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Modified Algorithm to Compute Adomian's Polynomial for Solving Non-Linear Systems of Partial Differential Equations

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Abstract: In this paper, we modified the method of computing Adomian's polynomial to find the numerical solution for non-linear systems of partial differential equations (PDEs) with less number of components, more accuracy and faster convergence when compared with the standard Adomian decomposition method (ADM).

Keywords Adomian Decomposition Method, Adomian's Polynomial, Non-linear system of Partial Differential Equations

1. Introduction

The partial differential equations (PDEs) have so many essential applications of science and engineering such as fluid mechanic, thermodynamic, heat transfer, physics, chemistry , micro electro mechanic system, etc. Most of these equations are nonlinear PDEs which are employed in modeling many physical and scientific phenomena such as thermal engineering, acoustic, electromagnetism, control, robotics, viscosity, diffusion, edge detection, turbulence, signal processing and many other processes. It is difficult to handle nonlinear part of these equations, although most of the scientists applied numerical methods to find the solution of these equations that based on linearization, perturbation, discretization.

In this paper, we compute numerical solutions to non-linear systems of PDEs by using ADM with modifying the method of computing Adomian's polynomial by formularizing it with another form to make the numerical solutions become easier and higher accuracy than the standard ADM. ADM has been proposed by the American mathematician, George Adomian (1923-1996) initially in 1980s with the aims to solve frontier physical problem[20], has been applied to a wide class of deterministic and stochastic problems, linear and nonlinear in physics, biology and chemical reactions etc. For nonlinear models, the method has shown reliable results in supplying analytical approximation that converges very rapidly [3, 16, 12, 6]. There are also some papers work on modified ADM by proposed new formula for the calculation of the Adomian's polynomials associated to nonlinear operator [1, 24, 25, 7, 10]. The convergence of the ADM was given by [11] who using fixed point theorem for abstract functional equations. There are many papers on the convergence of ADM were published, including the works of [1, 15, 23]. Furthermore, [8] introduced the order of convergence of ADM [10, 13] gave another view on the error analysis of ADM. also proposed a new definition for the technique to prove the convergence under suitable and reasonable hypotheses[23, 18, 20, 5].

2. An Analysis of ADM

It is a powerful method for solving nonlinear functional equations. This technique is based on a decomposition of a solution of a nonlinear functional equation in a series of functions, each term of this series is obtained from a polynomial generated by a power series expansion of an analytic function and it is very simple in an abstract formulation but the difficulty arises in calculating the polynomials and in proving the convergence of the series of functions [1, 2, 6, 10]. Consider the general non-linear coupled of PDEs written in an operators form [17]

$$L_t(u) + R_1(u) + M_1\,(u) + N_1\,(u,v) = f_1(x,t),$$
$$L_t(v) + R_2(v) + M_2\,(v) + N_2\,(u,v) = f_2(x,t), \qquad (2.1)$$

subject to initial conditions

$$u(x,0) = g_1(x)\ ,\ 0 \le x \le l_1 \quad \text{and} \quad v(x,0) = g_2(x)\ ,\ 0 \le x \le l_2,$$

where the notations $L_t = \frac{\partial}{\partial t}$, R_1 and R_2 symbolize the linear spatial differential operators, the notations M_1, M_2, N_1 and N_2 symbolize the nonlinear differential operators and $f_1(x,t),\ f_2\,(x,t)$ (f_1, f_2 for simplicity) are given functions.

2-1 The Standard ADM: The principal algorithm of ADM is given by applying the inverse operator, $L_t^{-1} = \int_0^t (\cdot)\,dt$ to both sides (2.1) yields [24, 3]:

$$u(x,t) = g_1(x) + L_t^{-1}\,f_1 - L_t^{-1}[R_1(u) + M_1\,(u) + N_1\,(u,v)],$$
$$v(x,t) = g_2(x) + L_t^{-1}\,f_2 - L_t^{-1}[R_2(v) + M_2\,(v) + N_2\,(u,v)], \qquad (2.2)$$

the standard ADM suggests that the linear functions $u(x,t)$ and $v(x,t)$ can be decomposed by an infinite series of components

$$u(x,t) = \sum_{n=0}^{\infty} u_n\,(x,t), \qquad v(x,t) = \sum_{n=0}^{\infty} v_n\,(x,t),$$

and the nonlinear operators M_1, M_2 , N_1 and N_2 by the infinite series

$M_1(u) = \sum_{n=0}^{\infty} A_n$, $N_1(u,v) = \sum_{n=0}^{\infty} B_n$, $M_2(v) = \sum_{n=0}^{\infty} C_n$, $N_2(u,v) = \sum_{n=0}^{\infty} D_n$, (2.3)

where $u_n(x,t)$ and $v_n(x,t)$, $n \geq 0$ are the components of $u(x,t)$ and $v(x,t)$ that will be elegantly determined and A_n , B_n, C_n and D_n are called Adomian's polynomials. For the nonlinear operators $M_1(u)$ and $N_1(u,v)$ the Adomian's polynomial can be defined by

$$A_n\,(u_0,\dots,u_n) = \frac{1}{n!}\,\frac{d^n}{d\lambda^n}\left[M_1\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0} ,$$

$$B_n\,(u_0,\dots,u_n; v_0,\dots,v_n) = \frac{1}{n!}\,\frac{d^n}{d\lambda^n}\left[N_1\left(\sum_{i=0}^{n} \lambda^i u_i\,, \sum_{i=0}^{n} \lambda^i v_i\right)\right]_{\lambda=0} , \qquad (2.4)$$

according to the decomposition method [16, 21, 22] the non-linear system (2.2) is constructed in a form of the following recursive relations:

$$u_0(x,t) = g_1(x) + {L_t}^{-1} f_1 \;, \qquad v_0(x,t) = g_2(x) + {L_t}^{-1} f_2 \;,$$

and,

$$u_{n+1}(x,t) = -{L_t}^{-1}[R_1(u) + A_n + B_n] \;,$$
$$v_{n+1}(x,t) = -{L_t}^{-1}[R_2(v) + C_n + D_n] . \qquad n \geq 0 \qquad (2.5)$$

It is worth noting that the zeroth components u_0 and v_0 are defined, then the remaining components u_n and v_n , n ≥ 1 can be completely determined by such way that each component is computed by using the previous terms. As a result, the components $u_0, u_1, u_2, \dots$ and $v_0, v_1, v_2 \dots$ are identified and the series solutions are thus entirely determined. For numerical comparison purposes, we constructed the solution $u(x,t)$ and $v(x,t)$ as follows:

$$\lim_{n\to\infty} \Phi_n\,(x,t) = u(x,t), \qquad \lim_{n\to\infty} \Psi_n\,(x,t) = v(x,t),$$

where,

$$\Phi_n(x,t) = \sum_{k=0}^{n} u_k\,(x,t), \quad \Psi_n(x,t) = \sum_{k=0}^{n} v_k\,(x,t). \;\; n \geq 0, \qquad (2.6)$$

the recurrence relation is given as in (2.5) and (2.6). Moreover, the decomposition series solutions generally converged very rapidly for real physical problems [3, 18].

2-2 The Modified ADM: We present an efficient modification of ADM that will facilitate the calculations and accelerate the convergence by decomposing the nonlinear term in two parts with taking into consideration not to repeat the term for more than time in computing the polynomials. So, that the polynomials for the nonlinear term $M_1(u) = \overline{M}_1\,(m_1, m_2)$ can be obtained as follows:

$\bar{A}_0(u_0) = \overline{M}_1\left({m_1}_0, {m_2}_0\right)$

$$\bar{A}_1(u_0,u_1) = \bar{M}_1\left(m_{1_0},m_{2_1}\right) + \bar{M}_1\left(m_{1_1},m_{2_0}\right) + \bar{M}_1\left(m_{1_1},m_{2_1}\right)$$
$$\bar{A}_2(u_0,u_1,u_2) = \bar{M}_1\left(m_{1_0},m_{2_2}\right) + \bar{M}_1\left(m_{1_2},m_{2_0}\right) + \bar{M}_1\left(m_{1_1},m_{2_2}\right) + \bar{M}_1\left(m_{1_2},m_{2_1}\right) + \bar{M}_1\left(m_{1_2},m_{2_2}\right)$$
$$\vdots$$

The $\bar{A}_n$ can be finally written as the following convenient relation

$$\bar{A}_i = \sum_{j=i}^{2i} \rho^j(\bar{M}_1(m_1,m_2)) = \sum_{\substack{j=i\\k+h=j}}^{2i} \bar{M}_1\left(m_{1_k},m_{2_h}\right), \quad i,k,h = 0,1,2,\dots \quad (2.7)$$

where ρ is decompose of the nonlinear term and $m_1,\ m_2$ represent the dependent variable u.

By the same procedure we get the polynomials for the nonlinear term $N_1(u,v) = \bar{N}_1\,(n_1,n_2)$ as follows:

$$\bar{B}_0(u_0,v_0) = \bar{N}_1\left(n_{1_0},n_{2_0}\right)$$
$$\bar{B}_1(u_0,u_1;v_0,v_1) = \bar{N}_1\left(n_{1_0},n_{2_1}\right) + \bar{N}_1\left(n_{1_1},n_{2_0}\right) + \bar{N}_1\left(n_{1_1},n_{2_1}\right)$$
$$\bar{B}_2(u_0,u_1,u_2;v_0,v_1,v_2) = \bar{N}_1\left(n_{1_0},n_{2_2}\right) + \bar{N}_1\left(n_{1_2},n_{2_0}\right) + \bar{N}_1\left(n_{1_1},n_{2_2}\right) + \bar{N}_1\left(n_{1_2},n_{2_1}\right) + \bar{N}_1\left(n_{1_2},n_{2_2}\right)$$
$$\vdots$$

The $\bar{B}_n$ can be finally written as the following convenient relation

$$\bar{B}_i = \sum_{j=i}^{2i} \rho^j(\bar{N}_1(n_1,n_2)) = \sum_{\substack{j=i\\k+h=j}}^{2i} \bar{N}_1\left(n_{1_k},n_{2_h}\right), \qquad i,k,h = 0,1,2,\dots \quad (2.8)$$

where n_1 and n_2 represent the dependent variable u or v. And so on, similarly for finding the polynomials $\bar{C}_n$ and $\bar{D}_n$.

3- Applications: To illustrate the modification some examples, are presented in the following.

<u>Example 1</u>: We consider the non-linear system [9]:

$$u_t = uu_x + vu_y\ , \qquad (3.1a)$$
$$v_t = vv_y + uv_x\ , \qquad (3.1b)$$

with the initial conditions

$$u(x,y,0) = x+y\,; \quad v(x,y,0) = x+y\ , \qquad 0 \le x\,,y \le 1$$

the exact solution given as:

$$u(x,y,t) = \frac{x+y}{1-2t} \quad , \quad v(x,y,t) = \frac{x+y}{1-2t} \quad . \ 0 \le t \le T \qquad (3.1c)$$

The numerical solution by standard ADM, when we use (2.2) to find ADM of (3.1a,b) which becomes:

$$u(x,y,t) = u(x,y,0) + {L_t}^{-1}[uu_x + vu_y],$$
$$v(x,y,t) = v(x,y,0) + {L_t}^{-1}[vv_y + uv_x],$$

and then by using (2.3) we obtain

$$u(x,y,t) = u(x,y,0) + {L_t}^{-1}[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n],$$
$$v(x,y,t) = v(x,y,0) + {L_t}^{-1}[\sum_{n=0}^{\infty} C_n + \sum_{n=0}^{\infty} D_n],$$

where

$$uu_x = \sum_{n=0}^{\infty} A_n, \quad vu_y = \sum_{n=0}^{\infty} B_n, vv_y = \sum_{n=0}^{\infty} C_n, \quad uv_x = \sum_{n=0}^{\infty} D_n,$$

the Adomian's polynomials A_n, B_n, C_n and D_n are generated according to (2.4) we can give the first few Adomian's polynomials of A_n and B_n respectively:

$$\mathrm{A}_0 = u_0 u_{0_x}, \qquad \mathrm{B}_0 = v_0 u_{0_y},$$
$$\mathrm{A}_1 = u_0 u_{1_x} + u_1 u_{0_x}, \qquad \mathrm{B}_1 = v_0 u_{1_y} + v_1 u_{0_y},$$
$$\mathrm{A}_2 = u_0 u_{2_x} + u_1 u_{1_x} + u_2 u_{0_x}, \qquad \mathrm{B}_2 = v_0 u_{2_y} + v_1 u_{1_y} + v_2 u_{0_y},$$
$$\vdots$$

and so on, the polynomials C_n and D_n can be constructed in the same way as in the above procedure. According to (2.5) the zeroth components u_0 and v_0 written as follows:

$$u_0(x,y,t) = u(x,y,0), \qquad v_0(x,y,t) = v(x,y,0),$$

and the recursive relation can be written as follows:

$$u_{n+1}(x,y,t) = {L_t}^{-1}[A_n + B_n],$$

$$v_{n+1}(x,y,t) = {L_t}^{-1}[C_n + D_n].$$

where, $n \geq 0$. So, we get the following components:

$$u_0 = x + y, \qquad v_0 = x + y,$$
$$u_1 = 2xt + 2yt, \qquad v_1 = 2xt + 2yt,$$
$$u_2 = \frac{1}{2}(8x + 8y)t^2, \qquad v_2 = \frac{1}{2}(8x + 8y)t^2,$$
$$\vdots$$

substituting all components in decomposition series (2.6) which is a Taylor series we obtain the closed form solutions (3.1c).

Now, if we use the modified method (2.7) and (2.8) for nonlinear operators M_1 and N_1 we will get the following form of $\bar{A}_n$ and $\bar{B}_n$ respectively:

$$M_1(u) = uu_x = \bar{M}_1(m_1, m_2), \qquad N_1(u,v) = vu_y = \bar{N}_1(n_1, n_2)$$
$$\bar{A}_0 = u_0 u_{0_x}, \qquad \bar{B}_0 = v_0 u_{0_y},$$
$$\bar{A}_1 = u_0 u_{1_x} + u_1 u_{0_x} + u_1 u_{1_x}, \qquad \bar{B}_1 = v_0 u_{1_y} + v_1 u_{0_y} + v_1 u_{1_y},$$

$\bar{A}_2 = u_0 u_{2_x} + u_2 u_{0_x} + u_1 u_{2_x} + u_2 u_{1_x} + u_2 u_{2_x},$
$\bar{B}_2 = v_0 u_{2_y} + v_2 u_{0_y} + v_1 u_{2_y} + v_2 u_{1_y} + v_2 u_{2_y}$
$\vdots$
and so on, similarly for the polynomials $\bar{C}_n$ and $\bar{D}_n$.

By using (2.5) and (2.6) it is worth mentioning that zeroth and first components are similar in two formulas of standard ADM but another components are different, for this reason the approximate solution has higher accuracy and faster convergence to the exact solution than the standard ADM where the another components can be written as follows:

$$u_2 = \frac{1}{3}(8x + 8y)\,t^3 + \frac{1}{2}(8x + 8y)\,t^2,$$
$$v_2 = \frac{1}{3}(8x + 8y)\,t^3 + \frac{1}{2}(8x + 8y)\,t^2$$
$$\vdots$$

so on, by substituting the above components in the decomposition series (2.6) we get the following numerical solutions of $u(x, y, t)$ and $v(x, y, t)$ which gives us the closed form solutions (3.1c). The numerical results are listed in Table 1.

Example 2: The mathematical models on many phenomena in applied sciences lead to non-linear PDEs such as the homogeneous form of the system of two-dimensional Burger's equations which is proposed as mathematical model of free turbulence [12, 14]

$$u_t + uu_x + vu_y = \frac{1}{R}(u_{xx} + u_{yy}), \qquad (3.2a)$$
$$v_t + vv_y + uv_x = \frac{1}{R}(v_{xx} + v_{yy}), \qquad (3.2b)$$

subject to the initial conditions

$$u(x, y, 0) = \frac{3}{4} - \frac{1}{4\left(1 + e^{R(y-x)/8}\right)}, \qquad v(x, y, 0) = \frac{3}{4} + \frac{1}{4\left(1 + e^{R(y-x)/8}\right)},$$

the exact solution [17]:

$$u(x, y, t) = \frac{3}{4} - \frac{1}{4\left(1 + e^{R(-t+4y-4x))/32}\right)},$$
$$v(x, y, t) = \frac{3}{4} + \frac{1}{4\left(1 + e^{R(-t+4y-4x))/32}\right)}, \quad (3.2c)$$

where R is Reynolds number. As the same procedure in the above example when we using equations (2.2)-(2.4) for the system (3.2a, b) and according to (2.5) we have the following zeroth components:

$$u_0 = \frac{3}{4} - \frac{1}{4(1 + e^{R(y-x))/8})}, \qquad v_0 = \frac{3}{4} + \frac{1}{4(1 + e^{R(y-x))/8})},$$

and the recursive relation can be written as follows:

$$u_{n+1}(x,y,t) = {L_t}^{-1}\left[\frac{1}{R}(u_{n_{xx}} + u_{n_{yy}}) - A_n - B_n\right],$$

$$v_{n+1}(x,y,t) = {L_t}^{-1}\left[\frac{1}{R}(v_{n_{xx}} + v_{n_{yy}}) - C_n - D_n\right],$$

where $n \geq 0$.

Substituting all components decomposition series (2.6) which is a Taylor series we obtain the closed form solutions (3.2c).

Now, if we use another formula for Adomian's polynomials $A_n,\ B_n,\ C_n$ and D_n by applying equation (2.7) and (2.8) for nonlinear operators M_1 , N_1 respectively we will get the forms of $\bar{A}_n$ and $\bar{B}_n$ as follows

$$M_1\,(u) = uu_x = \bar{M}_1(m_1, m_2) \quad , \quad N_1\,(u,v) = vu_y = \bar{N}_1(m_1, m_2)$$
$$\bar{A}_0 = u_0 u_{0_x} \quad , \quad \bar{B}_0 = v_0 u_{0_y},$$
$$\bar{A}_1 = u_0 u_{1_x} + u_1 u_{0_x} + u_1 u_{1_x}, \quad \bar{B}_1 = v_0 u_{1_y} + v_1 u_{0_y} + v_1 u_{1_y},$$
$$\bar{A}_2 = u_0 u_{2_x} + u_2 u_{0_x} + u_1 u_{2_x} + u_2 u_{1_x} + u_2 u_{2_x},$$
$$\bar{B}_2 = v_0 u_{2_y} + v_2 u_{0_y} + v_1 u_{2_y} + v_2 u_{1_y} + v_2 u_{2_y},$$
$$\vdots$$

and so on, the other the polynomials $\bar{C}_n$ and $\bar{D}_n$ can be constructed in a similar manner. This yields the modifier ADM which is differ from standard ADM from the second component and so on where the components forms are as follows:

$$u_2 = \frac{1}{196608}\,\frac{1}{\left(1 + e^{\frac{1}{8}R\,(y-x)}\right)^5}\left(R^2\, e^{\frac{1}{8}R\,(y-x)}\, t^2\left(-R\, e^{\frac{1}{4}R\,(y-x)}\, t\right.\right.$$

$$v_2 = \frac{1}{196608}\,\frac{1}{\left(1 + e^{\frac{1}{8}R\,(y-x)}\right)^5}\left(R^2\, e^{\frac{1}{8}R\,(y-x)}\, t^2\left(-R\, e^{\frac{1}{4}R\,(y-x)}\, t + R\, e^{\frac{1}{8}R\,(y-x)}\, t + 24 + 24\, e^{\frac{1}{8}R\,(y-x)} - 24\, e^{\frac{1}{4}R\,(y-x)} - 24\, e^{\frac{3}{8}R\,(y-x)}\right)\right)$$

$$\vdots$$

Then by using all these components and the initial conditions in the decomposition series (2.6) we find the numerical solution for the non-linear system (3.2a, b). The numerical results are listed in Tables 2a-2b.

<u>Example 3:</u> We consider three typical equations in the hierarchy which are a new generalized Hirota–Satsuma coupled KdV equation as [19]:

$$u_t = \frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x \qquad (3.3a)$$
$$v_t = -v_{xxx} + 3u\, v_x \qquad (3.3b)$$
$$w_t = -w_{xxx} + 3u\, w_x \qquad (3.3c)$$

with the initial conditions

$$u(x,0)=\frac{1}{3}(\beta-2k^2)+2k^2tanh^2[kx],$$
$$v(x,0)=-\frac{4k^2c_0(\beta+k^2)}{3{c_1}^2}+\frac{4k^2(\beta+k^2)}{3c_1}tanh[kx], \qquad (3.3d)$$
$$w(x,0)=c_{0+}+c_1tanh[kx],$$

the analytical solution

$$u(x,t)=\frac{1}{3}(\beta-2k^2)+2k^2\tanh^2[k(x+\beta t)] \quad ,$$
$$v(x,t)=-\frac{4k^2c_0(\beta+k^2)}{3{c_1}^2}+\frac{4k^2(\beta+k^2)}{3c_1}\tanh[k(x+\beta t)] \quad , \qquad (3.3e)$$
$$w(x,t)=\ c_0+c_1\tanh[k(x+\beta t)] \quad ,$$

where $k, c_0, c_1 \neq 0$ and β are arbitrary constants. To calculate the terms of the decomposition series (2.6) for $u(x,t), v(x,t)$ and $w(x,t)$, we shall apply (2.2) and (2.3) then substitute the initial conditions (3.3d). By using the Adomian's polynomials (2.4) correspond to standard ADM or Adomian's polynomials (2.7) and (2.8) correspond to the modifier ADM for the nonlinear terms $uu_x, (vw)_x, uv_x$ and uw_x where

$$M_1\,(u)=uu_x=\bar{M}_1(m_1,m_2), \qquad N_1\,(v,w)=(vw)_x=\left(\bar{N}_1(m_1,m_2)\right)_x,$$
$$N_2(u,v)=uv_x=\bar{N}_1(m_1,m_2) \quad \text{and} \quad N_3(u,w)=uw_x=\bar{N}_1(m_1,m_2).$$

We can give the first few Adomian's polynomials of $A_n, \bar{A}_n, B_n$ and $\bar{B}_n$ respectively:

$A_0=u_0{u_0}_x$, $\qquad B_0=(v_0w_0)_x$,
$A_1=u_0{u_1}_x+u_1{u_0}_x$, $\qquad B_1=(v_0w_1)_x+(v_1w_0)_x$,
$A_2=u_0{u_2}_x+u_2{u_0}_x+u_1{u_1}_x$, $\qquad B_2=(v_0w_2)_x+(v_2w_0)_x+(v_1w_2)_x$,
$\vdots$

while,

$\bar{A}_0=u_0{u_0}_x$, $\qquad \bar{B}_0=(v_0w_0)_x$,
$\bar{A}_1=u_0{u_1}_x+u_1{u_0}_x+u_1{u_1}_x$, $\qquad \bar{B}_1=(v_0w_1)_x+(v_1w_0)_x+(v_1w_1)_x$,
$\bar{A}_2=u_0{u_2}_x+u_2{u_0}_x+u_1{u_2}_x+u_2{u_1}_x+u_2{u_2}_x$,
$\bar{B}_2=(v_0w_2)_x+(v_2w_0)_x+(v_1w_2)_x+(v_2w_1)_x+(v_2w_2)_x$,
$\vdots$

and so on, and the same to other polynomials $C_n, \bar{C}_n, D_n$ and $\bar{D}_n$. When we use the standard ADM we will substitute A_n, B_n, C_n and D_n in (2.5) to get the closed solution (3.3e). While substitute $\bar{A}_n, \bar{B}_n, \bar{C}_n$ and $\bar{D}_n$ in (2.5) we have the numerical solutions $u(x,t)$, $v(x,t)$ and $w(x,t)$by using the modified Adomian's. The numerical results are listed in Tables 3a-3c.

4-Numerical Implementations of ADM

In order to verify numerically whether the proposed methodology leads to the accurate solutions, we evaluate the standard ADM, modified ADM solutions using the approximation for some examples of non-linear systems of PDEs solved in the previous sections. To show the efficiency of the present methods for our problems in comparison with the exact solution we report the absolute error(**Abs.error**). The calculations in this paper have been done using the MAPLE 11 and MATHEMATICA 7 software package for the standard and the modified ADM. The results are listed in Tables1-3 below.

Table 1: The numerical results in comparison with the analytical solutions for various values of x , y and t for test problem (3.1a, b).

t	*y*	*x*	*Exact(u)*	**Abs. error1**	**Abs. error2**
0.1	0.125	0.125	0.3125000000	3.2000e-008	3. 0000e-010
		0.785	1.137500000	1.1600e-007	0.0000e+000
	0.5	0.5	1.250000000	1.2800e-007	0.0000e+000
	0.875	0.125	1.250000000	1.2800e-007	0.0000e+000
		0.785	2.075000000	2.1200e-007	0.0000e+000
0.2	0.125	0.125	0.4166666667	4.3690e-005	4. 2700e-008
		0.785	1.516666667	1.5903e-004	1. 5900e-007
	0.5	0.5	1.666666667	1.7476e-004	1. 6600e-007
	0.875	0.125	1.666666667	1.7476e-004	1. 6600e-007
		0.785	2.766666667	2.9010e-004	2. 8600e-007
0.3	0.125	0.125	0.6250000000	3.7791e-003	1. 4145e-005
		0.785	2.275000000	1.3756e-002	5. 1480e-005
	0.5	0.5	2.500000000	1.5116e-002	5. 6574e-005
	0.875	0.125	2.500000000	1.5116e-002	5. 6574e-005
		0.785	4.150000000	2.5093e-002	9. 3897e-005

We note from the above results, the **Abs. error2** obtained by the proposed algorithm as compared with **Abs. error1** of the standard algorithm for Adomian's polynomials by 9-components given results more accurate than the results using 9-components by standard ADM and in some results we obtained the value of **Abs. error2** equals to zero in [4] they obtained this value by 30-components.

Table 2: The numerical results in comparison with the analytical solutions for various values of x and t for test problem (3.2a, b) when y = 1, R = 50 and R = 100 are respectively:

Table 2a

t	*X*	*Exact(u)*	**Abs. error1**	**Abs. error2**
0.2	0.1	0.7487736486	7.4512e-003	3.2300-007
	0.2	0.7477185907	1.3845e-003	5.1341e-007
	0.3	0.7457712717	2.5604e-003	6.6847e-007
	0.4	0.7422137941	4.6943e-003	3.7530e-007
	0.5	0.7358362425	8.4719e-003	1.5728e-006
0.5	0.1	0.7480460278	1.7787e-003	1.3929e-005
	0.2	0.7463741051	3.2970e-003	2.2084e-005
	0.3	0.7433101792	6.0701e-003	2.8536e-005
	0.4	0.7377855554	1.1040e-002	1.5022e-005
	0.5	0.7281090401	1.9654e-002	7.1873e-005

Table 2b

t	*X*	*Exact(v)*	**Abs. error1**	**Abs. error2**
0.2	0.1	0.7499939249	5.1579e-006	2.3540e-008
	0.2	0.7499787972	1.8001e-005	8.2046e-008
	0.3	0.7499260107	6.2817e-005	2.8493e-007
	0.4	0.7497419422	2.1906e-004	9.7710e-007
	0.5	0.7491015994	7.6234e-004	3.2014e-006
0.5	0.1	0.7499844874	1.8237e-005	1.1431e-006
	0.2	0.7499458640	6.3640e-005	3.9838e-006
	0.3	0.7498111486	2.2192e-004	1.3830e-005
	0.4	0.7493420815	7.7219e-004	4.7377e-005
	0.5	0.7477185907	2.6661e-003	1.5460e-004

The numerical results show that using of the modification for Adomian's polynomial by 3- components gives results more accurate than the results using 5- components by standard ADM which is presented in [14] and a new algorithm for 4- components is presented in [10] when compared with the exact solutions and this is clear from **Abs. error2** when y=1.

5. Convergence Analysis of ADM

The convergence of the decomposition series has been investigated by several authors[23, 18, 21, 5]. They obtained some results about the speed of convergence of this method. The convergence analysis for the decomposition method to non-linear systems of PDEs by considering Hilbert space $H = L^2(\Omega)$ where $\Omega = (a, b) \times [0, T]$, and using the set of applications:

$$u: \Omega \to \mathbb{R} \quad \text{with} \quad \int_\Omega u^2(x,s)dsd\tau < +\infty \ ,$$

the scalar product

$$\langle u, w \rangle = \int_\Omega u(x,s)w(x,s)dsd\tau \,,$$

and the associate norm

$$\|u\|^2 = \int_\Omega u^2(x,s)dsd\tau \ .$$

Let us recall that the ADM is convergent if the following two hypotheses are satisfied:

(H1) $\langle L_t(u) - L_t(w), u - w \rangle \geq K_1 \|u - w\|^2, \quad K_1 > 0\,, \qquad \forall\, u, w \in H$

(H2) Whatever may be $N > 0$, there exist a constant $C_1(N) > 0$ such that for $u, w \in H$ with $\|u\| \leq N$ and $\|w\| \leq N$ we have:

$$\langle L_t(u) - L_t(w), z_1 \rangle \leq C_1(N) \|u - w\| \, \|z_1\|, \qquad \text{for every } z_1 \in H$$

Theorem 1: (Sufficient condition of the convergence). The ADM applied to (3.1a, b) converges towards a particular solution.

Proof: To prove this theorem, firstly, we will verify the convergence hypothesis (H1) for the operators $L_t(u)$ and $L_t(v)$. From

$L_t(u) = \frac{\partial}{\partial t}u = u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u\,, \quad L_t(v) = \frac{\partial}{\partial t}v = u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v\,,$

we have:

$$\begin{aligned} L_t(u) - L_t(w) &= \left[u\frac{\partial}{\partial x}u - w\frac{\partial}{\partial x}w\right] + \left[v\frac{\partial}{\partial y}u - v\frac{\partial}{\partial y}w\right] \\ &= \frac{1}{2}\frac{\partial}{\partial x}[u^2 - w^2] + v\frac{\partial}{\partial y}[u - w]\,, \end{aligned}$$

therefore,

$$\begin{aligned} &\langle L_t(u) - L_t(w), u - w \rangle \\ &\qquad = \frac{1}{2}\langle\frac{\partial}{\partial x}[u^2 - w^2], u - w\rangle + \langle v\frac{\partial}{\partial y}[u - w], u - w\rangle \ , (5.1a) \end{aligned}$$

by the Schwartz inequality and the definition of the scalar product and the properties of the differential operator $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in *H*, then there exists a constant $\delta_1 > 0$ such that

$$\begin{aligned} \langle -\frac{\partial}{\partial x}[u^2 - w^2], u - w \rangle &\leq \left\|\frac{\partial}{\partial x}[u^2 - w^2]\right\| \|u - w\| \\ &\leq \delta_1 \|u^2 - w^2\| \, \|u - w\| \end{aligned}$$

$$= \delta_1 \left\| (u-w)(u+w) \right\| \left\| u-w \right\| \leq 2\,\delta_1 N \left\| u-w \right\|^2,$$

where $\|u\| \leq N$, $\|w\| \leq N$. Hence ([18], [21])

$$\langle -\frac{\partial}{\partial x}[u^2-w^2], u-w\rangle \leq 2\delta_1 N \left\| u-w \right\|^2 \Leftrightarrow$$

$$\langle \frac{\partial}{\partial x}[u^2-w^2], u-w\rangle \geq 2\delta_1 N \left\| u-w \right\|^2, (5.1b)$$

again by the Schwartz inequality there exists a constant $\delta_2 > 0$ such that

$$\begin{aligned}\langle -v\frac{\partial}{\partial y}[u-w], u-w\rangle &\leq \left\| v\frac{\partial}{\partial y}[u-w] \right\| \left\| u-w \right\| \\ &\leq \delta_2 \left\| v \right\| \left\| u-u \right\| \left\| u-w \right\| \\ &\leq \delta_2 N \left\| u-w \right\|^2,\end{aligned}$$

where, $\|v\| \leq N$,
similarly,

$$\langle -\frac{\partial}{\partial x}[u-w], u-w\rangle \leq \delta_2 N \left\| u-w \right\|^2 \Leftrightarrow$$

$$\langle \frac{\partial}{\partial x}[u-w], u-w\rangle \geq \delta_2 N \left\| u-w \right\|^2, (5.1c)$$

substituting $(5.1b)$ and $(5.1c)$ into equation $(5.1a)$ gives

$$\begin{aligned}\langle L_t(u)-L_t(w), u-w\rangle &\geq \delta_1 N \left\| u-w \right\|^2 + \delta_2 N \left\| u-w \right\|^2 \\ &= (\delta_1 N + \delta_2 N) \left\| u-w \right\|^2 \\ &= K_1 \left\| u-w \right\|^2,\end{aligned}$$

since, $K_1 = (\delta_1 N + \delta_2 N) > 0 \Rightarrow \delta_1 > -\delta_2$.

By the same method for (3.1b) there exist $\theta_1 > 0$ and $\theta_2 > 0$ we obtained that

$$\begin{aligned}\langle L_t(v)-L_t(w), v-w\rangle &\geq \theta_1 N \left\| v-w \right\|^2 + \theta_2 N \left\| v-w \right\|^2 \\ &= (\theta_1 N + \theta_2 N) \left\| v-w \right\|^2 \\ &= K_2 \left\| v-w \right\|^2,\end{aligned}$$

since, $K_2 = (\theta_1 N + \theta_2 N) > 0 \Rightarrow \theta_1 > -\theta_2$. So the hypothesis (H1) holds.

Now we verify the hypothesis (H2)

$$\begin{aligned}\langle L_t(u)-L_t(w), z_1\rangle &= \langle \frac{1}{2}\frac{\partial}{\partial x}[u^2-w^2] + v\frac{\partial}{\partial y}[u-w], z_1\rangle \\ &\leq \delta_1 N \left\| u-w \right\| \left\| z_1 \right\| + \delta_2 N \left\| u-w \right\| \left\| z_1 \right\| \\ &\leq (\delta_1 N + \delta_2 N) \left\| u-w \right\| \left\| z_1 \right\| \\ &\leq C_1(N) \left\| u-w \right\| \left\| z_1 \right\|,\end{aligned}$$

where, $C_1(N) = \delta_1 N + \delta_2 N$,

and, $\langle L_t(v) - L_t(w), z_2\rangle \leq \langle \frac{1}{2}\frac{\partial}{\partial y}[v^2 - w^2] + u\frac{\partial}{\partial x}[v - w], z_2\rangle$
this leads to

$$\langle L_t(v) - L_t(w), z_2\rangle \leq (\theta_1 N + \theta_2 N)\|v - w\|\,\|z_2\| \leq C_2(N)\|v - w\|\,\|z_2\|$$

where $C_2(N) = \theta_1 N + \theta_2 N$, hence the hypothesis (H2) holds.

Theorem 2 (Sufficient condition of the convergence). The ADM applied to couple Burgers equations (3.2a,b) converges towards a particular solution.

Proof: We will prove the convergence hypothesis (H1) for the operators $L_t(u)$ and $L_t(v)$.

From
$$L_t(u) = \frac{\partial}{\partial t}u = \frac{1}{R}\left(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u\right) - u\frac{\partial}{\partial x}u - v\frac{\partial}{\partial y}u,$$
$$L_t(v) = \frac{\partial}{\partial t}v = \frac{1}{R}\left(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v\right) - v\frac{\partial}{\partial y}v - u\frac{\partial}{\partial x}v,$$
we have:

$$L_t(u) - L_t(w) = \frac{1}{R}\frac{\partial^2}{\partial x^2}[u - w] + \frac{1}{R}\frac{\partial^2}{\partial y^2}[u - w] - \frac{1}{2}\frac{\partial}{\partial x}[u^2 - w^2] - v\frac{\partial}{\partial y}[u - w],$$

so by the Schwartz inequality and the definition of the scalar product and the properties of the differential operator $\frac{\partial}{\partial x}$ and $\frac{\partial^3}{\partial x^3}$ in *H*, therefore there exists a constant*s* $\delta_1, \delta_2, \delta_3$ and $\delta_4 > 0$ such that

$$\langle L_t(u) - L_t(w), u - w\rangle \geq \frac{1}{R}\delta_1\|u - w\|^2 + \frac{1}{R}\delta_2\|u - w\|^2 + \delta_3 N\ \|u - w\|^2 + \delta_4 N\ \|u - w\|^2$$
$$\langle L_t(u) - L_t(w), u - w\rangle \geq \left(\frac{1}{R}\delta_1 + \frac{1}{R}\delta_2 + \delta_3 N + \delta_4 N\right)\|u - w\|^2$$
$$\langle L_t(u) - L_t(w), u - w\rangle \geq K_1\|u - w\|^2,$$

since, $K_1 = \left(\frac{1}{R}\delta_1 + \frac{1}{R}\delta_2 + \delta_3 N + \delta_4 N\right) > 0 \Rightarrow \delta_1 > -\delta_2 - R\,\delta_3 N - R\delta_4 N.$

By the same method for (3.2b) there exist θ_1, θ_2, θ_3 and $\theta_4 > 0$ such that

$$\langle L_t(v) - L_t(w), v - w\rangle \geq \left(\frac{1}{R}\theta_1 + \frac{1}{R}\theta_2 + \theta_3 N + \theta_4 N\right)\|v - w\|^2$$

since, $K_2 = \left(\frac{1}{R}\theta_1 + \frac{1}{R}\theta_2 + \theta_3 N + \theta_4 N\right) > 0 \Rightarrow \theta_1 > -(\theta_2 + R\,\theta_3 N + R\theta_4 N)$

then the hypothesis (H1) holds. Now we verify the hypothesis (H2)

$$\langle L_t(u) - L_t(w), z_1\rangle$$
$$= \langle \frac{1}{R}\frac{\partial^2}{\partial x^2}[u-w] + \frac{1}{R}\frac{\partial^2}{\partial y^2}[u-w] - \frac{1}{2}\frac{\partial}{\partial x}[u^2 - w^2] - v\frac{\partial}{\partial y}[u-w], z_1\rangle$$
$$\leq \left(\delta_1\frac{1}{R} + \delta_2\frac{1}{R} + \delta_3 N + \delta_4 N\right)\|u \ - w\|\,\|z_1\|$$
$$\leq C_1(N)\|u-w\|\,\|z_1\|\ ,$$

where, $C_1(N) = \left(\delta_1\frac{1}{R} + \delta_2\frac{1}{R} + \delta_3 N + \delta_4 N\right)$,

and, $\langle L_t(v) - L_t(w), z_2\rangle = \langle -\frac{\partial^3}{\partial x^3}[v-w] - \left[u\frac{\partial}{\partial x}v - u\frac{\partial}{\partial x}w\right], z_2\rangle$

this yields

$$\langle L_t(v) - L_t(w), z_2\rangle \leq \left\| -\frac{\partial^3}{\partial x^3}[v-w]\right\|\,\|z_2\| + \left\| -\left[u\frac{\partial}{\partial x}v - u\frac{\partial}{\partial x}w\right]\right\|\,\|z_2\|$$
$$\leq \theta_1\|v-w\|\,\|z_2\| + \theta_2 N\|v-w\|\,\|z_2\|$$
$$\leq C_2(N)\|v-w\|\,\|z_2\|,$$

where $C_2(N) = \theta_1 + \theta_2 N$ and the hypothesis (H2) holds.

<u>Theorem</u> 3: (Sufficient condition of the convergence).The ADM applied to Hirota–Satsuma coupled KdV equation (3.3a, b, c) converges towards a particular solution.

<u>Proof:</u> We prove the convergence hypothesis (H1) for the operators $L_t(u)$ and $L_t(v)$, there exists $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$, $\forall\, u, v, w$ and $\bar{y} \in H$.

From $L_t(u) = \frac{\partial}{\partial t}u = \frac{1}{2}\frac{\partial^3}{\partial x^3}u - 3u\frac{\partial}{\partial x}u + 3\frac{\partial}{\partial x}(vw)$,

$$L_t(v) = \frac{\partial}{\partial t}v = -\frac{\partial^3}{\partial x^3}v + 3u\frac{\partial}{\partial x}v,$$
$$L_t(w) = \frac{\partial}{\partial t}w = -\frac{\partial^3}{\partial x^3}w + 3u\frac{\partial}{\partial x}w,$$

we have:

$$L_t(u) - L_t(\bar{y}) = \frac{1}{2}\frac{\partial^3}{\partial x^3}[u-\bar{y}] - \frac{3}{2}\frac{\partial}{\partial x}[u^2 - \bar{y}^2] + 3\frac{\partial}{\partial x}[vw - vw],$$

so by the Schwartz inequality and the definition of the scalar product and the properties of the differential operator $\frac{\partial}{\partial x}$ and $\frac{\partial^3}{\partial x^3}$ in H, therefore there exists a constant δ_1 and $\delta_2 > 0$ such that

$$\langle L_t(u) - L_t(\bar{y}), u - \bar{y}\rangle \geq \left(\tfrac{1}{2}\delta_1 + \ 3\delta_2 N\ \right)\|u-\bar{y}\|^2$$
$$\langle L_t(u) - L_t(\bar{y}), u - \bar{y}\rangle \geq K_1\|u-\bar{y}\|^2,$$

since, $K_1 = \left(\frac{1}{2}\delta_1 + \ 3\delta_2 N\ \right) > 0 \ \Rightarrow\ \delta_1 > -6\delta_2 N$,

by the same method for (3.3b) there exist θ_1 and $\theta_2 > 0$ such that

$$\langle L_t(v) - L_t(\bar{y}), v - \bar{y}\rangle \geq (\theta_1 + 3\theta_2 N)\|v - \bar{y}\|^2,$$

since, $K_2 = (\theta_1 + 3\theta_2 N) > 0 \Rightarrow \theta_1 > -3\theta_2 N,$

and for (3.4c) there exist α_1 and $\alpha_2 > 0$ such that

$$\langle L_t(w) - L_t(\bar{y}), w - \bar{y}\rangle \geq (\alpha_1 + 3\alpha_2 N)\|w - \bar{y}\|^2,$$

since, $K_3 = (\alpha_1 + 3\alpha_2 N) > 0 \Rightarrow \alpha_1 > -3\alpha_2 N,$

so the hypothesis (H1) holds.

Now we verify the hypothesis (H2)

$$\begin{aligned}\langle L_t(u) - L_t(\bar{y}), z_1\rangle &= \langle \frac{1}{2}\frac{\partial^3}{\partial x^3}[u - \bar{y}] - \frac{3}{2}\frac{\partial}{\partial x}[u^2 - \bar{y}^2], z_1\rangle \\ &\leq \left(\frac{1}{2}\delta_1 + 3\delta_2\right)\|u - \bar{y}\|\,\|z_1\| \\ &\leq C_1(N)\|u - \bar{y}\|\,\|z_1\|,\end{aligned}$$

where, $C_1(N) = \left(\frac{1}{2}\delta_1 + 3\delta_2\right),$

and,

$$\begin{aligned}\langle L_t(v) - L_t(\bar{y}), z_2\rangle &= \langle -\frac{\partial^3}{\partial x^3}[v - \bar{y}] + 3\left[u\frac{\partial}{\partial x}v - u\frac{\partial}{\partial x}\bar{y}\right], z_2\rangle \\ &\leq \theta_1\|v - \bar{y}\|\,\|z_2\| + 3\theta_2 N\|v - \bar{y}\|\,\|z_2\| \\ &\leq C_2(N)\|v - \bar{y}\|\,\|z_2\|,\end{aligned}$$

where, $C_2(N) = \theta_1 + 3\theta_2 N.$

For equation (3.3c)

$$\begin{aligned}\langle L_t(w) - L_t(\bar{y}), z_3\rangle &= \langle -\frac{\partial^3}{\partial x^3}[w - \bar{y}] + 3\left[u\frac{\partial}{\partial x}w - u\frac{\partial}{\partial x}\bar{y}\right], z_3\rangle \\ &\leq \alpha_1\|w - \bar{y}\|\,\|z_3\| + 3\alpha_2 N\|w - \bar{y}\|\,\|z_3\| \\ &\leq C_3(N)\|v - \bar{y}\|\,\|z_3\|,\end{aligned}$$

where, $C_3(N) = \alpha_1 + 3\alpha_2 N$ and the hypothesis (H2) holds.

6. Conclusions

In this paper, we have proposed efficient method for solving non-linear systems of PDEs, with high convergence and small error that ADM and we introduce modification on Adomian's polynomial which is straightforward, does not use the explicit forms of Adomian's polynomial and rapidly convergence by least number of component as compared than standard ADM. As seen in Tables 1–3, errors are very small and they have better results than other papers cited in this paper.

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