**Abstract**

In this paper, we introduce the concept relative $FI$-lifting modules relative a proper class of short exact sequences of modules as in [4]. It is well known, any finite direct sum of $FI$-lifting modules is again $FI$-lifting (see[9]), but this property is not true in general for infinite direct sums. In this paper we will show that it is true for relative $FI$-lifting modules.

**Keywords:** $FI$-Lifting modules; strong $FI$-lifting modules; fully invariant submodules; lifting modules; proper class.

**1 Introduction**

A module $M$ is called *(strong)* $FI$-lifting if every fully invariant submodule $N$ of $M$ contains a (fully invariant) direct summand $K$ of $M$, such that $N/K \ll M/K$. Lifting modules and their generalizations have been studied by many authors [see 1, 2, 4, 8]. *(strong)* $FI$-lifting modules is the proper generalization of lifting module. It is well known, any finite direct sum of $FI$-lifting modules is again $FI$-lifting (see[9]), but this property is not true in general for infinite direct sums of $FI$-lifting modules. In this article, we introduced (strongly) $E$-$FI$-lifting module relative a proper class of short exact sequence $s$ of modules as in [4] and characterize for any direct sums of copies of them are such relative $FI$-lifting modules

**2 Preliminary Notes**

Throughout this article, $R$ is an associative ring and with nonzero identity and all modules are unital right $R$-modules, morphisms will operate on the right. We use $N \leq M$ to indicate that $N$ is a submodule of $M$. Let $M$ be a skeletally small (full) subcategory of Mod-$R$. $Add(M)(add(M))$ denote
the class of modules which are isomorphic to direct summands of (finite) direct sums of modules of \( M \), \( \text{Prod}(M) \) denoted the class of modules which are isomorphic to direct summands of direct products of modules of \( M \), \( \text{Gen}(M) \) denoted the class of modules generated by \( M \), \( \text{Cogen}(M) \) denoted the class of modules was cogenerated by \( M \). \( \text{Cogen}'(M) \) denote the class of modules for which there exist a monomorphism from \( K \) to some \( M^{(i)} \), and the class \( \text{Prod}'(M) \) of modules \( K \) for which there is an \( E \)-monomorphism from \( K \) to some \( M^{(i)} \).

Let us consider two classes of modules related to \( M \) and to a proper class \( E \) of short exact sequences in \( \text{Mod}-R \). Denote by \( \text{Add}(M) \) the class of modules \( N \) for which there is an \( E \)-epimorphism \( \bigoplus_{i \in I} M_i \rightarrow N \) with each \( M_i \in M \) (or, equivalently, an \( E \)-epimorphism \( X \rightarrow N \) with \( X \in \text{Add}(M) \)), and by \( \text{Prod}(M)=\{K \in \text{Mod}-R|\text{there is an } E \text{-epimorphism } K \rightarrow \prod \limits_{i \in I} M_i, i \in I \text{ with each } M_i \in M\} \) (or, equivalently, an \( E \)-epimorphism \( X \rightarrow N \) with \( X \in \text{Prod}(M) \)).

**Definition 2.1** \([4]\) Let \( E \) be a class of short exact sequences in \( \text{Mod}-R \). If an exact sequence \( 0 \rightarrow K \rightarrow L \rightarrow N \rightarrow 0 \) belongs to \( E \), then \( f \) is called \( E \)-monomorphism and \( g \) is called an \( E \)-epimorphism. Also, \( \text{Im } f \) is called an \( E \)-submodule of \( L \) and \( N \) is called an \( E \)-homosphic image of \( L \).

The class \( E \) is called a proper class if it has the following properties:

- **P1.** \( E \) is closed under isomorphisms.
- **P2.** \( E \) contains all splitting short exact sequences.
- **P3.** The class of \( E \)-monomorphism is closed under composition; \( f, f' \) are monomorphisms and \( f'f \) is an \( E \)-monomorphism, then \( f \) is an \( E \)-monomorphism.
- **P4.** The class of \( E \)-epimorphism is closed under composition; \( f, f' \) are epimorphisms and \( f'f \) is an \( E \)-epimorphism, then \( f \) is an \( E \)-epimorphism.

**Definition 2.2** Let \( M \) be a module, \( X \leq M \) is called fully invariant (denote \( X \triangleright M \)), if for every \( h \in \text{End}_R(M) \), \( h(X) \subseteq X \).

**Definition 2.3** \([6]\) A module \( M \) is called (strongly) FI-lifting, if for every fully invariant submodules \( N \) of \( M \), there is a (fully invariant) direct summand \( D \) of \( M \) such that \( N/D \ll M/D \).

It is well known, if \( M \) is an FI-lifting, \( A \) is a fully invariant coclosed submodule of \( M \), then \( M/A \) are FI-lifting.

It is easy to proof that any direct summand of strongly FI-lifting is strongly FI-lifting. Clearly, indecomposable FI-lifting module are strongly FI-lifting.
module. Noted that (strong) $FI$-lifting modules is closed under its coclosed fully invariant submodules and its (indecomposable) factor modules. So we have:

**Lemma 2.4** [4] For an $R$-module $M$, the following are equivalent:
1. $M$ is $\Sigma$-$E$-direct injective.
2. For every $U \in \text{Cogen}'(M)$ and every $V \in E\text{Prod}'(M)$, every monomorphism $V \to U$ is an $E$-monomorphism.
3. For every $U, V \in E\text{Prod}'(M)$, every monomorphism $V \to U$ is an $E$-monomorphism.

**Lemma 2.5** [4] Let $M$ be a $E$-direct injective module. If $U \leq M$, and $V$ is an $E$-submodule of $M$, then every monomorphism $V \to U$ is an $E$-monomorphism.

For a module $M$, denote by $\sigma[M]$ the full subcategory of $\text{Mod-}R$ whose objects are submodules of $M$-generated modules.

**Definition 2.6** [6] A module $N \in \sigma[M]$ is called $M$-singular if $N \cong L/K$ for some $K \supseteq L$ and $L \in \sigma[M]$. A module $N \in \sigma[M]$ is called $M$-small if $N \ll L$ for some $L \in \sigma[M]$. The modules in the torsion class of the torsion theory in $\sigma[M]$ cogenerated by the $M$-small modules are called non-$M$-cosingular.

**Definition 2.7** [4] A module $M$ is called strong $E$-lifting, if it has coclosure property, and the coclosed submodule is an $E$-submodule.

Calling a module $M$ has coclosure property, if every submodule of $M$ has coclosed submodule.

### 3 Main Results

**Definition 3.1** A module $M$ is called $E$-$FI$-lifting, if the every fully invariant submodules $N$ of $M$, there is an $E$-submodule $D$ of $M$ such that $N/D \ll M/D$.

It is clearly that every fully invariant coclosed submodule of an $E$-$FI$-lifting module is $E$-submodule. If $M$ has coclosure property, the converse is true. The following theorem will show that $FI$-lifting module was generalized by $E$-$FI$-lifting module.

**Theorem 3.2** Let $M$ be a module, consider the following statements:
(a) $M$ is strongly $E$-$FI$-lifting module.
(b) $M$ is $E$-$FI$-lifting module.
(c) $M$ is FI-lifting module.

Then

(i) for every module $M$, $(a) \Rightarrow (b) \Rightarrow (c)$.
(ii) If the direct summand of $M$ coincide with fully invariant submodule then $(c) \Rightarrow (a)$.

**Proof** (i) $(a) \Rightarrow (b)$ is clearly. Let $E = E_S$ then $(b) \Rightarrow (c)$.

(ii) Since $M$ is FI-lifting module, let $N$ be a fully invariant submodule of $M$, then there exist a direct summand $K$ of $M$ such that $N/K \ll M/K$ by the hypothesis, $K$ is the fully invariant submodule of $M$, and clearly $K$ is $E$-submodules of $M$, so $M$ is strongly $E$-FI-lifting module.

A module $M$ is called **duo module** provide that every submodule of $M$ is fully invariant submodule. So we have the following:

**Corollary 3.3** If $M$ is a duo module, then the following statements are equivalent:

(a) $M$ is an strongly $E$-FI-lifting module.
(b) $M$ is an $E$-FI-lifting module.
(c) $M$ is FI-lifting module.

**Proof** By Theorem 3.2 above, the proof is clear.

In general an $E$-FI-lifting module is not $E$-lifting module, look at the following example:

**Example 3.4** [12] Let $p$ be a prime number, consider $Z$-module $M = (Z/pZ) \oplus (Z/p^3Z)$.

$M$ is $E_S$-FI-lifting module, but it is not $E_S$-lifting module by [11]. However, when $M$ is a duo module, $E$-FI-lifting module is an $E$-lifting.

**Definition 3.5** A module $M$ is called strongly FI-lifting, if for every fully invariant submodules $N$ of $M$, there is a fully invariant direct summand $D$ of $M$ such that $N/D \ll M/D$.

If $M$ is a duo module and has coclosure property, strongly $E$-lifting module is equivalent to strongly $E$-FI-lifting module.

We well known that if $M$ has the coclosure property, so does every coclosed submodule, every fully invariant (coclosed) submodule and every homomorphic image of $M$. What about the $E$-FI-lifting module?
Lemma 3.6 Let $A$ be a fully invariant coclosed submodule of an $E$-FI-lifting $B$, then $A$ is $E$-FI-lifting. If $B$ has coclosure property, then $B/A$ is $E$-FI-lifting.

Proof Let $B$ be an $E$-FI-lifting module, and $A$ be a coclosed submodule of $B$, let $C$ be a fully invariant submodule of $A$. Since $B$ is $E$-FI-lifting, $C$ contains a cosmall submodule $D$ of $C$ in $B$ such that $D$ is an $E$-submodule of $B$. Then $D$ is an $E$-submodule of $A$, since $A$ is the coclosed submodule of $B$, $D$ is a cosmall submodule of $C$ in $B$, it follows that $D$ is a cosmall submodule of $C$ in $A$ by [6, Lemma 3.9]. Let $C/A$ a fully invariant coclosed submodule of $B/A$, then $A$ is a fully invariant coclosed submodule of $B$, and $A$ is a fully invariant $E$-submodule of $B$, then $C/A$ is an $E$-submodule of $B/A$, so $B/A$ is an $E$-FI-lifting module.

Lemma 3.7 Let $A$ be a coclosed fully invariant submodule of a strong $E$-FI-lifting module $B$. If $B$ has coclosure property, then $A$ and $B/A$ are strongly $E$-FI-lifting modules. The proof is similarly to Lemma 3.6.

Theorem 3.8 Let $M = \bigoplus_{i \in I} M_i$. If $M_i(i \in I)$ are $E$-FI-lifting modules, then $M$ is $E$-FI-lifting module.

Proof Let $M_i(i \in I)$ are $E$-FI-lifting modules, $S \triangleright M$. There exist $e_i^2 = e_i E (= \operatorname{End}M)$ such that $M_i = Me_i$. Since $S \triangleright M$, $S = \bigoplus_{i \in I} (S \cap M_i) = \bigoplus_{i \in I} Se_i$. Clearly, $\operatorname{End}_R(Me_i) = e_i E e_i$. Because $(Se_i)e_i e_i e_i = Se_i E e_i \leq Se_i$, so $Se_i \triangleright Me_i = M_i$. Hence there exist $E$-submodules $D_i$ of $M_i$, such that $Se_i/D_i \ll M_i/D_i$, also, $Se_i/D_i \ll M/D_i$, $S/\bigoplus_{i \in I} D_i \ll M/\bigoplus_{i \in I} D_i$ by ([5], Lemma 2.5), $D_i$ are $E$-submodules of $M$, so does $\bigoplus_{i \in I} D_i$, hence $M$ is $E$-FI-lifting module.

By [9, Theorem 3.4], any finite direct sum of $FI$-lifting modules is again $FI$-lifting. The following two examples show that this property is not true in general for infinite direct sums of $FI$-lifting modules. Let $R$ be a discrete valuation ring with maximal ideal $m$. Let $M = \bigoplus_{i = 1}^{\infty} R/m^i$ or $M = R^{\mathbb{N}}$. By [10, Corollary 2, P.48], $\operatorname{Rad}(M)$ does not have a supplement in $M$. Since $\operatorname{Rad}(M)$ is a fully invariant submodule of $M$, $M$ is not $FI$-lifting. On the other hand, it is clear that $R/m^i(i \geq 1)$ and $R$ are lifting modules.

Corollary 3.9 Let $M = \bigoplus_{i \in I} M_i$. If $M_i(i \in I)$ are strongly $E$-FI-lifting modules, then $M$ is $E$-FI-lifting modules. The proof is immediately by Theorem
3.8 and Lemma 3.7 above.

A module $M$ is said to be $\sum P$ if every direct sum of copies of $M$ has the property P. Now we characterize relative (strongly $E$-$FI$)-lifting modules following from [4].

**Theorem 3.10** Let $M$ be a module, consider the following statements:

(a) $M$ is $\sum E$-$FI$-lifting.
(b) Every module in $\text{Add}(M)$ is $E$-$FI$-lifting.
(c) Every fully invariant $N \in \text{Cogen}'(M)$ has an $M$-small $E$-homomorphic image $N/Y$, such that $Y$ is in $E\text{Prod}'(M)$.
(d) Every fully invariant non-$M$-cosingular module in $\text{Cogen}'(M)$ is in $E\text{Prod}'(M)$.
(e) Every fully invariant non-$M$-cosingular module in $\text{Cogen}'(M)$ is in $E$-$FI$-lifting.

Then the following implications hold:

(1) For every module (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).
(2) If $M$ is $\sum E$-direct injective and has the $\sum$-coclosure property, then (d) $\Rightarrow$ (e).
(3) If $M$ is $\sum E$-direct injective and has the $\sum$-$FI$-coclosure property, and for every $U \subseteq N \subseteq F \in E\text{Prod}'(M)$, $K/U$ is $M$-small, implies $U$ is a cosmall submodule of $K$ in $F$, then (c) $\Rightarrow$ (a).
(4) If $M$ is non-$M$-cosingular, then (e) $\Rightarrow$ (a).

**Proof** (1)(a) $\Rightarrow$ (b), let $M$ be an $E$-$FI$-lifting module, by Theorem 2.8, $M^{(1)}$ is $E$-$FI$-lifting module.

(b)$\Rightarrow$ (a) is clear.

(b)$\Rightarrow$ (c) Let $N \in \text{cogen}'(M)$ and $N$ be a fully invariant submodule, take a monomorphism $f : N \rightarrow M^{(1)}$. $N \supset M$, it is easily to get $f(N) \supset M^{(1)}$.

Since $M^{(1)}$ is $E$-$FI$-lifting module, there exist an fully invariant submodule $E$-submodule $Y'$ of $M^{(1)}$ such that $Y' \subseteq f(N)$, and $f(N)/Y' \ll M^{(1)}/Y'$. Since $f(N)/Y' \ll M^{1}/Y'$, if $Y = f^{-1}(Y')$, then it follows that $N/Y$ is $M$-small, $Y \in E\text{Prod}'(M)$, also $N \in \text{cogen}'(M), Y$ is an $E$-submodule of $N$.

(c) $\Rightarrow$ (d) Clear.

(2) Assume that $M$ is $\sum E$-direct injective and has the $\sum$-coclosure property.

(d) $\Rightarrow$ (e) Let $N$ be a fully invariant and non-$M$-cosingular module in $\text{cogen}'(M)$, we say $N$ is $E$-$FI$-lifting. Consider a monomorphism $f : N \rightarrow M^{(1)}$. Let $L$ be a fully invariant submodule of a non-$M$-singular module of $N$. Then $f(L)$ has coclosure in $M^{(1)}$, say $K$. Hence $K$ is a coclosed in $f(N)$, and thus $K$ is a fully invariant submodule of a non-$M$-cosingular module. Since $K \in \text{Cogen}'(M)$, we have $K \in E\text{Prod}'(M)$ by hypothesis. Now by Lemma 1.8 the inclusion $K \rightarrow f(N)$ is an $E$-monomorphism. Then the inclusion $f^{-1}(K) \rightarrow N$ is an $E$-monomorphism. Since $f^{-1}(K)$ is coclosed in $N$, it follows that $N$ is an $E$-$FI$-lifting.
(3) If $M$ is $\Sigma$-$E$-direct injective and has the $\Sigma$-$FI$-coclosure property, and for every $U \subseteq N \subseteq F \in E\text{Prod}'(M)$, $K/U$ is $M$-small, implies $U$ is a cosmall submodule of $K$ in $F$, then (c) $\Rightarrow$ (a).

Let $N$ be a fully invariant submodule of $F = M^{(I)}$, where $I$ is any set. By (C) the module $N \subseteq \text{cogen}'(M)$ has an $M$-small homorphic image $N/Y$ such that $Y \in E\text{Prod}'(M)$. Then $Y \subseteq N \subseteq F \in \text{EProd}'(M)$ and $N/Y$ is $M$-small, hence $N/Y \leq F/Y$ by hypothesis, by Lemma 1.8 the inclusion $N \rightarrow F$ is an $E$-monomorphism. $M^{(I)}$ has $FI$-coclosure property, so $F$ is $E$-$FI$-lifting.

(4) (e) $\Rightarrow$ (a) If $M$ is non-$M$-cosingular, then every $M^{(I)}$ is non-$M$-cosingular, hence $E$-$FI$-lifting.

**Theorem 3.11** Let $M$ be a module, consider the following statements:

(a) $M$ is $\Sigma$-stronly $E$-$FI$-lifting.
(b) Every module in $\text{Add}(M)$ is strongly $E$-$FI$-lifting.
(c) Every module in $E\text{Prod}'(M)$ is strongly $E$-$FI$-lifting.
(d) Every non-$M$-Cosingular module in $\text{Cogen}'(M)$ is strongly $E$-$FI$-lifting.
(e) A module is in $E\text{Prod}'(M)$ if and only if it is in $\text{Cogen}'(M)$ and non-$M$-cosingular.

Then the following implications holds:

1. For every module (a)$\iff$ (b)$\iff$(c).
2. If $M$ is also non-$M$-cosingular module, then (a)$\Rightarrow$ (e) and (d) $\Rightarrow$ (a).
3. If $M$ is $\Sigma$-$E$-direct injective $\Sigma$-$E$-lifting, then (e) $\Rightarrow$ (c) and (e) $\Rightarrow$ (d).

**Proof** (1) (a)$\Rightarrow$ (c) Let $K \in \text{Prod}'(M)$. Then there is an $E$-monomorphism $K \rightarrow M^{(I)}$, by Lemma 2.6 $K$ is strongly $E$-$FI$-lifting;

(c) $\Rightarrow$ (b) $\Rightarrow$ (a) are clear.

(2) Assume $M$ is non-$M$-singular.

(a)$\Rightarrow$ (c) by Theorem 2.9, every fully invariant non-$M$-singular module in $\text{Cogen}'(M)$ is in $E\text{Prod}'(M)$. Conversely, let $K \in E\text{Prod}'(M)$ and take some $E$-monomorphism $K \rightarrow M^{(I)}$. Then $K$ is an $E$-submodule of $M^{(I)}$, have coclosed in $M^{(I)}$. Now since $M^{(I)}$ is non-$M$-singular, so is $K$.

(d) $\Rightarrow$ (a) If $M$ is non-$M$-singular, then every $M^{(I)}$ is non-$M$-singular, hence $M$ is strongly $E$-$FI$-lifting.

(3) Assume $M$ is $\Sigma$-$E$-direct injective $\Sigma$-$E$-$FI$-lifting.

(e)$\Rightarrow$ (c), (d) By hypothesis and Theorem 2.9, every module in $E\text{Prod}'(M)$ is $E$-$FI$-lifting. Now let $K \in E\text{Prod}'(M)$ and $L$ be an $E$-submodule of $K$, by an easy variation of Lemm1.9, $L \in E\text{Prod}'(M)$, hence $L$ is non-$M$-singular, so that $L$ is coclosed in $K$. Thus $K$ is strongly $E$-$FI$-lifting.

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