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Mininjective Dimensions

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Abstract

In this article, the mininjective dimensions are introduced, some equivalent characterizations of mininjective modules and mininjective rings are given, and some homology properties of mininjective modules have been studied.

Keywords: mininjective modules; min-coherent rings; dimensions

1 Introduction

The research of the dimensions of the modules plays a very important role to investigate the properties of the modules. It has been proven that every left R-module has a mininjective cover over a left min-coherent ring(see[4]), so every left R-module has a left mininjective resolution. On the other hand, every left R-module has a mininjective preenvelope over any ring R (see[5]), so every left R-module has a right mininjective resolution. Accordingly, in this article we introduce the concept of mininjective dimensions of modules, and some equivalent characterizations of mininjective modules and mininjective rings are given, some homology properties of mininjective modules have also been studied.

2 Preliminary Notes

Throughout this article, R is an associative ring with identity and all modules are unitary R-modules. We write ${}_R\mathrm{M}$ and MI for the categories of all left

R-module and all mininjective left R-module. For any left R-module M, the character module M^+ is defined by $M^+ = Hom_Z(M, Q/Z)$. A left R-module M is called mininjective in case $Ext^1_R(R/I, M) = 0$ for any simple left ideal I of R(see[3]). A right R-module M is called min-flat in case $Tor_1(M, R/I) = 0$ for any simple left ideal I of R(see[5]). As a generalization of coherent ring, the author in[5] introduced the concept of min-coherent ring, R is called a left min-coherent ring in case every simple left ideal of R is finitely presented. He also obtained the following theorem, Let R be a ring, then the following are equivalent: (1) R is a left min-coherent ring. (2) Any left R-module M is mininjective if and only if M^+ is min-flat. (3) Every right R-module M has a monic min-flat preenvelope. (4) Any direct product of min-flat right R-module is min-flat. Let R be a ring, then $({}^{\perp}\mathrm{MI}, \mathrm{MI})$ is a complete cotorsion theory(see[5]). If R is a left min-coherent ring, then every left R-module has a mininjective cover(see[4]). Recall that a ring R is a left mininjective ring if ${}_RR$ is mininjective[6].

3 Main Results

Definition 3.1 *The right* MI-*dimM of a left R-module M, denoted by right* MI-*dimM, is defined as* $inf\{n|$*there is a right* MI *of the form of M* $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots \longrightarrow F^n \longrightarrow 0$, *where* F^i *is mininjective,* $i = 0, \cdots, n\}$. *If there is no such n, set right* MI-*dimM* $= \infty$. *The global right* MI-*dimension of left R-module M, denoted by gl right* MI-*dimM, is defined to be sup*$\{$*right* MI-*dim M:M* $\in_R$ M$\}$. *The left versions can be defined similarly.*

As is mentioned in the introduction, every left R-module M has a mininjective cover over a left min-coherent ring, so every left R-module M has a left MI-resolution, that is, there is a Hom(MI,$-$) exact complex $\cdots \to F_1 \to F_0 \to M \to 0$(not necessarily exact), with each F_i is mininjective. For any left R-module N has a mininjective preenvelope over any ring, so N has a right MI-resolution, that is, there is a Hom($-$, MI) exact complex $0 \to N \to F^0 \to F^1 \to \cdots$ with each F^i is mininjective.

If R is a left min-coherent ring, then Hom($-$,$-$) is left balanced on ${}_R\mathrm{M} \times_R \mathrm{M}$ by MI $\times$ MI. Let $Ext_n(-,-)$ denote the *nth* left derived functor of Hom($-$,$-$) with respect to the pair MI $\times$ MI. Then, for two left R-module M and N, $Ext_n(M, N)$ can be computed using a right MI-resolution of M or a left MI-resolution of N. Let $0 \longrightarrow M \xrightarrow{g} F^0 \xrightarrow{f} F^1 \longrightarrow \cdots$ be a right MI-resolution of M, applying $Hom(-, N)$, we obtain the deleted complex

$$\cdots \longrightarrow Hom(F^1, N) \xrightarrow{f^*} Hom(F^0, N) \longrightarrow 0\,.$$

Then $Ext_n(M,N)$ is the nth homology of the complex above. There is a canonical map

$$\sigma : Ext_0(M,N) = Hom(F^0,N)/Im(f^*) \to Hom(M,N)$$

defined by $\sigma(\alpha + Im(f^*)) = \alpha g,\ \alpha \in Hom(F^0,N)$.

Proposition 3.2 *Let R be a left min-coherent ring, the following are equivalent for a left R-module M.*

(1) M is mininjective.

(2) The canonical map σ:$Ext_0(M,N) \to Hom(M,N)$ is an epimorphism for any R-module N.

(3) The canonical map σ:$Ext_0(M,M) \to Hom(M,M)$ is an epimorphism.

Proof *is obvious.*

(2)$\Longrightarrow$(3) is trivial.

(3)$\Longrightarrow$(1) By(3), there exist $\alpha \in Hom(F^0,M)$ such that $\sigma(\alpha + Im(f^*)) = \alpha g = 1_M$. Thus M is isomorphic to a direct summand of F^0, hence M is mininjective.

Proposition 3.3 *The following are equivalent for a left min-coherent ring.*

(1) R is a left mininjective ring.

(2) The canonical map σ:$Ext_0({}_RR,N) \to Hom({}_RR,N)$ is an epimorphism for any R-module N.

(3) The canonical map σ:$Ext_0({}_RR,{}_RR) \to Hom({}_RR,{}_RR)$ is an epimorphism.

(4) Every left R-module has an epic mininjective cover.

(5) Every right R-module is a submodule of a min-flat right R-module.

(6) Every injective right R-module is min-flat.

(7) Every right R-module has a monic min-flat preenvelope.

(8) Every flat left R-module is mininjective.

Proof *(1)$\Longleftrightarrow$* (2) *$\Longleftrightarrow$(3) follow from Proposition* 3.2.

(1)$\Longrightarrow$(4) Let M be a left R-module, M has a mininjective cover g:$N \to M$, on the other hand there is an exact sequence $F \to M \to 0$ with F is free. Since F is mininjective by (1), g is an epimorphism.

(4)$\Longrightarrow$(1) Let f:$N \to {}_RR$ be an epic mininjective cover. Then ${}_RR$ is isomorphism to a direct summand of N, so ${}_RR$ is mininjective.

(1)$\Longrightarrow$(5) Let N be any right R-module, there is an exact sequence $0 \to N \to \prod({}_RR^+)$, $\prod({}_RR^+)$ is min-flat, so (5) follows.

(5)$\Longrightarrow$(6) is obvious.

(6)$\Longrightarrow$(1) Since ${}_RR^+$ is injective, by (6) ${}_RR^+$ is min-flat, ${}_RR$ is mininjective, so R is a left mininjective ring.

(5)$\Longleftrightarrow$(7) is clear.

(6)$\Longrightarrow$(8) Let M be any flat left R-module, then M^+ is injective, by (6) M^+ is min-flat, M is mininjective.

(8)$\Longrightarrow$(1) is obvious

Proposition 3.4 *Let R be a left min-coherent ring, the following are equivalent for a left R-module M.*

(1) right MI-*dim*$M \leq 1$.

(2) The canonical map σ:$Ext_0(M,N) \to Hom(M,N)$ is a monomorphism for any left R-module N.

Proof *(1)$\Longrightarrow$(2) By (1), M has a right* MI*-resolution*

$0 \longrightarrow M \xrightarrow{g} F^0 \xrightarrow{f} F^1 \longrightarrow 0$, *thus we get an exact sequence*

$0 \longrightarrow Hom(F^1,N) \xrightarrow{f^*} Hom(F^0,N) \xrightarrow{g^*} Hom(M,N)$ *for any left R-module N. So σ is a monomorphism.*

(2)$\Longrightarrow$(1) Consider the exact sequence $0 \to M \to F^0 \to L^1 \to 0$, where $M \to F^0$ is a mininjective preenvelope of M. We only need to show that L^1 is mininjective, by [1,Theorem 8.2.3], we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & & Ext_0(L^1,L^1) & \longrightarrow & Ext_0(F^0,L^1) & \longrightarrow & Ext_0(M,L^1) & \longrightarrow & 0 \\
 & & {\scriptstyle\sigma_1}\downarrow & & {\scriptstyle\sigma_2}\downarrow & & {\scriptstyle\sigma_3}\downarrow & & \\
0 & \longrightarrow & Hom(L^1,L^1) & \longrightarrow & Hom(F^0,L^1) & \longrightarrow & Hom(M,L^1) & &
\end{array}$$

Note that σ_2 is an epimorphism by Proposition 3.2*, and σ_3 is a monomorphism by (2). Hence, σ_1 is an epimorphism by [2,Theorem 6.5]. Thus L^1 is mininjective by Proposition* 3.2*, and so (1) follows.*

Theorem 3.5 *Let R be a left min-coherent ring and an integer $n \geq 2$. The following are equivalent for a left R-module M.*

(1) gl right MI-*dim*$M \leq n$.

(2) $Ext_{n+k}(M,N) = 0$ for all left R-module N and $k \geq -1$.

(3) $Ext_{n-1}(M,N) = 0$ for all left R-module N.

Proof *(1)$\Longrightarrow$(2) Let $0 \to M \to F^0 \to F^1 \to \cdots \to F^n \to 0$ be a right* MI*-resolution of M, which induces an exact sequence $0 \to Hom(F^n,N) \to Hom(F^{n-1},N) \to Hom(F^{n-2},N)$ for any left R-module N. Hence, $Ext_n(M,N) = Ext_{n-1}(M,N) = 0$. Note that it is clear $Ext_{n+k}(M,N) = 0$ for $k \geq -1$.*

(2)$\Longrightarrow$(3) is trivial.

(3)$\Longrightarrow$(1) Let $0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-1} \to F^n$ be partial right MI*-resolution of M with $L^n = coker(F^{n-2} \to F^{n-1})$. We only need to show that L^n is mininjective. In fact, we have the following exact commutative*

diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & F^0 \arrow[r] & \cdots \arrow[r] & F^{n-2} \arrow[r, "f"] & F^{n-1} \arrow[r, "g"] \arrow[dr, "\pi"'] & F^n \arrow[r] & \cdots \\
& & & & & & L^n \arrow[u, "\lambda"'] \arrow[dr] & \\
& & & & & & 0 \arrow[u] & 0
\end{tikzcd}$$

where $\pi{:}F^{n-1} \to L^n$ *is an natural epimorphism,* $\lambda{:}L^n \to F^n$ *is a mininjective preenvelope, and* $g = \lambda\pi$. *By (3)* $Ext_{n-1}(M, L^n) = 0$*, thus the sequence* $Hom(F^n, L^n) \xrightarrow{g^*} Hom(F^{n-1}, L^n) \xrightarrow{f^*} Hom(F^{n-2}, L^n)$ *is exact. Since* $f^*(\pi) = \pi f = 0$*, then* $\pi \in ker(f^*) = im(g^*)$. *Thus there exists* $h \in Hom(F^n, L^n)$ *such that* $\pi = g^*(h) = hg = h\lambda\pi$*, and hence* $h\lambda = 1$ *since* π *is an epimorphism. Therefore* L^n *is mininjective.*

Theorem 3.6 *Let Let* R *be a left min-coherent ring,* M *is a left* R*-module, and an integer* $n \geq 0$. *Then right* MI-$dimM \leq n$ *if and only if for every left* MI-*resolution* $\cdots \to F_n \to F_{n-1} \cdots \to F_1 \to F_0 \to N \to 0$ *of every left* R*-module* N, $Hom(M, F_n) \to Hom(M, K_n)$ *is an epimorphism, where* $K_n = ker(F_{n-1} \to F_{n-2})$.

Proof *We proceed by induction on* n. *Let* $n = 0$*, if* M *is mininjective, it is clear that* $Hom(M, F_0) \to Hom(M, K_0)$ *is an epimorphism. Conversely, put* $M = N$*, then* $Hom(M, F_0) \to Hom(M, M)$ *is an epimorphism, and so* M *is mininjective.*

Let $n \geq 2$*, for any left* R*-module* M*, there is an exact sequence* $0 \to M \to E \to L \to 0$ *with* E *is mininjective and* $Ext^1_R(L, G) = 0$ *for any mininjective* G. *Then we have the following exact commutative diagrams:*

$$\begin{array}{ccccc}
Hom(E, F_n) & \longrightarrow & Hom(E, K_n) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
Hom(M, F_n) & \longrightarrow & Hom(M, K_n) & & \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

and

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Hom(L,K_n) \arrow[r] \arrow[d] & Hom(L,F_{n-1}) \arrow[r] \arrow[d] & Hom(L,K_{n-1}) \arrow[d] & \\
0 \arrow[r] & Hom(E,K_n) \arrow[r] \arrow[d] & Hom(E,F_{n-1}) \arrow[r] \arrow[d] & Hom(E,K_{n-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Hom(M,K_n) \arrow[r] & Hom(M,F_{n-1}) \arrow[r] \arrow[d] & Hom(M,K_{n-1}) & \\
 & & 0 & &
\end{tikzcd}
$$

where $K_{n-1} = ker(F_{n-2} \to F_{n-3})$. *Then right* MI-*dim*$M \leq n$ *if and only if right* MI-*dim*$L \leq n-1$ *if and only if* $Hom(L, F_{n-1}) \to Hom(L, K_{n-1})$ *is epic by induction if and only if* $Hom(E, K_n) \to Hom(M, K_n)$ *is an epimorphism by the second diagram if and only if* $Hom(M, F_n) \to Hom(M, K_n)$ *is an epimorphism by the first diagram.*

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