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$\sum$-FI-Lifting Modules

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Abstract

In this paper, we introduce the concept relative FI-lifting modules relative a proper class of short exact sequences of modules as in [4]. It is well known, any finite direct sum of FI-lifting modules is again FI-lifting (see[9]), but this property is not true in general for infinite direct sums. In this paper we will show that it is true for relative FI-lifting modules.

Keywords: FI-Lifting modules; strong FI-lifting modules; fully invariant submodules; lifting modules; proper class.

1 Introduction

A module M is called *(strong) FI-lifting* if every fully invariant submodules N of M contains a (fully invariant) direct summand K of M, such that $N/K \ll M/K$. Lifting modules and their generalizations have been studied by many authors [see 1, 2, 4, 8].*(strong) FI-lifting* modules is the proper generalization of lifting module. It is well known, any finite direct sum of FI-lifting modules is again FI-lifting (see[9]), but this property is not true in general for infinite direct sums of FI-lifting modules. In this article, we introduced (strongly) $\mathbb{E}$-FI-lifting module relative a proper class of short exact sequence s of modules as in [4] and characterize for any direct sums of copies of them are such relative FI-lifting modules

2 Preliminary Notes

Throughout this article, R is an associative ring and with nonzero identity and all modules are unital right R-modules, morphisms will operate on the right. We use $N \leq M$ to indicate that N is a submodule of M. Let M be a skeletally small (full) subcategory of Mod-R. Add(M)(add(M)) denote the class of modules which are isomorphic to direct summands of (finite) direct sums of modules of M, $Prod$(M) denoted the class of modules which are isomorphic to direct summands of direct products of modules of M, Gen(M) denoted the class of modules generated by M, $Cogen$(M) denoted the class of modules was cogenerated by M. $Cogen'$(M) denote the class of modules K for which there exist a monomorphism from K to some $M^{(I)}$, and the class $Prod'$(M) of modules K for which there is an $\mathbb{E}$-monomorphism from K to some $M^{(I)}$.

Let us consider two classes of modules related to M and to a proper class $\mathbb{E}$ of short exact sequences in Mod-R. Denote by $\mathbb{E}Add$(M) the class of modules N for which there is an $\mathbb{E}$-epimorphism $\bigoplus_{i\in I} M_i \longrightarrow N$ with each $M_i \in$ M(or, equivalently, an $\mathbb{E}$-epimorphism $X \longrightarrow N$ with $X \in Add$(M)), and by $\mathbb{E}Prod$(M)=$\{K \in$ Mod-R|there is an $\mathbb{E}$-epimorphism $K \twoheadrightarrow \prod M_i, i \in I$ with each $M_i \in$ M$\}$ (or, equivalently, an $\mathbb{E}$-epimorphism $X \longrightarrow N$ with $X \in Prod$(M)).

Definition 2.1 [4] *Let $\mathbb{E}$ be a class of short exact sequences in Mod-R. If an exact sequence 0$\rightarrow K \rightarrow L \rightarrow N \rightarrow 0$ belongs to $\mathbb{E}$, then f is called $\mathbb{E}$-monomorphism and g is called an $\mathbb{E}$-epimorphism. Also, Imf is called an $\mathbb{E}$-submodule of L and N is called an $\mathbb{E}$-homophic image of L.*

The class $\mathbb{E}$ is called a proper class if it has the following properties:

P1. $\mathbb{E}$ is closed under isomorphisms.

P2. $\mathbb{E}$ contains all splitting short exact sequences.

P3. The class of $\mathbb{E}$-monomorphism is closed under composition; f, f' are monomorphisms and $f'f$ is an $\mathbb{E}$-monomorphism, then f is an $\mathbb{E}$-monomorphism.

P4. The class of $\mathbb{E}$-epimorphism is closed under composition; f, f' are epimorphisms and $f'f$ is an $\mathbb{E}$-epimorphism, then f is an $\mathbb{E}$-epimorphism.

Definition 2.2 *Let M be a module, $X \leq M$ is called fully invariant (denote $X \rhd M$), if for every $h \in End_R(M), h(X) \subseteq X$.*

Definition 2.3 [6] *A module M is called (strongly) FI-lifting, if for every fully invariant submodules N of M, there is a (fully invariant) direct summand D of M such that $N/D \ll M/D$.*

It is well known, if M is an FI-lifting, A is a fully invariant coclosed submodule of M, then M/A are FI-lifting.

It is easy to proof that any direct summand of strongly FI-lifting is strongly FI-lifting. Clearly, indecomposable FI-lifting module are strongly FI-lifting module. Noted that (strong) FI-lifting modules is closed under its coclosed fully invariant submodules and its (indecomposable) factor modules. So we have:

Lemma 2.4 [4] *For an R-module M, the following are equivalent:*
(1) M is $\sum$-$\mathbb{E}$-direct injective.
(2) For every $U \in Cogen'(M)$ and every $V \in \mathbb{E}Prod'(M)$, every monomorphism $V \to U$ is an $\mathbb{E}$-monomorphism.
(3) For every $U, V \in \mathbb{E}Prod'(M)$, every monomorphism $V \to U$ is an $\mathbb{E}$-monomorphism.

Lemma 2.5 [4] *Let M be a $\mathbb{E}$-direct injective module. If $U \leq M$, and V is an $\mathbb{E}$-submodule of M, then every monomorphism $V \longrightarrow U$ is an $\mathbb{E}$-monomorphism.*

For a module M, denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are submodules of M-generated modules.

Definition 2.6 [6] *A module $N \in \sigma[M]$ is called* M-singular *if $N \cong L/K$ for some $K \trianglerighteq L$ and $L \in \sigma[M]$.A module $N \in \sigma[M]$ is called* M-small *if $N \ll L$ for some $L \in \sigma[M]$. The modules in the torsion class of the torsion theory in $\sigma[M]$ cogenerated by the M-small modules are called* non-M-cosingular.

Definition 2.7 [4] *A module M is called* strong $\mathbb{E}$-lifting, *if it has coclosure property, and the coclosed submodule is an $\mathbb{E}$-submodule.*

Calling a module M has coclosure property, if every submodule of M has coclosed submodule.

3 Main Results

Definition 3.1 *A module M is called $\mathbb{E}$-FI-lifting, if the every fully invariant submodules N of M, there is an $\mathbb{E}$-submodule D of M such that $N/D \ll M/D$.*

It is clearly that every fully invariant coclosed submodule of an $\mathbb{E}$-FI-lifting module is $\mathbb{E}$-submodule. If M has coclosure property, the converse is true.

The following theorem will show that FI-lifting module was generalized by $\mathbb{E}$-FI-lifting module.

Theorem 3.2 ***L**et M be a module, consider the following statements:*
(a) M is strongly $\mathbb{E}$-FI-lifting module.
(b) M is $\mathbb{E}$-FI-lifting module.

(c) M is FI-lifting module.

Then

(i) for every module M, (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

(ii) If the direct summand of M coincide with fully invariant submodule then (c)$\Rightarrow$ (a).

Proof *(i) (a) $\Rightarrow$ (b) is clearly. Let $\mathbb{E} = \mathbb{E}_{\mathbb{S}}$ then (b) $\Rightarrow$ (c).*

(ii) Since M is FI-lifting module, let N be a fully invariant submodule of M, then there exist a direct summand K of M such that $N/K \ll M/K$ by the hypothesis, K is the fully invariant submodule of M, and clearly K is $\mathbb{E}$-submodules of M, so M is strongly $\mathbb{E}$-FI-lifting module.

A module M is called *duo module* provide that every submodule of M is fully invariant submodule. So we have the following:

Corollary 3.3 *If M is a duo module, then the following statements are equivalent:*

(a) M is an strongly $\mathbb{E}$-FI-lifting module.

(b) M is an $\mathbb{E}$-FI-lifting module.

(c) M is FI-lifting module.

Proof *By Theorem 3.2 above, the proof is clear.*

In general an $\mathbb{E}$-FI-lifting module is not $\mathbb{E}$-lifting module, look at the following example:

Example 3.4 [12] ***L****et p be a prime number, consider Z-module $M = (Z/pZ)\bigoplus(Z/p^3Z)$. M is $\mathbb{E}_S$-FI-lifting module, but it is not $\mathbb{E}_{\mathbb{S}}$-lifting module by [11]. However, when M is a duo module, $\mathbb{E}$-FI-lifting module is an $\mathbb{E}$-lifting.*

Definition 3.5 *A module M is called strongly FI-lifting, if for every fully invariant submodules N of M, there is a fully invariant direct summand D of M such that $N/D \ll M/D$.*

If M is a duo module and has coclosure property, strongly $\mathbb{E}$-lifting module is equivalent to strongly $\mathbb{E}$-FI-lifting module.

We well known that if M has the coclosure property, so does every coclosed submodule, every fully invariant (coclosed) submodule and every homomorphic image of M. What about the $\mathbb{E}$-FI-lifting module?

Lemma 3.6 ***L****et A be a fully invariant coclosed submodule of an $\mathbb{E}$-FI-lifting B, then A is $\mathbb{E}$-FI-lifting. If B has coclosure property, then B/A is $\mathbb{E}$-FI-lifting.*

Proof *Let B be an $\mathbb{E}$-FI-lifting module, and A be a coclosed submodule of B, let C be a fully invariant submodule of A. Since B is $\mathbb{E}$-FI-lifting, C contains a cosmall submodule D of C in B such that D is an $\mathbb{E}$-submodule of B, Then D is an $\mathbb{E}$-submodule of A, since A is the coclosed submodule of B, D is a cosmall submodule of C in B, it follows that D is a cosmall submodule of C in A by [6, Lemma 3.9]. Let C/A a fully invariant coclosed submodule of B/A, then A is a fully invariant coclosed submodule of B, and A is a fully invariant $\mathbb{E}$-submodule of B, then C/A is an $\mathbb{E}$-submodule of B/A, so B/A is an $\mathbb{E}$-FI-lifting module.*

Lemma 3.7 ***L****et A be a coclosed fully invariant submodule of a strong $\mathbb{E}$-FI-lifting module B. If B has coclosure property, then A and B/A are strongly $\mathbb{E}$-FI-lifting modules. The proof is similarly to Lemma 3.6.*

Theorem 3.8 ***L****et $M = \bigoplus_{i\in I} M_i$. If $M_i (i \in I)$ are $\mathbb{E}$-FI-lifting modules, then M is $\mathbb{E}$-FI-lifting module.*

Proof *Let $M_i(i \in I)$ are $\mathbb{E}$-FI-lifting modules, $S \rhd M$. There exist $e_i^2 = e_i \epsilon E (= EndM)$ such that $M_i = Me_i$. Since $S \rhd M, S = \bigoplus_{i\in I}(S \bigcap M_i) = \bigoplus_{i\in I} Se_i$. Clearly, $End_R(Me_i) = e_iEe_i$. Because $(Se_i)e_iEe_i = Se_iEe_i \leqq Se_i$, so $Se_i \rhd Me_i = M_i$. Hence there exist $\mathbb{E}$-submodules D_i of M_i, such that $Se_i/D_i \ll M_i/D_i$, also, $Se_i/D_i \ll M/D_i$, $S/\bigoplus_{i\in I} D_i \ll M/\bigoplus_{i\in I} D_i$ by ([5], Lemma 2.5), D_i are $\mathbb{E}$-submodules of M, so does $\bigoplus_{i\in I} D_i$, hence M is $\mathbb{E}$-FI-lifting module.*

By [9, Theorem 3.4], any finite direct sum of FI-lifting modules is again FI-lifting. The following two examples show that this property is not true in general for infinite direct sums of FI-lifting modules. Let R be a discrete valuation ring with maximal ideal m. Let $M = \bigoplus_{i=1}^{\infty} R/m^i$ or $M = R^{\mathbb{N}}$. By [10, Corollary 2, P.48], $Rad(M)$ does not have a supplement in M. Since $Rad(M)$ is a fully invariant submodule of M, M is not FI-lifting. On the other hand, it is clear that $R/m^i (i \geq 1)$ and R are lifting modules.

Corollary 3.9 *Let $M = \bigoplus_{i\in I} M_i$. If $M_i(i \in I)$ are strongly $\mathbb{E}$-FI-lifting modules, then M is $\mathbb{E}$-FI-lifting modules. The proof is immediately by Theorem*

3.8 and Lemma 3.7 above.

A module M is said to be $\sum$P if every direct sum of copies of M has the property P. Now we characterize relative (strongly) $\mathbb{E}$-FI-lifting modules following from [4].

Theorem 3.10 *Let M be a module, consider the following statements:*
(a) M is $\sum$-$\mathbb{E}$-FI-lifting.
(b) Every module in $Add(M)$ is $\mathbb{E}$-FI-lifting.
(c) Every fully invariant $N \in Cogen'(M)$ has an M-small $\mathbb{E}$-homorphic image N/Y, such that Y is in $\mathbb{E}prod'(M)$.
(d) Every fully invariant non-M-cosingular module in Cogen'(M) is in $\mathbb{E}$Prod'(M).
(e) Every fully invariant non-M-cosingular module in Cogen'(M) is in $\mathbb{E}$-FI-lifting.
Then the following implications hold:
(1) For every module (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).
(2) If M is $\sum$-$\mathbb{E}$-direct injective and has the $\sum$-coclosure property, then (d) $\Rightarrow$ (e).
(3) If M is $\sum$-$\mathbb{E}$-direct injective and has the $\sum$-FI-coclosure property, and for every $U \subseteq N \subseteq F \in \mathbb{E}Prod'(M)$, K/U is M-small, implies U is a cosmall submodule of K in F, then (c) $\Rightarrow$ (a).
(4) If M is non-M-cosingular, then (e) $\Rightarrow$ (a).

Proof *(1)(a) $\Rightarrow$ (b), let M be an $\mathbb{E}$-FI-lifting module, by Theorem 2.8, $M^{(I)}$ is $\mathbb{E}$-FI-lifting module.*
(b)$\Rightarrow$ (a) is clear.
(b)$\Rightarrow$(c) Let $N \in$ cogen'(M) and N be a fully invariant submodule, take a monomorphism $f : N \longrightarrow M^{(I)}$. $N \rhd M$, it is easily to get $f(N) \rhd M^{(I)}$. Since $M^{(I)}$ is $\mathbb{E}$-FI-lifting module, there exist an fully invariant submodule $\mathbb{E}$-submodule Y' of $M^{(I)}$ such that $Y' \subseteq f(N)$, and $f(N)/Y' \ll M^{(I)}/Y'$. Since $f(N)/Y' \ll M^I/Y'$, if $Y = f^{-1}(Y')$, then it follows that N/Y is M-small, $Y \in \mathbb{E}Prod'(M)$, also $N \in$ cogen'(M), Y is an $\mathbb{E}$-submodule of N.
(c) $\Rightarrow$ (d) Clear.
(2) Assume that M is $\sum$-$\mathbb{E}$-direct injective and has the $\sum$-coclosure property.
(d) $\Rightarrow$ (e) Let N be a fully invariant and non-M-cosingular module in cogen'(M), we say N is $\mathbb{E}$-FI-lifting. Consider a monomorphism f: $N \longrightarrow M^{(I)}$. Let L be a fully invariant submodule of a non-M-singular module of N. Then f(L) has coclosure in $M^{(I)}$, say K. Hence K is a coclosed in $f(N)$, and thus K is a fully invariant submodule of a non-M-cosingular module. Since $K \in Cogen'(M)$, we have $K \in \mathbb{E}Prod'(M)$ by hypothesis. Now by Lemma 1.8 the inclusion $K \longrightarrow f(N)$ is an $\mathbb{E}$-monomorphism. Then the inclusion $f^{-1}(K) \longrightarrow N$ is an $\mathbb{E}$-monomorphism. Since $f^{-1}(K)$ is coclosed in N, it follows that N is an $\mathbb{E}$-FI-lifting.

(3) If M is $\sum$-$\mathbb{E}$-direct injective and has the $\sum$-FI-coclosure property, and for every $U \subseteq N \subseteq F \in \mathbb{E}Prod'(M)$, K/U is M-small, implies U is a cosmall submodule of K in F, then (c) $\Rightarrow$ (a).

Let N be a fully invariant submodule of $F = M^{(I)}$, where I is any set. By (C) the module $N \in cogen'(M)$ has an M-small homorphic image N/Y such that $Y \in \mathbb{E}Prod'(M)$. Then $Y \subseteq N \subseteq F \in \mathbb{E}Prod'(M)$ and N/Y is M-small, hence $N/Y \ll F/Y$ by hypothesis, by Lemma 1.8 the inclusion $N \to F$ is an $\mathbb{E}$-monomorphism. $M^{(I)}$ has FI-coclosure property, so F is $\mathbb{E}$-FI-lifting.
(4) (e) $\Rightarrow$ (a) If M is non-M-cosingular, then every $M^{(I)}$ is non-M-cosingular, hence $\mathbb{E}$-FI-lifting.

Theorem 3.11 ***L**et M be a module, consider the following statements:*
(a) M is $\sum$-strongly-$\mathbb{E}$-FI-lifting.
(b) Every module in $Add(M)$ is strongly $\mathbb{E}$-FI-lifting.
(c) Every module in $\mathbb{E}Prod'(M)$ is strongly $\mathbb{E}$-FI-lifting.
(d) Every non-M-Cogsingular module in Cogen'(M)is strongly $\mathbb{E}$-FI-lifting.
(e) A module is in $\mathbb{E}Prod'(M)$ if and only if it is in Cogen'(M) and non-M-cosingular.

Then the following implications holds:
(1) For every module (a)$\Longleftrightarrow$ (b)$\Longleftrightarrow$(c).
(2) If M is also non-M-cosingular module, then (a)$\Rightarrow$(e) and (d) $\Rightarrow$(a).
(3) If M is $\sum$-$\mathbb{E}$-direct injective $\sum$-$\mathbb{E}$-lifting, then (e)$\Rightarrow$(c) and (e) $\Rightarrow$ (d).

Proof *(1) (a)$\Rightarrow$(c) Let $K \in Prod'(M)$. Then there is an $\mathbb{E}$-monomorphism $K \to M^{(I)}$, by Lemma 2.6 K is strongly $\mathbb{E}$-FI-lifting;*
(c) $\Rightarrow$ (b) $\Rightarrow$(a) are clear.

(2) Assume M is non-M-singular.
(a)$\Rightarrow$ (e) by Theorem 2.9, every fully invariant non-M-singular module in $Cogen'(M)$ is in $\mathbb{E}Prod'(M)$. Conversely, let $K \in \mathbb{E}Prod'(M)$ and take some $\mathbb{E}$-monomorphism $K \to M^{(I)}$. Then K is an $\mathbb{E}$-submodule of $M^{(I)}$, have coclosed in $M^{(I)}$. Now since $M^{(I)}$ is non-M-singular, so is K.
(d) $\Rightarrow$ (a) If M is non-M-singular, then every $M^{(I)}$ is non-M-singular, hence M is strongly $\mathbb{E}$-FI-lifting.

(3) Assume M is $\sum$-$\mathbb{E}$-direct injective $\sum$-$\mathbb{E}$-FI-lifting.
(e)$\Rightarrow$ (c), (d) By hypothesis and Theorem 2.9, every module in $\mathbb{E}Prod'(M)$ is $\mathbb{E}$-FI-lifting. Now let $K \in \mathbb{E}Prod'(M)$and L be an $\mathbb{E}$-submodule of K, by an easy variation of Lemm1.9, $L \in \mathbb{E}Prod'(M)$, hence L is non-M-singular, so that L is coclosed in K. Thus K is strongly $\mathbb{E}$-FI-lifting.

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