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Expansion Mapping Theorems

in G – Metric Spaces

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Abstract: In this paper, we prove expansion mappings theorems in G-metric space. Also we introduce the concept of R-weak commutativity of type (A_f), (A_g) and (P) in G-metric.

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1. Introduction:

In 1992, Dhage[1] introduced the concept of D – metric space. Recently, Mustafa and Sims[5] shown that most of the results concerning Dhage's D – metric spaces are invalid. Therefore, they introduced a improved version of the generalized metric space structure and called it as G – metric space. For more details on G – metric spaces, one can refer to the papers [5]-[8].

Now we give basic definitions and some basic results ([5]-[8]) which are helpful for proving our main result.

In 2006, Mustafa and Sims[6] introduced the concept of G-metric spaces as follows:

Definition 1.1.[6] Let X be a nonempty set, and let $G: X \times X \times X \to R^+$ be a function satisfying the following axioms:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables) and

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality) then the function G is called a generalized metric, or, more specifically a G – metric on X and the pair (X, G) is called a G – metric space.

Definition 1.2.[6] Let (X, G) be a G–metric space, and let $\{x_n\}$ a sequence of points in X, a point 'x' in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G–convergent to x.

Proposition 1.1.[6] Let (X, G) be a G – metric space. Then the following are equivalent:
(1) $\{x_n\}$ is G-convergent to x,
(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,
(3) $G(x_n, x, x) \to 0$ as $n \to \infty$,
(4) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Definition 1.3.[6] Let (X, G) be a G – metric space. A sequence $\{x_n\}$ is called G – Cauchy if, for each $\in > 0$ there exists a positive integer N such that $G(x_n, x_m, x_l) < \in$ for all $n, m, l \geq N$; i.e. if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 1.2.[6] Let (X, G) be a G – metric space. Then the function G(x, y, z) is jointly continuous in all three of its variables.

Definition 1.4.[6] A G – metric space (X, G) is said to be G–complete if every G-Cauchy sequence in (X,G) is G-convergent in X.

Proposition 1.3.[6] Let (X, G) be a G – metric space. Then, for any x, y, z, a in X it follows that:
(i) If $G(x, y, z) = 0$, then $x = y = z$,
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(iii) $G(x, y, y) \leq 2G(y, x, x)$,
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(v) $G(x, y, z) \leq {}^{2}/_{3}\,(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
(vi) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

2. Main Results

There has been a considerable interest to study common fixed point for a pair (or family) of mappings satisfying contractive conditions in metric spaces. Several interesting and elegant results were obtained in this direction by various authors. It was the turning point in the "fixed point arena" when the notion of commutativity was used by Jungck[2] to obtain common fixed point theorems. This result was further generalized and extended in various ways by many authors. In particular, now we look in the context of common fixed point theorem in G- metric spaces.

In 1982, Sessa[11] introduced the concept of weakly commuting maps as follows:

Definition 2.1.[11] Two self mappings f and g of a metric space (X, d) is said to be weakly commuting if $d(fgx,gfx) \leq d(fx,gx)$ for all x in X.

In 1994, Pant[9] introduced the notion of R-weakly commuting mappings in metric spaces as follows:

Definition 2.2.[9] A pair of self mappings (f, g) of a metric space (X, d) is said to be R-weakly commuting if there exist some positive real number R such that
$d(fgx,gfx) \leq R\, d(fx,gx)$ for all x in X.
In 1997, Pathak, Cho and Kang[10] introduced the improved notions of R-weakly commuting mappings and called them R-weakly commuting mappings of type (A_f) and R-weakly commuting mappings of type (A_g).
Definition 2.3.[10] A pair of self maps **(**f, g) of a metric space (X, d) is said to be
(a) R-weakly commuting mappings of type (A_f) if there exists some positive real number R such that $d(fgx,ggx) \leq R\, d(fx,gx)$ for all x in X.
(b) R-weakly commuting mappings of type (A_g) if there exists some positive real number R such that $d(gfx,ffx) \leq R\, d(fx,gx)$ for all x in X.
In 1998, Jungck and Rhoades introduced the concept of weakly compatibility as follows:
Definition 2.4.[3] Two self-mappings f and g are said to be weakly compatible if they commute at coincidence points.
Recently, in 2009, Kumar and Garg[4] introduce R-weakly commuting mappings of type (P) in metric space as follows:
Definition 2.5.[4] A pair of self maps **(**f, g) of a metric space (X, d) is said to be
R-weakly commuting mappings of type (P) if there exists some positive real number R such that $d(ffx,ggx) \leq R\, d(fx,gx)$ for all x in X.
Now, we introduce the concept of weakly commuting, R-weakly commuting maps, R-weakly commuting mappings of type (A_f), (A_g) and (P) in G – metric space as follows:
Definition 2.6. A pair of self maps **(**f, g) of a G-metric space is said to be weakly commuting if $G(fgx,gfx,gfx) \leq G(fx,gx,gx)$ for all x in X.
Definition 2.7. A pair of self maps **(**f, g) of a G-metric space is said to be R-weakly commuting if there exists some positive real number R such that

$$G(fgx,gfx,gfx) \leq R\, G(fx,gx,gx) \text{ for all x in X.}$$

Remark 2.1*:* If $R < 1$, then R- weakly commuting maps are weakly commuting.
Definition 2.8. A pair of self maps **(**f, g) of a G–metric space (X, G) is said to be
(a) R-weakly commuting mappings of type (A_f) if there exists some positive real number R such that $G(fgx,ggx,ggx) \leq R\, G(fx,gx,gx)$ for all x in X.
(b) R-weakly commuting mappings of type (A_g) if there exists some positive real number R such that $G(gfx,ffx,ffx) \leq R\, G(fx,gx,gx)$ for all x in X.
(c) R-weakly commuting mappings of type (P) if there exists some positive real number R such that $G(ffx,ggx,ggx) \leq R\, G(fx,gx,gx)$ for all x in X.
Remark 2.2: We have suitable examples which show that R-weakly commuting mappings, R- weakly commuting of type (A_f), R-weakly commuting mappings of type (A_g) and R-weakly commuting mappings of type (P) are distinct.
Example 2.1*:* Let X = [-1, 1] the set of all real numbers with the G – metric defined as follows: $G(x, y, z) = (|x-y| + |y-z| + |z-x|)$, for all x, y, z in X. Define $f(x) = |x|$ and $g(x) = |x| - 1$. Then by a straightforward calculation, one can show that $|fx-gx| = 1$, $|fgx - gfx| = 2(1 - |x|)$, $|fgx - ggx| = 1$, $|gfx - ffx| = 1$ and $|ffx - ggx| = 2|x|$, for all x, y in X. Now we conclude the following:

(i) pair (f, g) is not weakly commuting, (ii) for R = 2, pair (f, g) is R-weakly commuting mappings, R- weakly commuting of type (A_f), R-weakly commuting mappings of type (A_g) and R-weakly commuting mappings of type (P), (iii) for R = $\frac{3}{2}$, pair (f, g) is R-weakly commuting of type (A_f), R-weakly commuting mappings of type (A_g) but not R-weakly commuting mappings of type (P) and R-weakly commuting mappings.

Example 2.2*:* Let X = [0 , 1] the set of all real numbers with the G – metric defined as follows: G (x , y , z) = (│x-y│ +│y-z│ + │z-x│) , for all x, y, z in X. Define f(x) = x and g(x) =x^2. Then by a straightforward calculation, one can show that │fx-gx│= =│x(x-1) │, │fgx - gfx│= 0, │fgx - ggx│= │$x^2(x^2-1)$│, │gfx - ffx│=│x(x-1)│, and │ffx - ggx│=│$(x^2 + x + 1)(x^2-x)$│, for all x, y in X. Therefore, we conclude that (i) pair (f, g) is R-weakly commuting for all positive real values of R, (ii) for R = 3, pair (f, g) is R- weakly commuting of type (A_f), R-weakly commuting of type (A_g) and R-weakly commuting mappings of type (P), (iii) for R = 2, pair (f, g) is R- weakly commuting of type (A_f), R-weakly commuting mappings of type (A_g) but not R-weakly commuting mappings of type (P) (for this take x = ¾).

Example 2.3: Let X = [½ , 2] the set of all real numbers with the G – metric defined as follows: G (x , y , z) = (│x-y│ +│y-z│ + │z-x│) , for all x, y, z in X. Let us define self maps f and g on X by f(x) =$\frac{x+1}{3}$. g(x) = $\frac{x+2}{5}$. We calculate the following:

│fx-gx│=$\frac{2x-1}{15}$, │fgx - gfx│= 0, │fgx - ggx│= $\frac{2x-1}{75}$, │gfx - ffx│=$\frac{2x-1}{45}$, and

│ffx - ggx│=$\frac{8}{225}(2x-1)$, for all x, y in X. Now, we conclude the following:

The pair (f, g) is R-weakly commuting for all positive real values of R. For R ≥ $\frac{8}{15}$, it is R- weakly commuting of type (A_f), R-weakly commuting of type (A_g) and R-weakly commuting of type (P). For $\frac{1}{3}$ ≤ R <$\frac{8}{15}$, it is R- weakly commuting of type (A_f), R-weakly commuting of type (A_g) and R-weakly commuting of type (P).

For $\frac{1}{5}$≤ R < $\frac{1}{3}$, pair (f, g) is R- weakly commuting of type (A_f) but not R-weakly commuting of type (A_g) and R-weakly commuting of type (P).

Moreover, such mappings commute at their coincidence points. It is obvious that f and g can fail to be pointwise R- weakly commuting only if there exists some x in X such that fx = gx but fgx ≠ gfx, that is , only if they possess a coincidence point at which they do not commute. Therefore, the notion of pointwise R-weak commutativity type mapping is equivalent to commutativity at coincidence points.

Now we prove our main result:

Theorem 2.1. Let f and g be weakly compatible self maps of a G-metric space (X, G) satisfying the following conditions:

(2.1) f(X) $\subseteq$g(X);

(2.2) any one of the subspace f(X) or g(X) is complete;

(2.3) G(fx, fy, fz) ≤ q G(gx, gy, gz) for all x, y, z in X and 0≤ q < 1.

Then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (2.1), one can choose a point x_1 in X such that $fx_0 = gx_1$, In general choose x_{n+1} such that $y_n = fx_n = gx_{n+1}$.

Now, we prove $\{y_n\}$ is a G-Cauchy sequence in X.

From (2.3), take $x = x_n$, $y = x_{n+1}$, $z = x_{n+1}$ we have

$G(fx_n, fx_{n+1}, fx_{n+1}) \le q\, G(gx_n, gx_{n+1}, gx_{n+1}) = q\, G(fx_{n-1}, fx_n, fx_n)$

Continuing in the same way, we have

$G(fx_n, fx_{n+1}, fx_{n+1}) \le q^n\, G(fx_0, fx_1, fx_1) \Rightarrow G(y_n, y_{n+1}, y_{n+1}) \le q^n\, G(y_0, y_1, y_1)$

Therefore, for all n, m $\in$ N (set of natural numbers), n < m, we have by G(5)

$$\begin{aligned} G(y_n, y_m, y_m) &\le G(y_n, y_{n+1}, y_{n+1})+G(y_{n+1}, y_{n+2}, y_{n+2})+G(y_{n+2}, y_{n+3}, y_{n+3})+\cdots+ G(y_{m-1}, y_m, y_m) \\ &\le (q^n + q^{n+1} + q^{n+2} + \cdots + q^{m-1})\, G(y_0, y_1, y_1) \\ &\le (q^n + q^{n+1} + q^{n+2} + \cdots)\, G(y_0, y_1, y_1) \\ &\le \frac{q^n}{(1-q)} G(y_0, y_1, y_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $\{y_n\}$ is a G – Cauchy sequence in X. Since either f(X) or g(X) is complete, for definiteness assume that g(X) is complete subspace of X then the subsequence of $\{y_n\}$ must get a limit in g(X). Call it be 'z'. Let $u \in g^{-1}z$. Then gu = z. As $\{y_n\}$ is a G-Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ also convergent implying thereby the convergence of subsequence of the convergent sequence.

Now we show that fu = z. On setting x = u, $y = x_n$ and $z = x_n$, in (2.3), we have

$$G(fu, fx_n, fx_n) \le q\, G(gu, gx_n, gx_n).$$

Letting as $n \to \infty$, we have $G(fu, z, z) \le q\, G(gu, z, z)$ implies fu = z.

Therefore, fu = gu = z. i.e. 'u' is coincident point of f and g. Since f and g are weakly compatible, it follows that fgu = gfu i.e. fz = gz.

We now show that fz = z. Suppose that fz $\neq$ z, therefore G(fz, z, z) > 0.

From (2.3), on setting x = z, y = u, z = u, we have

$G(fz, z, z) = G(fz, fu, fu) \le q\, G(gz, gu, gu) = q\, G(fz, z, z) < G(fz, z, z)$ a contradiction, therefore fz = z. Thus, fz = gz = z i.e. 'z' is common fixed point of f and g.

Uniqueness. We assume that z_1 ($\neq$z) be another common fixed point of f and g.

Then $G(z, z_1, z_1) > 0$ and

$G(z, z_1, z_1) = G(fz, fz_1, fz_1)\,) \le q\, G(gz, gz_1, gz_1) = q\, G(z, z_1, z_1) < G(z, z_1, z_1)$,

a contradiction, therefore $z = z_1$. Hence uniqueness follows.

Example 2.4. Let X = [-1, 1] and let G: X × X × X $\to R^+$ be the G – metric defined as follows: G (x , y , z) = (|x-y| + |y-z| + |z-x|) , for all x, y, z in X. Then (X, G) is a G – metric space. Define f (x) = $\frac{x}{6}$ and g (x) = $\frac{x}{2}$.

Here we note that, (1) f (X) $\subseteq$ g(X), (2) Both f(X) and g(X) is complete,

(3) G (fx , fy, fz)) $\le$ q G(gx, gy, gz), holds for all x, y , z $\in$ X , $\frac{1}{3} \le q < 1$. However, the maps f and g are weakly compatible because f and g commute at coincidence point i.e. at x = 0 and x = 0 is the unique common fixed point of f and g. Thus all the conditions of the theorem 2.1 are satisfied.

Theorem 2.2: Theorem 2.1 remains true if a 'weakly compatible property' is replaced by any one (retaining the rest of the hypotheses) of the following:
(i) R- weakly commuting property,
(ii) R- weakly commuting property of type (A_f),
(iii)R-weakly commuting property of type (A_g),
(iv) R-weakly commuting property of type (P) and
(v) Weakly commuting property.
Proof: Since all the conditions of Theorem 2.1 are satisfied, then the existence of coincidence points for both the pairs is insured. Let 'x' be an arbitrary point of coincidence for the pair (f, g), then using R- weak commutativity one gets

$$G(fgx,gfx,gfx) \leq R\, G(fx,gx,gx) = 0 \text{ for all } x \text{ in } X,$$

which amounts to say that fgx = gfx. Thus the pair (f, g) is weakly compatible. Now applying Theorem 2.1, one concludes that f and g have a unique common fixed point.
In case (f, g) is R- weakly commuting pair of type (A_f), then

$G(fgx,ggx,ggx) \leq R\, G(fx,gx,gx) = 0$, which amounts to say that fgx = ggx.
Now, by using (G5), $G(fgx, gfx, gfx) \leq G(fgx, ggx, ggx) + G(ggx, gfx, gfx)$
$= 0 + G(gfx, gfx, gfx) = 0$, yielding thereby fgx = gfx.
In case (f, g) is R- weakly commuting pair of type (A_g), then
$G(gfx,ffx,ffx) \leq R\, G(fx,gx,gx) = 0$, which amounts to say that gfx = ffx.
Now, by using (G5), $G(fgx, gfx, gfx) \leq G(fgx, ffx, ffx) + G(ffx, gfx, gfx)$
$= G(ffx, ffx, ffx) + 0 = 0$, yielding thereby fgx = gfx.
Similarly, if pair (f, g) is R- weakly commuting of type (P) or weakly commuting, then (f, g) also commutes at their points of coincidence. Now, in view of Theorem 2.1, in all cases, f and g have a unique common fixed point. This completes the proof.

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