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The Euler-Maclaurin Formula and Sums of Powers Revisited

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Abstract

Using the Euler-Maclaurin summation formula the strictly increasing convergence

$$\lim_{m \to \infty} \sum_{j=1}^{m} \left(\frac{j}{m}\right)^m = \frac{e}{e-1}$$

is demonstrated.

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1 Introduction

Recently, in [7, pp. 63–64] the author tried to demonstrate the equality

$$\lim_{m\to\infty} \sum_{k=1}^{m-1} \left(\frac{k}{m}\right)^m = \frac{1}{e-1}\,. \tag{1}$$

In his demonstration he supposed that, for integers m and $3 \le k \le m+1$, the product $m^{1-k} \cdot m(m-1)\cdots(m-k+2) \equiv (1-\frac{1}{m})(1-\frac{2}{m})\cdots(1-\frac{k-2}{m})$ is of the form $1 + O(\frac{1}{m})$ as $m \to \infty$. Unfortunately, this is not true.

At the end of the year 2008 we offered to Mathematics Magazine the correction of this fault. However, the Editor rejected the publication of the proof as regular paper. He suggested the publication only in a form of the Letter to the Editor. Recently, Spivey [8] published in Mathematics Magazine the correction of his article. In the same issue, as Spivey's Letter, another contribution considering the limit in question presenting two alternative ways how to confirm (1), was also published, Holland [2]. Here we offer our original and correct demonstration following the steps presented in [7]. First we deduce simple estimate of the product in question. Then, using the Euler-Maclaurin formula, we prove (1).

2 Estimating the product $\frac{1}{m^k}\prod_{i=0}^{k-1}(m-i)$

For integers m and k such that $1 \le k \le m$ let us define

$$Q(m,k) := m^{-k}\prod_{i=0}^{k-1}(m-i) \equiv \prod_{i=1}^{k-1}\left(1-\frac{i}{m}\right) \tag{2}$$

and

$$\Delta(m,k) := 1 - Q(m,k). \tag{3}$$

Obviously $0 < \Delta(m,k) < 1$ for $2 \le k \le m$. Furthermore, $\Delta(m,1) \equiv 0$, $\Delta(m,2) \equiv 1/m$, $\Delta(m,3) \equiv 3/m - 2/m^2 < 3/m$ and $\Delta(m,4) \equiv 6/m - 11/m^2 + 6/m^3 < 12/m$. Thus, in these cases $Q(m,k) = O(\frac{1}{m})$. Similar results we obtain for k from any bounded set of positive integers. But, for those k which are not far from m we find that $Q(m,k) \approx 0$. Indeed, according to Stirling's factorial formula [1, 6.1.38; p. 257] we have

$$\begin{aligned} Q(m,m) &\equiv m^{-m}\, m! = m^{-m} \cdot \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \exp\left(\frac{\Theta}{12m}\right) \\ &= \frac{\sqrt{2\pi m}}{e^m}\exp\left(\frac{\Theta}{12m}\right) \end{aligned}$$

for some $\Theta = \Theta(m) \in (0,1)$. Consequently, using the identity

$$Q(m,m-j) \equiv \frac{Q(m,m)}{\prod\limits_{i=m-j}^{m-1}(1-\frac{i}{m})},$$

we get

$$\lim_{m\to\infty} Q(m,m-j) = 0$$

for any j from any bounded set of positive integers. These contrasts are illustrated in Figure 1 where are plotted the graphs of the sequences $k \mapsto Q(m,k)$ for $m = 30$ and $m = 100$.

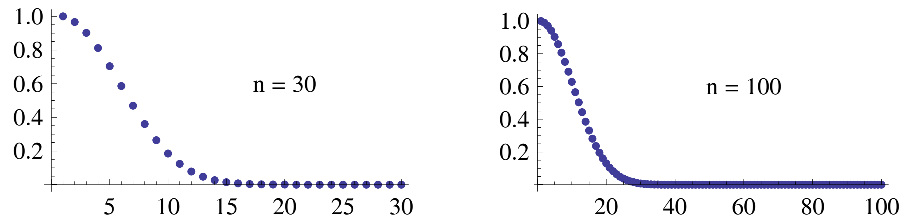

Figure 1: The graphs of the sequences $k \mapsto Q(m,k)$.

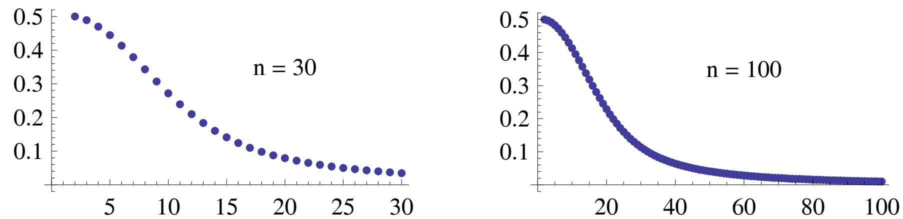

Figure 2: The graphs of the sequences $k \mapsto m\,\frac{\Delta(m,k)}{k(k-1)}$.

Furthermore, the graphs of the sequences $k \mapsto m\,\frac{\Delta(m,k)}{k(k-1)}$ with $m = 30$ and $m = 100$, depicted in Figure 2, suggest the estimate

$$0 < \Delta(m,k) < \frac{(k-1)k}{m} \quad \text{for } 2 \leq k \leq m. \tag{4}$$

Indeed, this relation has been already verified for $k \in \{2,3,4\}$. Moreover, if (4) is true for some $k < m$ it is also true for $k+1$:

$$\begin{aligned} \Delta(m,k+1) &\equiv 1 - Q(m,k)\cdot\left(1-\frac{k}{m}\right) \\ &\equiv \Delta(m,k) + Q(m,k)\cdot\frac{k}{m} \\ &\leq \Delta(m,k) + \frac{k}{m} \\ &< \frac{(k-1)k}{m} + \frac{k}{m} < \frac{k(k+1)}{m}\,. \end{aligned}$$

Remark. Using the recurrence and Stirling's formula for the Gamma function [1, 6.1.15, 6.1.38; pp. 256–257], we have, for $1 \leq k \leq m-1$,

$$\begin{aligned} Q(m,k) &= m^{-k}\cdot\frac{\Gamma(m+1)}{\Gamma(m-k+1)} \\ &= m^{-k}\cdot\sqrt{2\pi m}\left(\frac{m}{e}\right)^m \exp\left(\frac{\vartheta_1}{12m}\right) \\ &\quad\cdot\left[\sqrt{2\pi(m-k)}\left(\frac{m-k}{e}\right)^{m-k}\exp\left(\frac{\vartheta_2}{12(m-k)}\right)\right]^{-1} \end{aligned}$$

$$= \left(1 - \frac{k}{m}\right)^{-\frac{1}{2}} \left[e^{-k} \left(\frac{m}{m-k} \right)^{m-k} \right] \exp\left(\frac{\vartheta_1}{12m} - \frac{\vartheta_2}{12(m-k)} \right) \quad (5)$$

for some ϑ_1, $\vartheta_2 \in (0,1)$. Now, setting $h = -k/m$, $t = m-k > 0$ and $\varepsilon = k/m$ into the relation

$$\exp\left(ht - \frac{h^2 t}{2(1-\varepsilon)} \right) < (1+h)^t < \exp\left(ht - \frac{h^2 t}{2(1+\varepsilon)} \right),$$

presented in [5, (6)] and applied for $\left(\frac{m}{m-k}\right)^{m-k} = \left(1 - \frac{k}{m}\right)^{-(m-k)}$, we obtain from (5) the estimate

$$\exp\left(\frac{k^2}{2m} - k \right) < \left(1 - \frac{k}{m}\right)^{m-k} < \exp\left(\frac{k(3k^2 - km - 2m^2)}{2m(k+m)} \right).$$

Consequently, after taking the reciprocal values and multiplying the last relation by $\exp(-k)$, we find the estimate

$$Q_1(m,k) < Q(m,k) < Q_2(m,k), \quad (6)$$

for $1 \le k \le m-1$, where

$$Q_1(m,k) \equiv \left(1 - \frac{k}{m}\right)^{-\frac{1}{2}} \exp\left(-\frac{k^2}{2m}\left(1 + \frac{2k}{k+m}\right) - \frac{1}{12(m-k)} \right)$$

$$Q_2(m,k) \equiv \left(1 - \frac{k}{m}\right)^{-\frac{1}{2}} \exp\left(-\frac{k^2}{2m} + \frac{1}{12m} \right).$$

The estimate (6) is illustrated in Figure 3 showing the graphs of continuous approximates $k \mapsto Q_1(50,k)$ and $k \mapsto Q_2(50,k)$ indicated by continuous lines and bounding the graph of the sequence $k \mapsto Q(50,k)$.

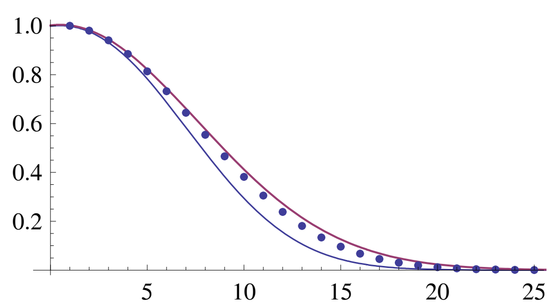

Figure 3: The graphs of continuous approximates $k \mapsto Q_1(50,k)$ and $k \mapsto Q_2(50,k)$ bounding the graph of the sequence $k \mapsto Q(50,k)$.

We remark that $Q(m,k) = m^{-k} \cdot m_{(k)}$, where $m_{(k)}$ is lower Pochhammer factorial estimated more accurately in [6].

3 The Euler-Maclaurin summation

For integers n_1 and n_2 satisfying $1 \le n_1 < n_2$, for positive integer p and for $f \in C^p[1,\infty)$ we have the Euler-Maclaurin summation formula[1] [4, p. 117]

$$\sum_{k=n_1}^{n_2-1} f(k) = \int_{n_1}^{n_2} f(x)\,dx + \sum_{j=1}^{p} \frac{B_j}{j!}\left[f^{(j-1)}(n_2) - f^{(j-1)}(n_1)\right] - \frac{1}{p\,!}\int_{n_1}^{n_2} P_p(-x)\,f^{(p)}(x)\,dx. \tag{7}$$

Let us use this equation for polynomial f, $f(x) \equiv x^m$, m being positive integer. Referring to (2), we have $f^{(k)}(x) \equiv m^k\,Q(m,k)\,x^{m-k}$ for $1 \le k \le m$, and $f^{(m+1)}(x) \equiv 0$. Therefore, for positive integer $m \ge 2$, setting $n_1 = 0$, $n_2 = m$ and $p = m+1$ into the equation (7), we obtain

$$\sum_{k=0}^{m-1} k^m = \frac{m^{m+1}}{m+1} + B_1\,m^m + \sum_{j=2}^{m} \frac{B_j}{j!}\,m^{j-1}\,Q(m,j-1)\,m^{m-j+1}.$$

Hence, referring to (3) and considering that $B_0 = 1$, we have

$$\sum_{k=1}^{m-1}\left(\frac{k}{m}\right)^m = \frac{m}{m+1} - 1 + \sum_{j=0}^{m}\frac{B_j}{j!} - \sum_{j=2}^{m}\frac{B_j}{j!}\,\Delta(m,j-1)\,. \tag{8}$$

The Bernoulli numbers $B_k := B_k(0)$ and polynomials $B_k(x)$ may be defined using the well known technique of generating function [1, 23.1.1; p. 804]

$$\frac{te^{xt}}{e^t-1} = \sum_{j=0}^{\infty} B_j(x)\frac{t^j}{j!} \tag{9}$$

for every x and for $|t| < 2\pi$. The power series on the right is absolutely convergent inside the disk of convergence where it can be differentiated termwise any-times. Thus, setting $x = 0$ and $t = 1$, the series

$$\sum_{j=0}^{\infty}\frac{B_j}{j!} = \frac{1}{e-1} \quad \text{and} \quad \sum_{j=2}^{\infty}\frac{B_j}{(j-2)!} = \left[\frac{d^2}{dt^2}\left(\frac{te^{xt}}{e^t-1}\right)\right]_{x=0,t=1} \tag{10}$$

are both absolutely convergent; the latter is obtained by differentiating (9) two times termwisely. Therefore, according to (4),

$$\left|\sum_{j=2}^{m}\frac{B_j}{j!}\,\Delta(m,j-1)\right| \le \sum_{j=2}^{m}\frac{|B_j|}{j!}\cdot\frac{j(j-1)}{m} \le \frac{1}{m}\sum_{j=2}^{\infty}\frac{|B_j|}{(j-2)!}\,.$$

[1] It is not possible to set $p = \infty$ in (7) since the series $\sum_{j=1}^{\infty}\frac{B_j}{j!}\left[f^{(j-1)}(x)\right]_m^n$ turns out to diverge for almost all functions $f(x)$ that occur in applications, regardless of m and n [3, p. 525].

Hence, considering (8) and (10), the equation (1) is verified. Convergence (1) is illustrated in Figure 4 where is depicted the sequence $m \mapsto S_m := \sum_{k=1}^{m-1} \left(\frac{k}{m}\right)^m$.

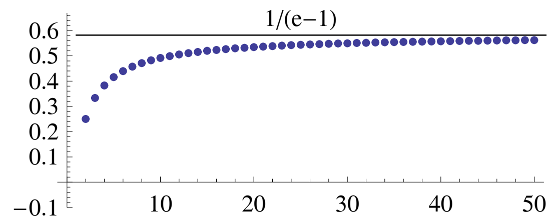

Figure 4: The graph of the sequence $m \mapsto \sum_{k=1}^{m-1} \left(\frac{k}{m}\right)^m$.

To prove that the sequence $(S_m)_{m\in\mathbb{N}}$ is strictly increasing we change the order of summation

$$S_m \equiv \sum_{k=1}^{m-1} \left(\frac{k}{m}\right)^m \equiv \sum_{j=1}^{m-1} \left(\frac{m-j}{m}\right)^m \equiv \sum_{j=1}^{m-1} \left(1 + \frac{-j}{m}\right)^m .$$

Since the function $x \mapsto (1 + j/x)^x$ is strictly increasing [5] for $j \neq 0$, the sequence $(S_m)_{m\in\mathbb{N}}$ is also strictly increasing.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, 9th edn, Dover Publications, New York, (1974).

[2] F. Holland, $\lim_{m\to\infty} \sum_{k=0}^{m} (k/m)^m = e/(e-1)$, Math. Mag. 83(2010), pp. 51–54.

[3] K. Knopp, *Theory and Applications of Infinite Series*, New York, Hafner, 1971.

[4] V. Lampret, *The Euler-Maclaurin and Taylor Formulas:Twin, Elementary Derivations*, Math. Mag. 74(2001), pp. 109–122.

[5] V. Lampret, *Estimating Powers with Base Close to Unity and Large Exponents*, Divulg. Mat. 13 (2005), pp. 21–34.

[6] V. Lampret, *Approximating real Pochhammer products: a comparison with powers*, Cent. Eur. J. Math. 7(2009), pp. 493–505.

[7] M.Z. Spivey, *The Euler-Maclaurin Formula and Sums of Powers*, Math. Mag. 79(2006), pp. 61–65.

[8] M.Z. Spivey, *Letter to the Editor*, Math. Mag. 83(2010), pp. 54–55.

[9] S. Wolfram, *Mathematica*, version 6.0, Wolfram Research, Inc., 1988–2008.

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