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Totally Real Warped Product Submanifolds in Generalized Complex Space Forms

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Abstract

In this article, we establish an inequality between the warping function and the squared mean curvature for totally real warped product submanifolds in generalized complex space forms. Some applications are also discussed.

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1. Introduction. The notion of warped product plays some important roles in differential geometry as well as in Physics [**7**]. In [**5**] B.Y. Chen studied warped CR-submanifolds in Kaehler manifolds. Afterward in [**8**] he obtained an inequality for warped products in complex hyperbolic spaces. Several geometers, studied warped product submanifolds in contact metric manifolds.

A. Mihai [**14**] studied warped submanifolds in complex space forms and established an inequality between the warping function f and the squared mean curvature $\|H\|^2$ and the holomorphic sectional curvature c for warped product submanifolds $M_1 \times_f M_2$ in a complex space form $\tilde{M}(c)$. He also studied [**15**] warped product submanifolds in generalized complex space forms. In the present paper, we establish a similar inequality for totally real warped product submanifolds in a generalized complex space form.

2. Preliminaries. Let $\tilde{M}$ be an almost Hermitian manifold with an almost complex structure (J, g). If J is integrable, i.e., the Nijenhuis tensor $[J, J]$ of J vanishes then $\tilde{M}$ is called a Hermitian manifold. The fundamental 2-form Ω of $\tilde{M}$ is defined by

(2.1) $$\Omega(X,Y) = g(X,JY),$$

for all $X, Y \in T\tilde{M}$.

An almost Hermitian manifold $\tilde{M}$ is called an almost Kaehler manifold if the fundamental 2-form Ω is closed and it is Kaehler manifold if $\tilde{\nabla}J = 0$.

An almost Hermitian manifold with J-invariant Riemannian curvature tensor $\tilde{R}$, i.e.,

(2.2) $$\tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W),$$

for any vector fields $X, Y, Z, W \in T\tilde{M}$

is called an RK-manifold[**20**]. An almost Hermitian manifold $\tilde{M}$ is said to have (pointwise) constant type if for each $p \in \tilde{M}$ and for all vector fields $X, Y, Z, W \in T\tilde{M}$ such that

(2.3) $$g(X,Y) = g(JX,Y) = g(X,Z) = g(JX,Z) = 0,$$

$$g(Y,Y) = g(Z,Z) = 1,$$

we have

(2.4) $$\tilde{R}(X,Y,X,Y) - \tilde{R}(X,Y,JX,JY) = \tilde{R}(X,Z,X,Z) - \tilde{R}(X,Z,JX,JZ).$$

An RK-manifold $\tilde{M}$ has (pointwise) constant type if and only if there is a differentiable function α on $\tilde{M}$ such that

(2.5) $$\tilde{R}(X,Y,X,Y) - \tilde{R}(X,Y,JX,JY) = \alpha[g(X,X)g(Y,Y) - g^2(X,Y) - g^2(X,JY)],$$

for all vector fields $X, Y \in T\tilde{M}$.

Furthermore, $\tilde{M}$ has global constant type if α is constant. The function α is called the constant type of $\tilde{M}$.

An RK-manifold of constant holomorphic sectional curvature c and constant type α is called a generalized complex space form $\tilde{M}(c,\alpha)$. The Ricci curvature tensor of generalized complex space $\tilde{M}(c,\alpha)$ has the following expression [**20**]:

(2.6)
$$\begin{aligned}\tilde{R}(X,Y,Z,W) = &\tfrac{c+3\alpha}{4}[g(X,Z)g(Y,W)-g(X,W)g(Y,Z)]\\ &+\tfrac{c-\alpha}{4}[g(JX,Z)g(JY,W)-g(JX,W)g(JY,Z)\\ &+2g(X,JY)g(Z,JW)],\end{aligned}$$

for any vector fields $X,Y,Z,W \in T\tilde{M}$.

If $c=\alpha$, then $\tilde{M}(c,\alpha)$ is a space of constant curvature. A complex space form $\tilde{M}(c)$ (a Kaehler manifold of constant holomorphic sectional curvature c) belongs to the class of almost Hermitian manifold $\tilde{M}(c,\alpha)$ (with constant type zero).

Let M be an n-dimensional submanifold in a generalized complex space form $\tilde{M}(c,\alpha)$ of dimension 2m. We denote $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$, and ∇ be a Riemannian connection of M, respectively. Also, let h be the second fundamental form and R the Riemannian curvature tensor of M.

Then the equation of Gauss is given by

(2.7)
$$\begin{aligned}\tilde{R}(X,Y,Z,W) = &R(X,Y,Z,W)+g(h(X,W),h(Y,Z))\\ &-g(h(X,Z),h(Y,W)),\end{aligned}$$

for any vector fields X,Y,Z,W tangent to M.

Let $p \in M$ and $\{e_1,.....,e_n,......,e_{2m}\}$ be an orthonormal basis of the tangent space $T_p\tilde{M}$, such that $\{e_1,.....,e_n\}$ are tangent to M.

We denote by $H(p)$ the mean curvature vector at $p \in M$, i.e.,

(2.8)
$$H(p) = \tfrac{1}{n}\sum_{i=1}^{n} h(e_i,e_i).$$

Also, we set

(2.9)
$$h_{ij}^{r} = g(h(e_i,e_j),e_r),\ i,j \in \{1,.....,n\},\ r \in \{n+1,.....,2m\}$$

and

(2.10)
$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)).$$

For any vector X tangent to M, we put $JX = PX + FX$, where PX and FX are the tangential and normal parts of JX, respectively.

We denote by

$$\textbf{(2.11)} \qquad \|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j).$$

Let M be a Riemannian n-manifold and $\{e_1, ..., e_n\}$ be an orthonormal frame field on M. For differentiable function f on M, the Laplacian Δf of f is defined by,

$$\textbf{(2.12)} \qquad \Delta f = \sum_{j=1}^{n} \{(\Delta_{e_j} e_j) f - e_j e_j f\}.$$

Now, we recall the following result:-

Lemma 2.1. Let $n \geq 2$ and $a_1, ..., a_n, b$ be (n+1) real numbers such that

$$\textbf{(2.13)} \qquad (\sum_{i=1}^{n} a_i)^2 = (n-1)(\sum_{i=1}^{n} a_i^2 + b)$$

Then $2a_1a_2 \geq b$ with equality holding if and only if

$$a_1 + a_2 = a_3 = ... = a_n.$$

3. Warped Product Submanifolds

Chen established a sharp relationship between the warping function function f of warped product $M_1 \times_f M_2$ isometrically immersed in a real space form $\tilde{M}(c)$ and the squared mean curvature $\|H\|^2$[**6**]. A. Mihai obtained a similar inequality for warped product submanifolds of a complex space form.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1. The warped product of M_1 and M_2 is the Riemannian manifold

$$M_1 \times_f M_2 = (M_1 \times M_2, g),$$

where $g = g_1 + f^2 g_2$ [**6**].

Let $x : M_1 \times_f M_2 \to \tilde{M}(c, \alpha)$ be an isometric immersion of a warped product $M_1 \times_f M_2$ into a generalized complex space form $\tilde{M}(c, \alpha)$. We denote by h the second fundamental form of x and $H_i = \frac{1}{n_i}$trace h_i where trace h_i is the trace of h restricted to M_i and n_i =dim M_i, $(i = 1, 2)$.

For a warped product $M_1 \times_f M_2$, we denote by D_1 and D_2 the distributions given by the vectors tangent to leaves and fibers, respectively. Thus, D_1 is obtained from the tangent vectors of M_1 via the horizontal lift and D_2 by the tangent vectors of M_2 via the vertical lift.

In this section, we study warped product submanifolds with $JD_1 \perp D_2$ in a generalized complex space form.

Theorem 3.1. *Let $x : M_1 \times_f M_2 \to \tilde{M}(c, \alpha)$ be an isometric immersion of an n-dimensional totally real warped product with $JD_1 \perp D_2$ into a 2m-dimensional generalized complex space form $\tilde{M}(c, \alpha)$ of constant holomorphic sectional curvature c and of constant type α. Then*

$$\textbf{(3.1)} \qquad \frac{\Delta f}{f} \le \frac{n^2}{4n_2} \| H \|^2 + n_1 \frac{c + 3\alpha}{4}$$

where n_i =dim M_i, $i = 1, 2$ and Δ is the Laplacian operator of M_i.

Moreover, the equally case of **(3.1)** *holds identically if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where H_i, $i = 1, 2$ are the partial mean curvature vectors.*

Proof. Let $M_1 \times_f M_2$ be a warped product submanifold of a generalized complex space form $\tilde{M}(c, \alpha)$ of constant holomorphic sectional curvature c and of constant type α.

We recall the following general formula on a warped product [**6**]

$$\textbf{(3.2)} \quad \Delta_X Z = \Delta_Z X = \frac{1}{f}(Xf)Z,$$

for any vector fields X, Z tangent to M_1 and M_2 respectively.

If X and Z are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$\textbf{(3.3)} \qquad K(X \wedge Z) = g(\Delta_Z \Delta_X X - \Delta_X \Delta_Z X,\ Z) = \frac{1}{f}\{(\Delta_X X)f - X^2 f\}.$$

We choose a local orthonormal frame $\{e_1, ..., e_n, ..., e_{2m}\}$, such that $e_1, ..., e_{n_1}$ are tangent to M_1 and $e_{n_1+1}, ..., e_n$ are tangent to M_2, e_{n+1} is parallel to the mean curvature vector H.

Then using **(3.3)**, we obtain

(3.4) $$\frac{\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s), \text{ for each } s \in \{n_1+1, ...,n\}.$$

On the other hand, the Gauss equation implies

(3.5) $$n^2 \| H \|^2 = 2\tau + \| h \|^2 - n(n-1)\frac{c+3\alpha}{4}.$$

If we put

(3.6) $$\epsilon = 2\tau - \frac{n^2}{2} \| H \|^2 - n(n-1)\frac{c+3\alpha}{4}.$$

Then, **(3.6)** can be written as,

(3.7) $$n^2 \| H \|^2 = 2(\epsilon + \| h \|^2).$$

With respect to orthogonal frame, **(3.7)** takes the following form:

(3.8) $$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = 2\{\sum_{r=n+1}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \epsilon\}.$$

Or equivalently,

(3.9) $$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = 2\{\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \epsilon\}.$$

If we put $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1}$ and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$, the above equation becomes

(3.10) $$(\sum_{i=1}^{3} a_i)^2 = 2\{\sum_{i=1}^{3} a_i^2 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - \sum_{2\leq j\neq k\leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1\leq s\neq t\leq n} h_{ss}^{n+1} h_{tt}^{n+1} + \epsilon\}.$$

Thus, $a_1,\ a_2,\ a_3$ satisfy the Lemma 2.1, (for $n = 3$), i.e.,

$$(\sum_{i=1}^{3} a_i)^2 = 2(b + \sum_{i=1}^{3} a_i^2),$$

with $$b = \epsilon + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - \sum_{2\leq j\neq k\leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1\leq s\neq t\leq n} h_{ss}^{n+1} h_{tt}^{n+1}.$$

Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3$.

In the case under consideration, this means

(3.11)
$$\sum_{1\leq j<k\leq n_1} h^{n+1}_{jj}h^{n+1}_{kk} + \sum_{n_1+1\leq s<t\leq n} h^{n+1}_{ss}h^{n+1}_{tt} \geq \tfrac{\epsilon}{2} + \sum_{1\leq\alpha<\beta\leq n}(h^{n+1}_{\alpha\beta})^2 + \tfrac{1}{2}\sum_{r=n+2}^{2m}\sum_{\alpha,\beta}^{n}(h^r_{\alpha\beta})^2.$$

Equality holds if and only if,

(3.12)
$$\sum_{i=1}^{n_1} h^{n+1}_{ii} = \sum_{t=n_1+1}^{n} h^{n+1}_{tt}.$$

Using again the Gauss equation, we have

(3.13)
$$\begin{aligned} n_2\tfrac{\Delta f}{f} &= \tau - \sum_{1\leq j<k\leq n_1} K(e_j\wedge e_k) - \sum_{n_1+1\leq s<t\leq n} K(e_s\wedge e_t) \\ &= \tau - \tfrac{n_1(n_1-1)(c+3\alpha)}{8} - \sum_{r=n+1}^{2m}\sum_{1\leq j<k\leq n_1}(h^r_{jj}h^r_{kk} - (h^r_{jk})^2) \\ &\quad - \tfrac{n_2(n_2-1)(c+3\alpha)}{8} - \sum_{r=n+1}^{2m}\sum_{n_1+1\leq s<t\leq n}(h^r_{ss}h^r_{tt} - (h^r_{st})^2) \end{aligned}$$

Combining **(3.11)** and **(3.13)**, and using **(3.4)** we obtain

(3.14)
$$\begin{aligned} n_2\tfrac{\Delta f}{f} &\leq \tau - \tfrac{n(n-1)(c+3\alpha)}{8} + n_1n_2\tfrac{c+3\alpha}{4} - \tfrac{\epsilon}{2} \\ &\quad - \sum_{1\leq j\leq n_1;n_1+1\leq t\leq n}(h^{n+1}_{jt})^2 - \tfrac{1}{2}\sum_{r=n+2}^{2m}\sum_{\alpha,\beta=1}^{n}(h^r_{\alpha\beta})^2 \\ &\quad + \sum_{r=n+2}^{2m}\sum_{1\leq j<k\leq n_1}((h^r_{jk})^2 - h^r_{jj}h^r_{kk}) \\ &\quad + \sum_{r=n+2}^{2m}\sum_{n_1+1\leq s<t\leq n}((h^r_{st})^2 - h^r_{ss}h^r_{tt}) \\ &= \tau - \tfrac{n(n-1)(c+3\alpha)}{8} + n_1n_2\tfrac{c+3\alpha}{4} - \tfrac{\epsilon}{2} - \sum_{r=n+1}^{2m}\sum_{1\leq j\leq n_1;n_1+1\leq t\leq n}(h^r_{jt})^2 \\ &\quad - \tfrac{1}{2}\sum_{r=n+2}^{2m}(\sum_{j=1}^{n_1}h^r_{jj})^2 - \tfrac{1}{2}\sum_{r=n+2}^{2m}(\sum_{t=n_1+1}^{n}h^r_{tt})^2 \\ &\leq \tau - \tfrac{n(n-1)(c+3\alpha)}{8} + n_1n_2\tfrac{c+3\alpha}{4} - \tfrac{\epsilon}{2}. \end{aligned}$$

Since we assume that $JD_1\perp D_2$, **(3.14)** implies

(3.15)
$$n_2\tfrac{\Delta f}{f} \leq \tau - \tfrac{n(n-1)(c+3\alpha)}{8} + n_1n_2\tfrac{c+3\alpha}{4} - \tfrac{\epsilon}{2}$$

$$= \frac{n^2}{4} \| H \|^2 + n_1 n_2 \frac{c+3\alpha}{4}$$

which implies **(3.1)**.

The equality case of **(3.14)** holds if and only if

(3.16) $$h^r_{jt} = 0,\ 1 \leq j \leq n_1,\ n_1 + 1 \leq t \leq n,\ n + 1 \leq r \leq 2m,$$

and

(3.17) $$\sum_{i=1}^{n_1} h^r_{ii} = \sum_{t=n_1+1}^{n} h^r_{tt} = 0,\ n + 2 \leq r \leq 2m.$$

Obviously **(3.16)** is equivalent to the mixed totally geodesicness of the warped product $M_1 \times_f M_2$ and **(3.12)** and **(3.17)** implies $n_1 H_1 = n_2 H_2$.

As applications, we derive certain obstructions to the existence of minimal warped product submanifolds in generalized complex space forms.

Corollary 3.2. Let $M_1 \times_f M_2$ be a warped product whose warping function f is Harmonic. Then,

(i) $M_1 \times_f M_2$ admits no minimal immersion with $JD_1 \perp D_2$ into a generalized complex space form $\tilde{M}(c, \alpha)$ with $c < -3\alpha$.

(ii) Every minimal immersion with $JD_1 \perp D_2$ of $M_1 \times_f M_2$ in the standard Euclidean space R^{2m} is a warped product immersion.

Proof. We assume that f is a Harmonic function on M_1 and $M_1 \times_f M_2$ admits a minimal immersion with $JD_1 \perp D_2$ in a generalized complex space form $\tilde{M}(c, \alpha)$. Then, the inequality **(3.1)** becomes $c \geq -3\alpha$. If $c = -3\alpha$, the equality case of **(3.1)** holds. By theorem **(3.1)**, it follows that $M_1 \times_f M_2$ is mixed totally geodesic and $H_1 = H_2 = 0$.

A well known result of Nolker [**17**] implies that the immersion is a warped product immersion.

Corollary 3.3. If the warping function f of a warped product $M_1 \times_f M_2$ is an eigen function of the Laplacian on M_1 with corresponding eigenvalue $\lambda > 0$, then $M_1 \times_f M_2$ does not admit a minimal immersion with $JD_1 \perp D_2$ into a generalized complex space form $\tilde{M}(c, \alpha)$ with $c \leq -3\alpha$.

Proof. If f is an eigen function of the Laplcian on M_1 with eigenvalue $\lambda > 0$, then inequality **(3.1)** implies that $\frac{n_1(c+3\alpha)}{4} \geq \lambda > 0$.

References

[**1**] K. Arslan, R. Ezentas, I. Mihai and G. Murathan, Contact CR-warped product submanifolds in Kenmotsu space forms, J. Korean Math. Soc., 42(5)(2005), 1101-1110.

[**2**] R.L. Bishop and B. O'Niell, Manifolds of negative curvature, Trans. Amer. Math. Soc., 145(1969), 1-49.

[**3**] B.Y. Chen, Some pinching and classification theorems for minimal submanifolds, Arch. Math., 60(1993), 568-578.

[**4**] B.Y. Chen, Geometry of slant submanifolds, K.U.Leuven, 1990.

[**5**] B.Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds, Monatash. Math., 133(2001), 177-195; 134(2001), 103-119.

[**6**] B.Y. Chen, On isomertic minimal immersions from warped products into real space forms, Proc. Edinburgh Math. Soc., 45(2002), 579-587.

[**7**] B.Y. Chen, Geometry of warped products as Riemannian submanifolds and related problems, Soochow J. Math., 28(2002), 125-156.

[**8**] B.Y. Chen, Non immersions theorems for warped products in complex hyperbolic spaces, Proc. Japan Acad., 78(2002), 96-100.

[**9**] B.Y. Chen, A general optimal inequality for warped products in complex projective spaces and its applications, Proc. Japan Acad. Ser. A Math. Sci., 79(2003), 89-94.

[**10**] A. Gray, nearly-Kaehler manifold, J. Differential Geometry, 4(1970), 283-309.

[**11**] I. Hasegawa and I. Mihai, Contact CR-warped product submanifolds in Sasakian manifolds, Kluwer Academic Publishers, Geometriae Dedicata, 102(2003), 143-150.

[**12**] Y.H. Kim and D.W. Yoon, Inequality for totally real warped products in locally conformal Kaehler space forms, Kyungpook Math. J., 44(2004), 585-592.

[**13**] K. Matsumoto and I. Mihai, Warped product submanifolds in Sasakian space forms, SUT J. Math., 38(2002), 135-144.

[**14**] A. Mihai, Warped product submanifolds in complex space forms, Acta. Sci. Math., 70(2004), 419-427.

[**15**] A. Mihai, Warped product submanifolds in generalized complex space forms, Acta Math. Acad. Paed. Nyis., 21(2005), 79-87.

[**16**] A. Mihai, Sharp operator A_H for slant submanifolds in generalized complex space forms, Turk. J. Math., 27(2003), 509-523.

[**17**] S. Nolker, Isometric immersions of warped product, Differential Geom. Appl., 6(1996), 1-30.

[**18**] B. Sahin, Non-existence of warped product semi-slant submanifolds of Kaehler manifolds, Geometriae Dedicata, 117(2006), 195-202.

[**19**] M.M. Tripathi, Some characterizations of CR-submanifolds of generalized complex space forms, Kuwait J. Sci. and Eng., 23(2)(1996), 133-138.

[**20**] F. Urbano, CR-Submanifolds of Nearly-Kaehler Manifolds, Doctoral Thesis, Granada, 1980.

[**21**] K. Yano and M. Kon, Structures on Manifolds, World Scientific, Singapore, 1984.

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