Int. J. Contemp. Math. Sciences, Vol. 5, 2010, no. 50, 2463 - 2472

Fixed Point Theorem for Expansive Mappings in G-Metric Spaces

Zead Mustafa, Fadi Awawdeh and Wasfi Shatanawi

Department of Mathematics, The Hashemite University
P.O. 330127, Zarqa 13115, Jordan
zmagablh@hu.edu.jo, awawdeh@hu.edu.jo, swasfi@hu.edu.jo

Abstract. In this paper, we define the expansive mapping in the setting of G-metric space, also several fixed point theorems for a class of expansive mappings defined on a complete G-metric space are studied.

Mathematics Subject Classification: Primary 47H10, Secondary 46B20

Keywords: Metric space, generalized metric space, G-metric space, 2-metric space, expansive mapping

1. Introduction

In 2005, a new structure of generalized metric spaces was introduced by Zead Mustafa and Brailey Sims as appropriate notion of generalized metric space called G-metric spaces (see [3]) as follows.

Definition 1. ([3]) *Let X be a nonempty set, and let $G : X \times X \times X \to \boldsymbol{R}^+$, be a function satisfying the following properties:*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$,

(G2) $0 < G(x, x, y)$; *for all* $x, y \in X$, *with* $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$, *with* $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, *(symmetry in all three variables), and*

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$, *(rectangle inequality).*

Then the function G is called generalized metric*, or, more specifically* G-metric *on X, and the pair (X, G) is called a* G-metric space.*(Throughout this paper we denote $\boldsymbol{R}^+$ the set of all positive real numbers and $\mathbf{N}$ the set of all natural numbers).*

Definition 2. ([3]) *Let (X, G) be a G-metric space, let (x_n) be a sequence of points of X, a point $x \in X$ is said to be the* limit *of the sequence (x_n) if*

$\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, *and one say that the sequence* (x_n) *is* G-convergent *to* x.

Thus, that if $x_n \longrightarrow 0$ *in a* G*-metric space* (X, G), *then for any* $\epsilon > 0$, *there exists* $N \in \mathbf{N}$ *such that* $G(x, x_n, x_m) < \epsilon$, *for all* $n, m \geq N$.

Proposition 1. ([3]) *Let* (X, G) *be a* G*-metric space. Then the following are equivalent.*

(1) (x_n) *is* G*-convergent to* x.
(3) $G(x_n, x_n, x) \to 0$, *as* $n \to \infty$.
(4) $G(x_n, x, x) \to 0$, *as* $n \to \infty$.
(5) $G(x_m, x_n, x) \to 0$, *as* $m, n \to \infty$.

Definition 3. ([3]) *Let* (X, G) *be a* G*-metric space. A sequence* (x_n) *is called* G*-Cauchy if given* $\epsilon > 0$, *there is* $N \in \mathbf{N}$ *such that* $G(x_n, x_m, x_l) < \epsilon$, *for all* $n, m, l \geq N$, *that is, if* $G(x_n, x_m, x_l) \longrightarrow 0$ *as* $n, m, l \longrightarrow \infty$.

Proposition 2. ([3]) *If* (X, G) *is a* G*-metric space, then the following are equivalent.*

1. *The sequence* (x_n) *is* G*-Cauchy.*
2. *For every* $\epsilon > 0$, *there exists* $N \in \mathbf{N}$ *such that* $G(x_n, x_m, x_m) < \epsilon$, *for all* $n, m \geq N$.

Definition 4. ([3]) *Let* (X, G) *and* (X', G') *be two* G*-metric spaces, and let* $f : (X, G) \to (X', G')$ *be a function. Then* f *is said to be* G-continuous at a point $a \in X$ *if and only if given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $x, y \in X$; *and* $G(a, x, y) < \delta$ *implies* $G'(f(a), f(x), f(y)) < \epsilon$. *A function* f *is* G*-continuous on* X *if and only if it is* G*-continuous at all* $a \in X$.

Proposition 3. ([3]) *Let* (X, G) *and* (X', G') *be two* G*-metric spaces. Then a function* $f : X \longrightarrow X'$ *is* G*-continuous at a point* $x \in X$ *if and only if it is* G*-sequentially continuous at* x; *that is, whenever* (x_n) *is* G*-convergent to* x *we have* $(f(x_n))$ *is* G*-convergent to* $f(x)$.

Proposition 4. ([3]) *Let* (X, G) *be a* G*-metric space. Then the function* $G(x, y, z)$ *is jointly continuous in all three of its variables.*

Definition 5. ([3]) A G-metric space (X, G) is said to be G-*complete* (or complete G-metric) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Definition 6. ([3]) *A* G*-metric space* (X, G) *is called symmetric* G*-metric space if* $G(x, y, y) = G(y, x, x)$ *for all* $x, y \in X$.

The following fixed point theorem for a contractive mapping on G-metric space has been proved in [2].

Theorem 1.1. ([2]) *Let* (X, G) *be a complete* G*-metric space and* $T : X \to X$ *be a mapping satisfies the following condition for all* $x, y, z \in X$

$$G(Tx, Ty, Tz) \leq kG(x, y, z) \tag{1.1}$$

where $k \in [0,1)$. *Then* T *has a unique fixed point.*

Theorem 1.2. ([2]) *Let* (X,G) *be a complete* G*-metric space and* $T : X \to X$ *be a mapping satisfies the following condition for all* $x, y \in X$

$$G(Tx,Ty,Ty) \leq kG(x,y,y) \tag{1.2}$$

where $k \in [0,1)$. *Then* T *has a unique fixed point.*

In [2] we showed that a mapping satisfies the Condition (1.1) will satisfy Condition (1.2) when $k \in [0,1)$, where the converse is true only when $k \in [0, \frac{1}{2})$.

However, when $k \in [\frac{1}{2}, 1)$, we showed in a counter example that Condition (1.2) need not imply Condition (1.1)(for details see [2]).

Definition 7. *Let* (X,G) *be a* G*-metric space and* T *be a self mapping on* X. *Then* T *is called expansive mapping if there exists a constant* $a > 1$ *such that for all* $x, y, z \in X$, *we have*

$$G(Tx,Ty,Tz) \geq aG(x,y,z).$$

The following example shows that expansive mapping on G-metric space need not be G-continuous.

Example 1. *Let* $T : (\boldsymbol{R}, G) \longrightarrow (\boldsymbol{R}, G)$ *be defined by*

$$T(x) = \left\{ \begin{array}{ll} 5x & ; \quad if\ x \leq 3 \\ 5x+2; & if\ x > 3 \end{array} \right\}$$

where $G(x,y,z) = \max\{|x-y|, |y-z|, |x-z|\}$. *Then* $(\boldsymbol{R}, G)$ *is a complete* G*-metric space and* T *is expansive mapping where* T *is not* G*-continuous.*

2. Main Results

We start our work by proving the following theorem:

Theorem 2.1. *Let* (X,G) *be a complete* G*-metric space. If there exists a constant* $a > 1$ *and a surjective self mapping* T *on* X, *such that for all* $x, y, z \in X$

$$G(Tx,Ty,Tz) \geq aG(x,y,z), \tag{2.1}$$

then T *has a unique fixed point.*

Proof. Under the assumption, if $Tx = Ty$, then $0 = G(Tx,Ty,Ty) \geq aG(x,y,y)$, which implies that $G(x,y,y) = 0$, and hence $x = y$. So, T is injective and invertible.

Let h be the inverse mapping of T. Then

$$G(x,y,z) = G(T(hx),T(hy),T(hz)) \geq aG(hx,hy,hz).$$

Thus, for all $x, y, z \in X$, we have $G(hx,hy,hz) \leq kG(x,y,z)$, where $k = \frac{1}{a}$. Applying Theorem (1.1), we conclude that the inverse mapping h has a unique fixed point $u \in X$; $h(u) = u$. But, $u = T(h(u)) = T(u)$. This gives that u is also a fixed point of T.

Suppose there exists another fixed point $v \neq u$ such that $Tv = v$, then $Tv = v = T(h(v)) = h(Tv)$, so Tv is another fixed point for h. By uniqueness we conclude that $u = Tv = v$, which implies that u is a unique fixed point of T. □

Theorem 2.2. *Let (X, G) be a complete G-metric space. If there exists a constant $c > 1$ and a surjective self mapping T on X, such that for all $x, y \in X$*

$$G(Tx, Ty, Ty) \geq cG(x, y, y), \tag{2.2}$$

then T has a unique fixed point.

Proof. Under the assumption, we see that T is injective, and hence T is invertible. Let h be the inverse mapping of T. So,

$$G(x, y, y) = G(T(hx), T(hy), T(hy)) \geq cG(hx, hy, hy).$$

Then, for all $x, y \in X$ we have $G(hx, hy, hy) \leq kG(x, y, y)$, where $k = \frac{1}{c}$. Applying Theorem (1.2) on the inverse mapping h, and use argument similar to that in proof Theorem (2.1), we conclude that T has unique fixed point □

Corollary 1. *Let (X, G) be a complete G-metric space. If there exists a constant $k > 1$ and a surjective self mapping on X, such that for all $x, y, z \in X$*

$$G(Tx, Ty, Tz) \geq k\{G(x, z, z) + G(y, z, z)\}, \tag{2.3}$$

then T has a unique fixed point.

Proof. Follows from Theorem (2.2), by taking $z = y$ in Condition (2.3). □

Theorem 2.3. *Let (X, G) be a complete G-metric space, and let $T : X \longrightarrow X$ be a surjective mapping satisfying the following condition for all $x, y, z \in X$*

$$G(T(x), T(y), T(z)) \geq k \max \left\{ \begin{array}{c} G(x, z, z) + G(y, z, z), \\ G(z, y, y) + G(x, y, y), \\ G(z, x, x) + G(y, x, x) \end{array} \right\} \tag{2.4}$$

where $k > 1$. Then T has a unique fixed point.

Proof. Condition (2.4) implies that T is injective and therefor invertible.

Let h be the inverse mapping of T. By Condition (2.4) for all $x, y, z \in X$, we have,

(2.5)

$$G(x, y, z) = G(T(hx), T(hy), T(hz)) \geq k \max \left\{ \begin{array}{c} G(hx, hz, hz) + G(hy, hz, hz), \\ G(hz, hy, hy) + G(hx, hy, hy), \\ G(hz, hx, hx) + G(hy, hx, hx) \end{array} \right\}.$$

But, by (G5) we have,

$$\max \left\{ \begin{array}{c} G(hx, hz, hz) + G(hy, hz, hz), \\ G(hz, hy, hy) + G(hx, hy, hy), \\ G(hz, hx, hx) + G(hy, hx, hx) \end{array} \right\} \geq G(hx, hy, hz). \tag{2.6}$$

Thus, Equation (2.5) implies that,

$$G(hx, hy, hz) \leq aG(x, y, z), \tag{2.7}$$

where $a = \frac{1}{k}$.

Applying Theorem (1.1) on Condition (2.7), we conclude that the inverse mapping h has a unique fixed point $u \in X$ such that $h(u) = u$. But, $u = T(h(u)) = T(u)$, which shows that u is also a fixed point of T.

To show u is unique fixed point, use an argument similar to that in Theorem (2.1). □

Theorem 2.4. *Let (X, G) be a complete nonsymmetric G-metric space and let $T : X \longrightarrow X$ be a surjective mapping satisfying the following condition for all $x, y, z \in X$*

$$G(T(x), T(y), T(z)) \geq k \max \left\{ \begin{array}{c} G(x, y, y) + G(y, x, x), \\ G(x, z, z) + G(z, x, x), \\ G(z, y, y) + G(y, z, z) \end{array} \right\}, \tag{2.8}$$

where $k > \frac{2}{3}$. Then T has a unique fixed point.

Proof. Suppose T satisfies Condition (2.8), then T is injective and has an inverse function. In Condition (2.8), let $z = y$. Then for all $x, y, \in X$, we have

$$G(Tx, Ty, Ty) \geq k\{G(x, y, y) + G(y, x, x)\}, \tag{2.9}$$

but (G5) implies that $G(x, y, y) \leq 2G(y, x, x)$. Therefore $\frac{1}{2}G(x, y, y) + G(x, y, y) \leq G(y, x, x) + G(x, y, y)$, then

$$\frac{3}{2}G(x, y, y) \leq G(y, x, x) + G(x, y, y). \tag{2.10}$$

In this line, Equations (2.9) and (2.10) leads to,

$$G(Tx, Ty, Ty) \geq \frac{3k}{2}G(x, y, y). \tag{2.11}$$

Let h be the inverse mapping of T, therefore Equation (2.11), implies that,

$$G(x, y, y) = G(T(hx), T(hy), T(hy)) \geq \frac{3k}{2}G(hx, hy, hy). \tag{2.12}$$

for all $x, y \in X$. Using that $k > \frac{2}{3}$, we have

$$G(hx, hy, hy) \leq cG(x, y, y) \tag{2.13}$$

where $c = \frac{2}{3k}$ and $c < 1$. Then, Theorem (1.2) implies that the inverse mapping h has a unique fixed point $u \in X$ such that $h(u) = u$, but, $u = T(h(u)) = T(u)$, which shows that u is a fixed point of T.

To prove uniqueness, suppose that $v \neq u$ is such that $T(v) = v$, then (2.8) implies that

$$G(u, v, v) = G(Tu, Tv, Tv) \geq k\{G(u, v, v) + G(v, u, u)\} \geq \frac{3k}{2}G(u, v, v),$$

but $\frac{3k}{2} > 1$, thus $G(u,v,v) > G(u,v,v)$, this contradiction implies that $u = v$. □

Theorem 2.5. *Let (X,G) be a complete nonsymmetric G-metric space and let $T : X \longrightarrow X$ be a surjective mapping satisfying the following condition for all $x, y \in X$*

$$G(T(x),T(y),T(y)) \geq k\ \max\{G(x,y,y), G(y,x,x)\} \tag{2.14}$$

where $k > 1$. Then T has a unique fixed point.

Proof. Since $\max\{G(x,y,y), G(y,x,x)\} \geq G(x,y,y)$, then from (2.14), we deduce that

$$G(T(x),T(y),T(y)) \geq k\, G(x,y,y), \text{ for all } x,y \in X. \tag{2.15}$$

From (2.15), it is clear that Theorem (2.2) implies that T has a unique fixed point. □

Corollary 2. *Let (X,G) be a complete nonsymmetric G-metric space, and let $T : X \longrightarrow X$ be a surjective mapping satisfying the following condition for all $x, y, z \in X$*

$$G(T(x),T(y),T(z)) \geq k\ \max\left\{\begin{array}{c} G(x,y,y),\ G(y,x,x), \\ G(x,z,z),\ G(z,x,x), \\ G(z,y,y),\ G(y,z,z) \end{array}\right\} \tag{2.16}$$

where $k > 1$. Then T has a unique fixed point.

Proof. Follows from Theorem (2.5) by taking $z = y$. □

Corollary 3. *Let (X,G) be a complete G-metric space and let $T : X \longrightarrow X$ be a surjective mapping satisfying the following condition for all $x, y, z \in X$*

$$G(T(x),T(y),T(z)) \geq k\,\{G(x,Tx,Tx) + G(Tx,y,z)\} \tag{2.17}$$

where $k > 1$. Then T has a unique fixed point.

Proof. From (G5), we have $G(x,Tx,Tx) + G(Tx,y,z) \geq G(x,y,z)$. Then Condition (2.17) becomes $G(T(x),T(y),T(z)) \geq kG(x,y,z)$ for all $x,y,z \in X$, and the proof follows from Theorem (2.1). □

Theorem 2.6. *Let (X,G) be a complete G-metric space and let $T : X \longrightarrow X$ be an onto mapping satisfying the following condition for all $x, y, z \in X$,*

(2.18)
$$G(T(x),T(y),T(z)) \geq aG(x,y,z) + bG(x,x,Tx) + cG(y,y,Ty) + dG(z,z,Tz)$$

where $a + b + c + d > 1$ and $b + c < 1$. Then T has a fixed point.

Proof. Let $x_0 \in X$, since T is onto then there exists an element x_1 satisfying $x_1 \in T^{-1}(x_0)$. By the same argument we can pickup $x_n \in T^{-1}(x_{n-1})$ where $(n = 2, 3, 4, 5, \dots)$. If $x_m = x_{m-1}$ for some m, then x_m is a fixed point of T. Assume $x_n \neq x_{n-1}$ for every n, then from (2.18) we have,

$G(x_{n-1}, x_{n-1}, x_n) = G(Tx_n, Tx_n, Tx_{n+1}) \geq aG(x_n, x_n, x_{n+1}) +$
$(b+c)G(x_n, x_{n-1}, x_{n-1}) + dG(x_{n+1}, x_n, x_n)$. So,
$(1-(b+c))G(x_{n-1}, x_{n-1}, x_n) \geq (a+d)G(x_{n+1}, x_n, x_n)$, therefor

$$G(x_{n+1}, x_n, x_n) \leq \frac{1-(b+c)}{a+d} G(x_{n-1}, x_{n-1}, x_n). \tag{2.19}$$

Let $q = \frac{1-(b+c)}{a+d}$. Then $q < 1$ and by repeated application of (2.19), we have

$$G(x_{n+1}, x_n, x_n) \leq q^n G(x_1, x_0, x_0). \tag{2.20}$$

Then, for all $n, m \in \mathbf{N}; n < m$, we have, by repeated use of the rectangle inequality and Equation (2.20) that
$G(x_m, x_n, x_n) \leq \ G(x_m, x_{m-1}, x_{m-1}) + G(x_{m-1}, x_{m-2}, x_{m-2})$
$+ G(x_{m-2}, x_{m-3}, x_{m-3}) + \cdots + G(x_{n+1}, x_n, x_n)$
$\leq (q^{m-1} + q^{m-2} + \cdots + q^n)G(x_0, x_1, x_1) \leq \frac{q^n}{1-q} G(x_0, x_1, x_1)$. So,
$\lim G(x_m, x_n, x_n) = 0$, as $n, m \longrightarrow \infty$ and (x_n) is G-Cauchy a sequence. By the completeness of (X, G), there exists $u \in X$ such that (x_n) is G-converges to u.

Let $y \in T^{-1}(u)$. For infinitely many n, $x_n \neq u$. For such n, we have
$G(x_n, u, u) = G(Tx_{n+1}, Ty, Ty) \geq aG(x_n, y, y) +$
$bG(x_{n+1}, x_{n+1}, x_n) + (c+d)G(y, y, u)$.

Since $G(x_n, u, u) \longrightarrow 0$, as $n \longrightarrow \infty$, we have $(c+d)G(y, y, u) = 0$, and $aG(x_n, y, y) \longrightarrow 0, (n \longrightarrow \infty)$. It is impossible that both $c+d = 0$ and $a = 0$. Therefor,

1. If $c+d \neq 0$, then $G(y, y, u) = 0$, which implies that $u = y$.
2. If $a \neq 0$, then $aG(x_n, y, y) \longrightarrow 0, (n \longrightarrow \infty)$, which implies $x_n \longrightarrow y$.

Hence in both cases we have $u = y$, but $T(y) = u$, so, $T(y) = u = y$, therefor, u is a fixed point of T.

We see that if $a < 1$, then a fixed point of T is not unique, since the identity mapping will satisfy Condition (2.18). However, if $a > 1$ this fixed point is unique. □

Corollary 4. *Let (X, G) be a complete G-metric space and let $T : X \longrightarrow X$ be an onto mapping satisfying the following condition for all $x, y, z \in X$*

$$G(T(x), T(y), T(z)) \geq \alpha G(x, y, z) + \beta\{G(x, x, Tx) + G(y, y, Ty) + G(z, z, Tz)\} \tag{2.21}$$

where $\alpha + 3\beta > 1$ and $\beta < \frac{1}{2}$. Then T has a fixed point.

Proof. In Theorem (2.6), If $a = \alpha$, and $b = c = d = \beta$, then the condition (2.18) reduced to Condition (2.21), so the proof follows from Theorem (2.6). □

Theorem 2.7. *Let (X, G) be a complete G-metric space and let $T : X \longrightarrow X$ be an onto and G-continuous mapping satisfying the following condition for all*

$x, y, z \in X$

$$G(T(x), T(y), T(z)) \geq k \min \begin{Bmatrix} G(x, z, z), G(y, z, z), G(z, y, y), \\ G(x, y, y), G(z, x, x), G(y, x, x) \end{Bmatrix} \tag{2.22}$$

where $k > 2$. Then T has a unique fixed point.

Proof. As in Theorem (2.6), there is a sequence (x_n) with $x_{n-1} \neq x_n$ and $T(x_n) = x_{n-1}$. Then from (2.22) we have,

$$G(x_n, x_{n-1}, x_{n-1}) = G(Tx_{n+1}, Tx_n, Tx_n)$$

$$\geq k \min \begin{Bmatrix} G(x_{n+1}, x_n, x_n), G(x_n, x_n, x_n), \\ G(x_n, x_n, x_n), G(x_{n+1}, x_n, x_n), \\ G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}) \end{Bmatrix}. \tag{2.23}$$

From G-metric property we have $G(x_{n+1}, x_n, x_n) \leq 2G(x_{n+1}, x_{n+1}, x_n)$, and so, (2.23) becomes

$G(x_n, x_{n-1}, x_{n-1}) \geq k \min\{G(x_{n+1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} \geq \frac{k}{2} G(x_{n+1}, x_n, x_n)$. This implies that

$$G(x_{n+1}, x_n, x_n) \leq \frac{2}{k} G(x_n, x_{n-1}, x_{n-1}).$$

Let $q = \frac{2}{k}$, then $q < 1$. By the same argument in the proof of Theorem (2.6), we see that the sequence (x_n) is G-Cauchy and by completeness of (X, G), the sequence (x_n) G-converges to a point $u \in X$.

Since T is G-continuous, then $T(x_n) = x_{n-1} \longrightarrow T(u)$, as $n \longrightarrow \infty$. Hence $Tu = u$ which shows that u is a fixed point of T.

To prove uniqueness, suppose that $v \neq u$ is such that $T(v) = v$, then (2.22) implies that $G(u, v, v) \geq k \min\{G(u, v, v), G(v, u, u)\}$, thus $G(u, v, v) \geq kG(v, u, u)$ again by the same we will find $G(v, u, u) \geq kG(u, v, v)$, hence

$$G(u, v, v) \geq k^2 G(u, v, v)$$

which implies that $u = v$, since $k > 2$. □

Theorem 2.8. *Let (X, G) be a complete G-metric space, and $T : X \longrightarrow X$ be an onto and G-continuous mapping satisfying the following condition for all $x \in X$*

$$G(T(x), T^2(x), T^3(x)) \geq aG(x, Tx, T^2(x)) \tag{2.24}$$

where $a > 1$. Then T has a fixed point.

Proof. Similar to that in Theorem (2.6), there is a sequence (x_n) with $x_{n-1} \neq x_n$ and $T(x_n) = x_{n-1}$. Then (2.24) implies that

(2.25)

$$G(x_{n-1}, x_{n-2}, x_{n-3}) = G(Tx_n, T^2x_n, T^3x_n) \geq aG(x_n, Tx_n, T^2x_n) = aG(x_n, x_{n-1}, x_{n-2}),$$

and therefor,

$$G(x_n, x_{n-1}, x_{n-2}) \leq \frac{1}{a} G(x_{n-1}, x_{n-2}, x_{n-3}).$$

Let $q = \frac{1}{a}$, then $q < 1$. By the same argument in the proof of Theorem (2.6), we see that the sequence (x_n) is G-Cauchy and by completeness of (X, G), the sequence (x_n) G-converges to a point $u \in X$.

Since T is G-continuous, then $T(x_n) = x_{n-1} \longrightarrow T(u)$, as $n \longrightarrow \infty$. Hence $Tu = u$ which shows that u is a fixed point of T. □

Theorem 2.9. *Let (X, G) be a complete G-metric space and let $T : X \longrightarrow X$ be an onto mapping satisfying the following condition for all $x, y, z \in X$*

(2.26)

$$G(T(x), T(y), T(z)) \geq k \max\{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}$$

where $k > 1$. Then T has a fixed point.

Proof. As in Theorem (2.6), there is a sequence (x_n) with $x_{n-1} \neq x_n$ and $T(x_n) = x_{n-1}$. Then (2.26) implies that

(2.27)

$$G(x_n, x_{n-1}, x_{n-1}) = G(Tx_{n+1}, Tx_n, Tx_n) \geq$$
$$k \max\{G(x_{n+1}, x_n, x_n), G(x_n, x_{n-1}, x_{n-1}), G(x_n, x_{n-1}, x_{n-1})\} = kG(x_{n+1}, x_n, x_n)$$

and as a result we get,

$$G(x_{n+1}, x_n, x_n) \leq \frac{1}{k} G(x_n, x_{n-1}, x_{n-1}).$$

Let $q = \frac{1}{k}$. Then $q < 1$. By the same argument in the proof of Theorem (2.6), we see that the sequence (x_n) is G-Cauchy and by completeness of (X, G), the sequence (x_n) G-converges to a point $u \in X$.

Let $y \in T^{-1}(u)$. For infinitely many n, $(x_n) \neq u$. For such n, from (2.26) we have,

$G(x_n, u, u) = G(Tx_{n+1}, Ty, Ty) \geq k \max\{G(x_{n+1}, x_n, x_n), G(y, Ty, Ty)\} = k \max\{G(x_{n+1}, x_n, x_n), G(y, u, u)\}$.

Since $G(x_n, u, u) \longrightarrow 0$, as $n \longrightarrow \infty$, we have $kG(y, u, u) = 0$ and $kG(x_{n+1}, x_n, x_n) \longrightarrow 0, (n \longrightarrow \infty)$. Therefor, $G(y, u, u) = 0$, which implies that $u = y$.

But $T(y) = u$, so, $T(y) = u = y$. This shows that u is a fixed point of T.

□

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Received: April, 2010