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Fixed Point Theorem for Weakly Compatible Maps Using E.A. Property in Fuzzy Metric Spaces Satisfying Contractive Condition of Integral Type

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Abstract

In this paper, we prove common fixed point theorem in fuzzy metric space for weakly compatible mappings satisfying integral type contractive condition and E.A. property. Our theorem generalizes the result of P.Vijayaraja and Z.M.I Sajath [5] recently appeared in this journal.

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1. Introduction:

Aamri and El. Moutawakil[1] generalized the concept of non compatibility by defining the notion of E.A. property and proved common fixed point theorems under strict contractive conditions. In this paper, we extend this concept to fuzzy metric spaces and establish the existence of common fixed points for a couple of pairs under general contractive condition of integral type using E.A. property.

2. Definitions:

Definition 2.1: A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:
(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0, 1]$;
(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.
Example 2.1: $a * b = \min\{a,b\}$ and $a * b = a . b$ are t-norms.

I. Kramosil and J. Michalek [3] introduced the concept of fuzzy metric spaces as follows:
Definition 2.2: The 3-tuple $(X, M, *)$ is called a fuzzy metric space (shortly, FM-space) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:
(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x. t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$ (Triangular inequality) and
(FM-5) $M(x, y, .) : [0, 1) \to [0, 1]$ is left continuous $\forall$ $x, y, z \in X$ and $s, t > 0$.
Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We can fuzzify examples of metric spaces into fuzzy metric spaces in a natural way:
Let (X, d) be a metric space. Define $a * b = a + b$ for all a, b in X.
Define $M(x, y, t) = t /(t + d(x, y))$ for all x, y in X and $t > 0$.Then $(X,M, *)$ is a fuzzy metric space and this fuzzy metric induced by a metric d is called the **Standard fuzzy metric**.
Definition 2.3: Let $(X, M, *)$ be fuzzy metric space. Then
(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$.
(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$, $\lim_{n\to\infty} M(x_n, x, t) = 1$.
Definition 2.4: An fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.
Example 2.2: Let $X = \{1/n : n \in N\} \cup \{0\}$ and let $*$ be the continuous t-norm and defined by $a * b = ab$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ and $x, y \in X$, define M, by

$$M(x, y, t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0 \\ 0 & t = 0 \end{cases}.$$

Clearly, $(X, M, *)$ is complete fuzzy metric space.
Definition 2.5: A pair of self mappings (f , g) of a fuzzy metric space $(X, M, *)$ is said to be commuting if $M(fgx , gfx , t) = 1$ and for all $x \in X$.
Definition 2.6: A pair of self mappings (f , g) of a fuzzy metric space $(X, M, *)$ is said to satisfy the E.A. property if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} M(fx_n, gx_n, t) = 1$.
Example 2.3: Let $X = [0, \infty)$.Let us consider $(X, M, *)$ be an fuzzy metric space as in example 2.2. Define $T, S : X \to [0,\infty)$ by $Tx = x/5$ and $Sx = 2x/5$ for all $x \in X$. For a sequence $\{x_n = 1/n\}$, $\lim_{n\to\infty} M(Sx_n, Tx_n,t) = 1$ .Therefore, T, S satisfies E.A. property.

Definition 2.7: A pair of self mappings (f, g) of a fuzzy metric space (X, M, *) is said to be weakly compatible if they commute at the coincidence points i.e. Tu = Su for some u in X , then TSu = STu.

It is easy to see that two compatible maps are weakly compatible.

Branciari-Integral contractive type condition [2]: For a given $\in >0$, there exists a real number $c \in (0, 1)$ and a locally Lebesgue-integrable function $g: [0, \infty) \to [0, \infty)$ such that $\int_0^{d(fx, fy)} g(t)\,dt \le c \int_0^{d(x,y)} g(t)dt$ and $\int_0^{\in} g(t)dt > \mathbf{0}$ for all x, y in X and for each $\in > 0$.
Also, Branciari-Integral contractive type condition is a generalization of Banach contraction map if g(t) = 1 for all $t \geq 0$.
Recently, P. Vijayraju and Z.M.I Sajath.[5], established the following theorem:

Theorem 2.1: Let A, B, S and T be self maps of fuzzy metric space (X, M, *) such that

(1) $A(X) \subset T(X)$, $B(X) \subset S(X)$,
(2) The pairs (A, S) or (B, T) satisfies E.A. property,
(3) For all $x \neq y$ in X, t > 0 such that
$M(Ax,By,t) > F(\min\{M(Sx,Ty,t),M(Sx,By,t),M(Ty,By,t)\})$ where
$F:[0, 1] \to [0, 1]$ is a increasing function satisfying F(t) > t for all t in (0 , 1] and F(1) = 1,
(4) (A, S) and (B, T) are weakly compatible,

If one of A(X), B(X), S(X) or T(X) is complete subspace of X then A, B, S and T have unique common fixed point in X.
We shall use the following lemma for the proof of our main result:
Lemma 2.1[4]: Let (X, M, *) be fuzzy metric space and for all x , y in X , t > 0 and if for a number k in (0 , 1), $M(x, y, kt) \geq M(x, y, t)$ .Then x = y.

3. Main Result:

We, now prove a common fixed point theorem using E.A. property in fuzzy metric space which generalizes the result of P.Vijayaraja and Z.M.I Sajath[5].
Theorem 3.1: Let A, B, S and T be self maps of fuzzy metric space (X, M, *) with continuous t-norm defined by a * b = min {a , b} for all a, b in [0 , 1] satisfying the following conditions:

(a) $A(X) \subset T(X)$, $B(X) \subset S(X)$,
(b) The pairs (A , S) or (B , T) satisfies E.A. property ,
(c) For all x, y in X , k in (0, 1) , t > 0 such that

$$\int_0^{M(Ax,By,kt)} \phi(t)dt \geq \int_0^{m(x,y,t)} \phi(t)dt$$

where ϕ : $R^+ \to R$ is a Lebesgue-integrable mapping which is summable, nonnegative and such that $\int_0^{\epsilon} \phi(t)dt > 0$ for each $\epsilon > 0$,where

m(x,y,t) = M(Sx, Ty, t) * M(Ax, Sx, t) * M(Bx, Ty, t) * M(By, Sx, 2t) * M(Ax, Ty, t)
for all x, y in X and t > 0 .If one of A(X), B(X), S(X) or T(X) is complete subspace of X then (A, S) and (B, T) have coincidence point. Further, if (A, S) and (B, T) are weakly compatible then A, B, S and T have unique common fixed point in X.

Proof: Suppose the pair (B, T) satisfies the E.A. property. Then there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = p$ for some p in X. (1)

Since $B(X) \subset S(X)$, there exists a sequence $\{y_n\}$ in X such that

$Bx_n = Sy_n$. (2)

Hence $\lim_{n\to\infty} Sy_n = p$. (3)

Now we claim that $\lim_{n\to\infty} Ay_n = p$.

By condition (c) of the theorem, take $x = y_n$, $y = x_n$, we have

$$\int_0^{M(Ay_n, Bx_n, kt)} \phi(t)dt \geq \int_0^{m(y_n, x_n, t)} \phi(t)dt, \text{ where} \quad (4)$$

$m(y_n, x_n, t) = M(Sy_n, Tx_n\ y, t)*M(Ay_n, Sy_n, t)*M(By_n, Ty_n, t)*M(Bx_n, Sy_n, 2t)*M(Ay_n, Tx_n, t)$

Taking limit as $n \to \infty$ and using (1), (2), (3) and definition of fuzzy metric space, we get $m(y_n, x_n, t) = M(Ay_n, p, t)$.

From (4), as $n \to \infty$, we get

$$\int_0^{M(Ay_n, p, kt)} \phi(t)dt \geq \int_0^{M(Ay_n, p, t)} \phi(t)dt$$

Using Lemma 2.1, we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sx_n = p$.

Now, suppose that S(X) is a complete subspace of X. Then p = Su for some u in X and subsequently, we have

$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = p = Su$ (5)

We shall now show that Au = Su.

Taking x = u, $y = x_n$ in condition (c), we have

$$\int_0^{M(Au, Bx_n, kt)} \phi(t)dt \geq \int_0^{m(u, x_n, t)} \phi(t)dt, \text{ where} \quad (6)$$

$m(u, x_n, t) = M(Su, Tx_n, t) * M(Au, Su, t) * M(Bx_n, Tx_n, t) * M(Bx_n, Su, 2t) * M(Au, Tx_n, t)$

Taking limit as $n \to \infty$ and using (1), (2) and (3), we get $m(u, x_n, t) = M(Au, Su, t)$.

From (6), as $n \to \infty$, we get $\int_0^{M(Au, Su, kt)} \phi(t)dt \geq \int_0^{M(Au, Su, t)} \phi(t)dt$

by using Lemma 2.1, we have Au = Su. Therefore (A, S) have coincidence point. The weak compatibility of A and S implies that ASu = SAu and thus

$$AAu = ASu = SAu = SSu. \quad (7)$$

As $A(X) \subset T(X)$, there exists v in X such that Au = Tv. (8)

Now, we claim that Tv = Bv.

Taking x = u, y = v in condition (c) we have

$$\int_0^{M(Au, Bv, kt)} \phi(t)dt \geq \int_0^{m(u, v, t)} \phi(t)dt, \text{ where} \quad (9)$$

m(u, v, t) = M(Su, Tv, t) * M(Au, Su, t) * M(Bv, Tv, t) * M(Bv, Su, 2t) * M(Au, Tv, t)

by using (7) and (8) , we get m(u, v, t) = M(Au, Bv, t)

From (9), we get $\int_0^{M(Au,Bv,kt)} \phi(t)dt \geq \int_0^{M(Au,Bv,t)} \phi(t)dt$

by using Lemma 2.1, we have Au = Bv.

Hence, Tv = Bv.

Thus we have Au = Su = Tv = Bv.

The weak compatibility of B and T implies that BTv = TBv = TTv = BBv. (10)

We shall now show that Au is the common fixed point of A, B, S and T.

Again taking x = Au, y = v in condition (c), we get

$$\int_0^{M(Au,AAv,kt)} \phi(t)dt = \int_0^{M(AAu,Bv,kt)} \phi(t)dt \geq \int_0^{m(Au,v,t)} \phi(t)dt, \quad \text{where} \tag{11}$$

m(Au, v , t) = M(SAu, Tv, t)*M(AAu, SAu, t)*M(Bv, Tv, t)*M(Bv, SAu, 2t)* M(AAu, Tv, t)
by using (7), (8) and (10), we obtain m(Au, v, t) = M(AAu, Bv, t).

From (11), we get $\int_0^{M(AAu,Bv,kt)} \phi(t)dt \geq \int_0^{M(AAu,Bv,t)} \phi(t)dt$.

Now the use of Lemma 2.1 gives AAu = Bv = Au and thus AAu = Au. Therefore, Au = AAu = SAu is the common fixed point of A and S.

Similarly, we prove that Bv is the common fixed point of B and T. Since Au = Bv, Au is common fixed point of A, B, S, and T. The proof is similar when T(X) is assumed to be a complete subspace of X. The cases in which A(X) or B(X) is a complete subspace of X are similar to the cases in which T(X) or S(X), respectively is complete subspace of X as A(X) $\subset$ T(X) and B(X) $\subset$ S(X).

Finally, we show the uniqueness of the common fixed point. If possible, let x_0 and y_0 be two common fixed points of A, B, S, and T. Then by taking x = x_0 and y = y_0 in condition (c), we have

$$\int_0^{M(Ax_0,By_0,kt)} \phi(t)dt \geq \int_0^{m(x_0,y_0,t)} \phi(t)dt, \quad \text{where} \tag{12}$$

m(x_0, y_0,t) = M(Sx_0, Ty_0, t) * M(Ax_0, Sx_0, t) * M(By_0, Ty_0, t)*M(By_0, Sx_0, 2t)* M(Ax_0, Ty_0, t)

By using definition of fixed point and fuzzy metric space, we get

m(x_0, y_0, t) = M(x_0, y_0, t)

Therefore, from (12) we get $\int_0^{M(x_0,y_0,kt)} \phi(t)dt \geq \int_0^{M(x_0,y_0,t)} \phi(t)dt$.

which implies, by the use of Lemma 2.1 that x_0 = y_0 and thus the mappings A, B, S, and T have a unique common fixed point.

Now, we give an example to validate theorem 3.1.

Example 3.1: Let (X, M, *) be a fuzzy metric space with X = [0, 1], t-norm * defined by a * b = min{a, b} where a, b$\in$ [0, 1], respectively. Let M(x,y,t) is the fuzzy set on $X^2 \times (0, \infty)$, defined by

$$M(x, y, t) = \begin{cases} \left(\exp\left(\frac{|x-y|}{t} \right) \right)^{-1} & \text{for all } x,y \in X \text{ and } t > 0 \\ 0 & \text{for all } x,y \in X \text{ and } t = 0. \end{cases}$$

Then it is well known that (X, M, *,) is a fuzzy metric space. Let us define self maps

A, B, S, and T on X such that Ax = $\frac{x}{64}$, Tx = $\frac{x}{2}$, Bx = $\frac{x}{32}$, Sx = $\frac{x}{4}$, then

for $k \in \left[\frac{1}{16,}, 1\right)$,

$$M(Ax, By, kt) = \left(\exp\left(\frac{\left|\frac{x}{64} - \frac{y}{32}\right|}{kt}\right)\right)^{-1} \geq \left(\exp\left(\frac{\left|\frac{x}{4} - \frac{y}{2}\right|}{t}\right)\right)^{-1} = M(Sx, Tx, t)$$

$\geq$ M(Sx, Ty, t) * M(Ax, Sx, t) * M(Bx, Ty, t)*M(By, Sx, 2t) * M(Ax, Ty, t)
Clearly,
(i) for ϕ(t) = 1 for all t > 0 ,condition (c) of above theorem-3.2 holds,
(ii) for sequence { $x_n = \frac{1}{n}$ } , pairs (A, S) or (B, T) satisfies E.A. property,
(iii) A(X) $\subset$ T(X), B(X) $\subset$ S(X),
(iv) one of A(X), B(X), S(X) or T(X) is complete subspace of X,
(v) the pairs (A , S) and (B , T) are weakly compatible at x = 0 which is the coincident point of the maps A , B, S and T.
Thus all the conditions of Theorem-3.1 are satisfied and also x = 0 is the unique common fixed point of A, B, S and T.

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