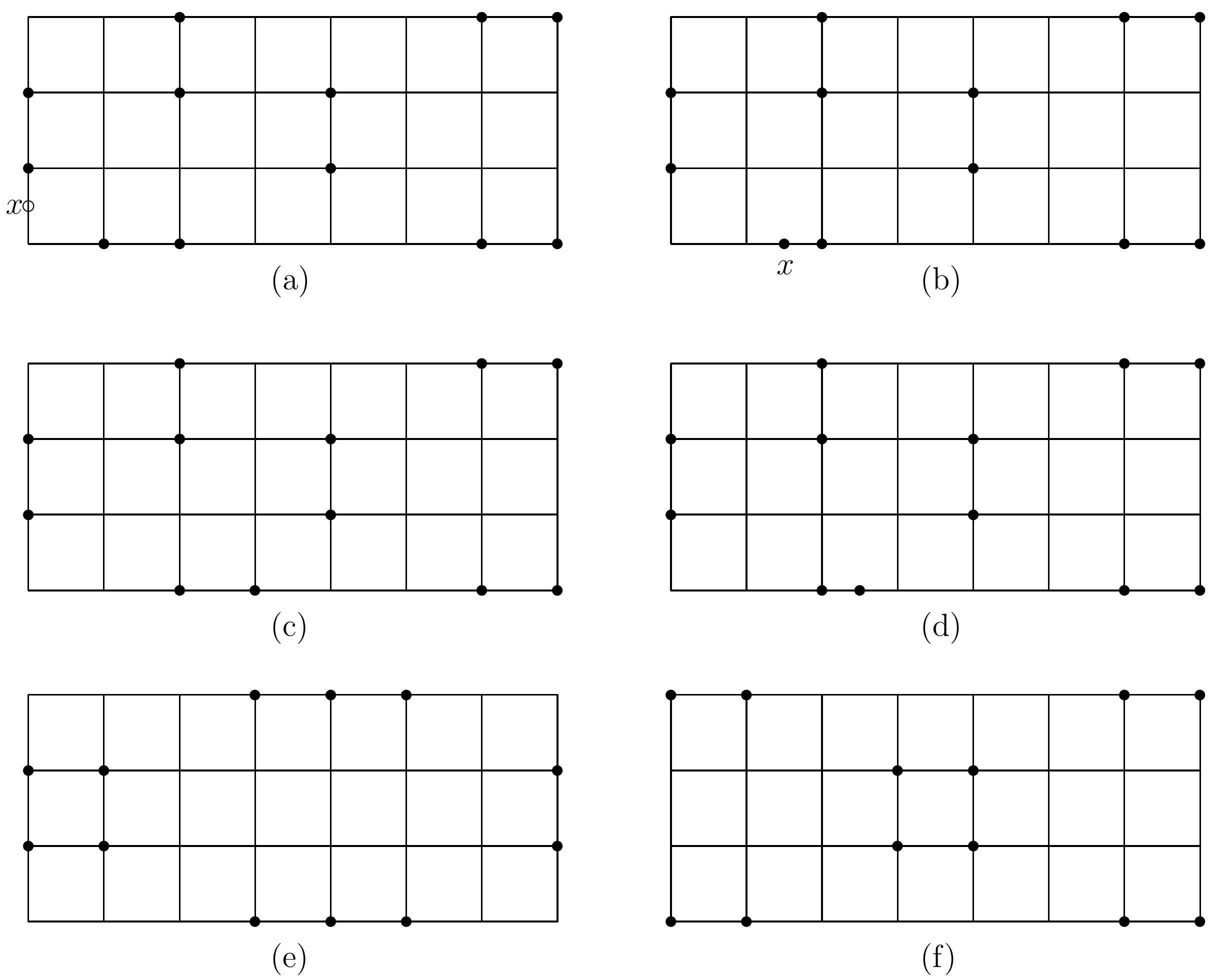


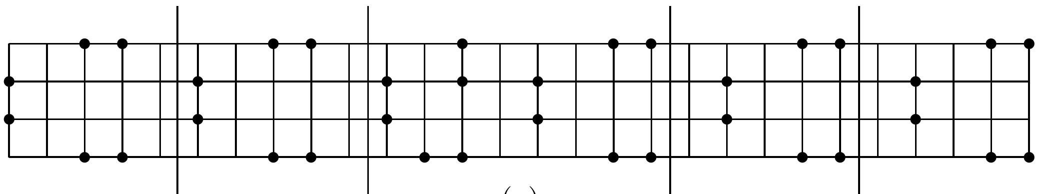
Figure 2

Now let $n \ge 13$, $n = 5q + 3$, and let the arbitrary subdivided edge e be $x_{i,j}x_{i,j+1}$ or $x_{i,j}x_{i+1,j}$ where $j = 5l + r$, $0 \le r \le 4$. We give a total dominating set S of $P^1_{4,n}$ with cardinality $\gamma_t(P_{4,n})$ as follows: We partition (split) the set of columns of $P^1_{4,n}$ into blocks B_i, $B_i \cong P_{4,5}$ for $i = 1, \dots, l-1, l+1, \dots, q$ and into block B_l, $B_l \cong P^1_{4,8}$ and dominate each such block B_i as follows. If the edge e is in one of the cases (i), (ii), (iii) or (iv) we dominate B_l by a set isomorphic to set S_1, S_2, S_3, or S_4, respectively and we dominate each B_i, $i = 1, \cdots l-1$ by a set isomorphic to set $P_1 = \{x_{2,1}, x_{3,1}, x_{4,3}, x_{4,4}, x_{1,3}, x_{1,4}\}$ and we dominate each B_i, $i = l + 1, \dots, q$ by a set isomorphic to set $P_2 = \{x_{2,2}, x_{3,2}, x_{4,4}, x_{4,5}, x_{1,4}, x_{1,5}\}$, (see Figure 3(a)).

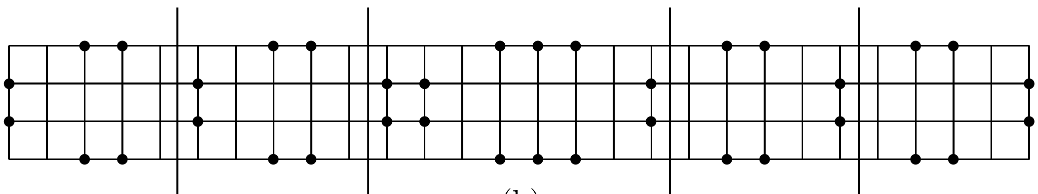
If the edge e is in the case (v), dominate B_l by a set isomorphic to set S_5 and dominate each B_i, $i = 1, \dots, l-1$ by a set isomorphic to set P_1

and we dominate each B_i, $i = l+1, \dots, q$ by a set isomorphic to set $P_3 = \{x_{4,2}, x_{4,3}, x_{1,2}, x_{1,3}, x_{2,5}, x_{3,5}\}$, (see Figure 3(b)).

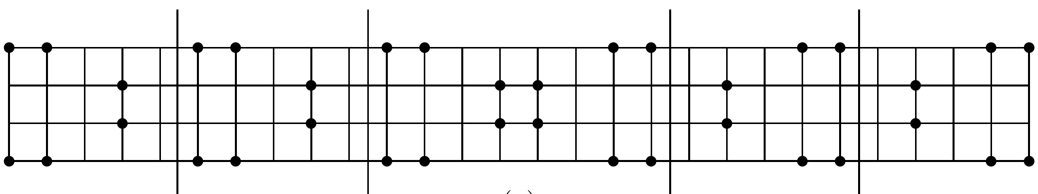
If the edge e is in the case (vi), dominate B_l by a set isomorphic to set S_6 and dominate each B_i, $i = 1, \cdots l-1$ by a set isomorphic to set $P_4 = \{x_{1,1}, x_{1,2}, x_{4,1}, x_{4,2}, x_{2,4}, x_{3,4}\}$ and dominate each B_i, $i = l+1, \cdots q$ by a set isomorphic to set P_2, (see Figure 3(c)).



(a)



(b)



(c)

Figure 3

Therefore $sd_{\gamma_t}(P_{4,n}) \geq 2$. We show next that $sd_{\gamma_t}(P_{4,n}) \leq 2$. Let ${P^{2^*}}_{4,n}$ be obtained from $P_{4,n}$ by adding new vertices x and y subdividing the edges $x_{1,1}x_{1,2}$ and $x_{2,1}x_{2,2}$, respectively. By induction on n, we show that $\gamma_t({P^{2^*}}_{4,n}) > \gamma_t(P_{4,n})$ ($sd_{\gamma_t}(P_{4,n}) \leq 2$). For the smallest n, $n = 8$, we show that $sd_{\gamma_t}(P_{4,8}) \leq 2$. By Theorem E, $\gamma_t(P_{4,8}) = 12$. The last five columns of ${P^{2^*}}_{4,8}$ is a block B, $B \cong P_{4,5}$. To totally dominate $P_{4,5}$ we need at least 8 vertices, since by Theorem E, $\gamma_t(P_{4,5}) = 8$. The first two columns of ${P^{2^*}}_{4,8}$ is a block B', $B' \cong {P^{2^*}}_{4,2}$. To totally dominate ${P^{2^*}}_{4,2}$ we need at least 5 vertices, since by case 3 of Theorem 3.1, $sd_{\gamma_t}(P_{4,2}) = 2$. At most the third column $(P_4)_3$ of ${P^{2^*}}_{4,8}$ can be totally dominated from the adjacent columns $(P_4)_2$ and $(P_4)_4$ (from the adjacent blocks B and B'). Therefore to totally dominate ${P^{2^*}}_{4,8}$ we need at least 13 vertices. This implies $sd_{\gamma_t}(P_{4,8}) = 2$.

Assume $n \geq 13$ and that the statement is true for $n-5$ ($\gamma_t({P^{2^*}}_{4,n-5}) > \gamma_t(P_{4,n-5})$). Then by induction assumption, Lemma 2.2, and equation 3, the statement is true for n as follows:

$$\gamma_t({P^{2^*}}_{4,n}) \geq \gamma_t({P^{2^*}}_{4,n-5}) + 6 > \gamma_t(P_{4,n-5}) + 6 = \gamma_t(P_{4,n}).$$

Therefore $sd_{\gamma_t}(P_{4,n}) = 2$.

Case 3: for $n \equiv 2 \pmod 5$, $sd_{\gamma_t}(P_{4,n}) = 1$.

Let $P^1_{4,n}$ be obtained from $P_{4,n}$ by adding a new vertex x subdividing the edge $x_{1,4}x_{2,4}$. By induction on n, we show that $\gamma_t(P^1_{4,n}) > \gamma_t(P_{4,n})$ ($sd_{\gamma_t}(P_{4,n}) = 1$). For the base case $n = 7$, we show that $sd_{\gamma_t}(P_{4,7}) = 1$. By Theorem E, $\gamma_t(P_{4,7}) = 10$. Let S be a total dominating set on $P^1_{4,7}$. Last two columns of $P^1_{4,7}$ is a block B, $B \cong P_{4,2}$. To totally dominate $P_{4,2}$ we need at least 4 vertices, since by Theorem C, $\gamma_t(P_{4,2}) = 4$, then $|S \cap B| \geq 4$. The first four columns of $P^1_{4,7}$ is a block B', $B' \cong P^1_{4,4}$. To totally dominate $P^1_{4,4}$ we need at least 7 vertices, since by Case 1, $sd_{\gamma_t}(P_{4,4}) = 1$, then $|S \cap B'| \geq 7$. At most the fifth column $(P_4)_5$ of $P^1_{4,7}$ can be totally dominated from the adjacent columns $(P_4)_4$ and $(P_4)_6$ (blocks B and B'). Therefore $|S| \geq 11$ and we can see that to totally dominate $P^1_{4,7}$ we need at least 11 vertices. This implies $sd_{\gamma_t}(P_{4,7}) = 1$. The rest of the proof of this case is very similar to the Case 1 and is therefore omitted.

Case 4: for $n \equiv 1 \pmod 5$, $sd_{\gamma_t}(P_{4,n}) = 1$.

Let $P^1_{4,n}$ be obtained from $P_{4,n}$ by adding a new vertex x subdividing the edge $x_{1,1}x_{1,2}$. By induction on n, we show that $\gamma_t(P^1_{4,n}) > \gamma_t(P_{4,n})$ ($sd_{\gamma_t}(P_{4,n}) = 1$). For the base case $n = 6$, we show that $sd_{\gamma_t}(P_{4,6}) = 1$. By Theorem E, $\gamma_t(P_{4,6}) = 8$. Let S be a total dominating set on $P^1_{4,6}$. Last two columns of $P^1_{4,6}$ is a block B, $B \cong P_{4,2}$. To totally dominate $P_{4,2}$ we need at least 4 vertices, since by Theorem C, $\gamma_t(P_{4,2}) = 4$, then $|S \cap B| \geq 4$. The first three columns of $P^1_{4,6}$ is a block B', $B' \cong P^1_{4,3}$. To totally dominate $P^1_{4,3}$ we need at least 5 vertices, since by Theorem 3.2, $sd_{\gamma_t}(P_{4,3}) = 1$. At most the fourth column $(P_4)_4$ of $P^1_{4,6}$ can be totally dominated from the adjacent columns $(P_4)_3$ and $(P_4)_5$ (from the adjacent blocks B and B'). Therefore $|S| \geq 9$ and we can see that to totally dominate $P^1_{4,6}$, we need at least 9 vertices. This implies $sd_{\gamma_t}(P_{4,6}) = 1$. The rest of the proof of this case is very similar to the Case 1 and is therefore omitted.

Case 5: for $n \equiv 0 \pmod 5$, $sd_{\gamma_t}(P_{4,n}) = 2$.

Let $P^1_{4,n}$ be obtained from graph $P_{4,n}$ by adding a vertex x subdividing an arbitrary edge e. By induction on n, we show that $\gamma_t(P^1_{4,n}) = \gamma_t(P_{4,n})$ for $n \equiv 0 \pmod 5$. For the smallest n, $n = 5$, we show that $\gamma_t(P^1_{4,5}) = \gamma_t(P_{4,5})$. For this purpose by symmetry on $P_{4,5}$ it is enough to consider subdividing of the edge e in following cases.

(i) If $e = x_{1,1}x_{2,1}, x_{1,1}x_{1,2}, x_{1,2}x_{2,2}, x_{2,1}x_{2,2}, x_{1,3}x_{2,3}, x_{2,3}x_{3,3}, x_{3,2}x_{3,3}$. Take $S_1 = \{x_{2,1}, x_{3,1}, x_{1,2}, x_{1,3}, x_{3,3}, x_{4,3}, x_{2,5}, x_{3,5}\}$, (see Figure 4(a)).

(ii) If $e = x_{1,2}x_{1,3}$. Take $S_2 = S_1 - \{x_{1,2}\} \cup \{x\}$, (see Figure 4(b)).

(iii) If $e = x_{2,1}x_{3,1}$, or $x_{2,2}x_{3,2}$. Take $S_3 = \{x_{2,1}, x_{3,1}, x_{2,2}, x_{3,2}, x_{4,4}, x_{4,5}, x_{1,4}, x_{1,5}\}$, (see Figure 4(c)).

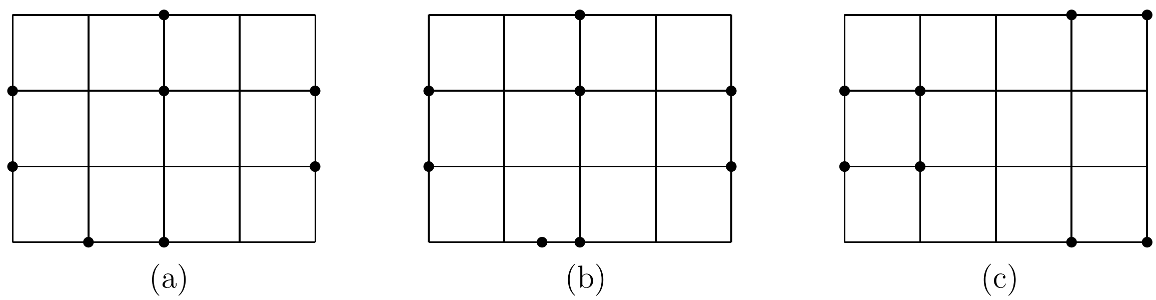
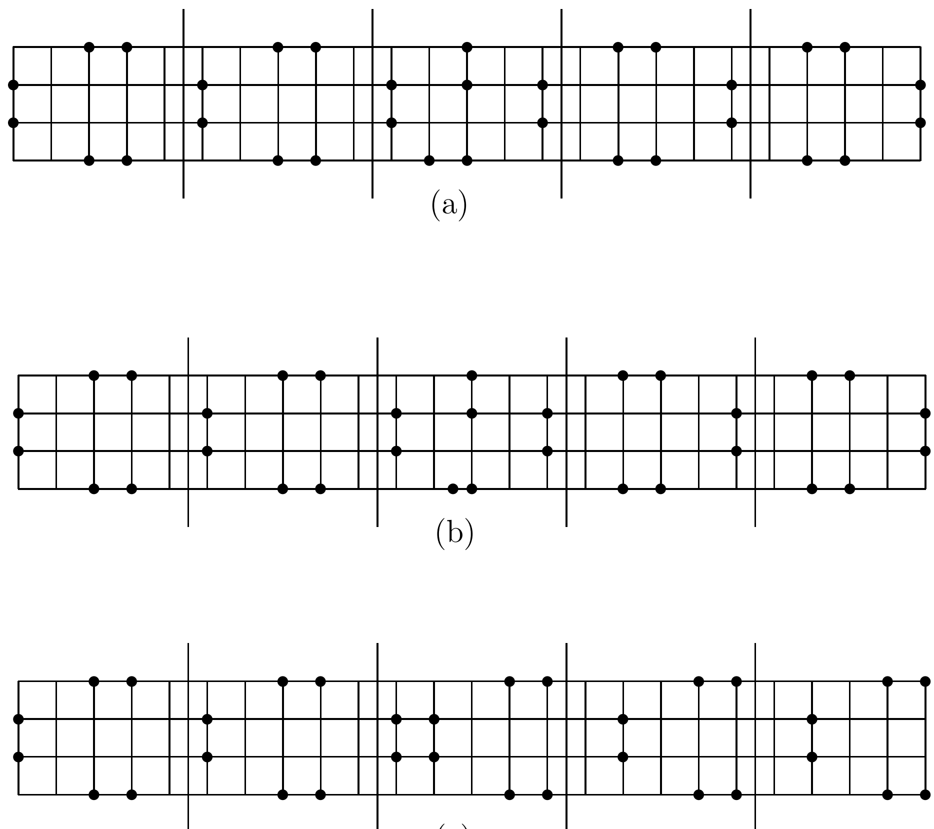


Figure 4

In all above cases S_i, $(1 \leq i \leq 3)$ is a total dominating set of size 8 for $P^1_{4,5}$. Now let $n \geq 10$, $n = 5q$, and let the arbitrary subdivided edge e be $x_{i,j}x_{i,j+1}$ or $x_{i,j}x_{i+1,j}$ where $j = 5l + r$, $0 \leq r \leq 4$. We give a total dominating set S of $P^1_{4,n}$ with cardinality $\gamma_t(P_{4,n})$ as follows: We partition (split) the set of columns of $P^1_{4,n}$ into blocks B_i, $B_i \cong P_{4,5}$ for $i = 1, \ldots, l-1, l+1, \ldots, q$ and into block B_l, $B_l \cong P^1_{4,5}$ and we dominate each such block B_i as follows. If the edge e is in one of the cases (i) or (ii) dominate B_l by a set isomorphic to set S_1, or S_2, respectively and dominate each B_i, $i = 1, \ldots, l-1$ by a set isomorphic to set $P_1 = \{x_{2,1}, x_{3,1}, x_{4,3}, x_{4,4}, x_{1,3}, x_{1,4}\}$ and dominate each B_i, $i = l+1, \ldots, q$ by a set isomorphic to set $P_2 = \{x_{4,2}, x_{4,3}, x_{1,2}, x_{1,3}, x_{2,5}, x_{3,5}\}$ (see Figure 4(a), 4(b)).

If the edge e is in the case (iii), dominate B_l by a set isomorphic to set S_3 and dominate each B_i, $i = 1, \ldots, l-1$ by a set isomorphic to set P_1 and dominate each B_i, $i = l+1, \ldots, q$ by a set isomorphic to set $P_3 = \{x_{2,2}, x_{3,2}, x_{1,4}, x_{1,5}, x_{4,4}, x_{4,5}\}$ (see Figure 4(c)).



(c)

Figure 5

Therefore $sd_{\gamma_t}(P_{4,n}) \geq 2$. We show next that $sd_{\gamma_t}(P_{4,n}) \leq 2$. Let $P^{2^*}_{4,n}$ be obtained from $P_{4,n}$ by adding new vertices x and y subdividing the edges $x_{1,1}x_{1,2}$ and $x_{2,1}x_{2,2}$, respectively. By induction on n, we show that $\gamma_t(P^{2^*}_{4,n}) > \gamma_t(P_{4,n})$ ($sd_{\gamma_t}(P_{4,n}) \leq 2$). For the smallest n, $n = 5$, we show that $sd_{\gamma_t}(P_{4,5}) \leq 2$. By Theorem E, $\gamma_t(P_{4,5}) = 8$. Let S be a total dominating set on $P^2_{4,5}$. The last two columns of $P^{2^*}_{4,5}$ is a block B, $B \cong P_{4,2}$. To totally dominate $P_{4,2}$ we need at least 4 vertices, since by Theorem C, $\gamma_t(P_{4,2}) = 4$, then $|S \cap B| \geq 4$. The first two columns of $P^{2^*}_{4,5}$ is a block B', $B' \cong P^{2^*}_{4,2}$. To totally dominate $P^{2^*}_{4,2}$ we need at least 5 vertices, since by Case 3 of Theorem 3.1, $sd_{\gamma_t}(P_{4,2}) = 2$, then $|S \cap B'| \geq 5$. At most the third column $(P_4)_3$ of $P^{2^*}_{4,5}$ can be totally dominated from the adjacent columns $(P_4)_2$ and $(P_4)_4$ (from the adjacent blocks B and B'). Therefore $|S| \geq 9$ and we can see that to totally dominate $P^{2^*}_{4,5}$ we need at least 9 vertices. This implies $sd_{\gamma_t}(P_{4,5}) = 2$. The rest of the proof of this case is very similar to the Case 2 and is therefore omitted. ■

THEOREM 3.4 *For $n, m \geq 3$, $sd_{\gamma_t}(P_{m,n}) \leq 3$.*

PROOF. Let $P^3_{m,n}$ be obtained from $P_{m,n}$ by adding new vertices x, y and z subdividing of the edges $x_{1,1}x_{1,2}$, $x_{1,1}x_{2,1}$ and $x_{2,1}x_{3,1}$, respectively and let S' be a $\gamma_t(P^3_{m,n})$- set and $D = S' \cap \{x, y, z\}$. Clearly S' contains at least one of x and y to dominate $x_{1,1}$. Hence $|D| \geq 1$. Also to totally dominate y, we have $|S' \cap \{x_{1,1}, x_{2,1}\}| = 1$. (If $x_{1,1}, x_{2,1} \in S'$, then $S' \setminus D$ is a total dominating set for $P_{m,n}$ and the proof is complete). We consider two cases.

Case 1. $x_{1,1} \in S'$. If $|D| \geq 2$, then let $S = (S' \cup \{x_{2,1}\}) \setminus D$. Assume that $|D| = 1$. If $x \in S'$, then $x_{2,2} \in S'$. Let $S = (S' \cup \{x_{2,1}\}) \setminus \{x_{1,1}, x\}$. If $y \in S'$, then $x_{3,1} \in S'$. Let $S = (S' \cup \{x_{2,1}\}) \setminus \{x_{1,1}, y\}$.

Case 2. $x_{2,1} \in S'$. If $|D| \geq 2$, then let $S = (S' \cup \{x_{1,1}\}) \setminus D$. Assume that $|D| = 1$. If $x \in S'$, then $x_{2,2}, x_{1,2} \in S'$. Let $S = S' \setminus D$. If $y \in S'$, then $x_{1,2} \in S'$. Let $S = (S' \cup \{x_{2,2}\}) \setminus \{y, x_{2,1}\}$.

In all cases, S is a total dominating set for $P_{m,n}$ and $|S| < |S'|$. This implies that $\gamma_t(P_{m,n}) < \gamma_t(P^3_{m,n})$ and the proof is complete. ■

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