Totally Real Warped Product Submanifolds in Generalized Complex Space Forms

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Abstract

In this article, we establish an inequality between the warping function and the squared mean curvature for totally real warped product submanifolds in generalized complex space forms. Some applications are also discussed.

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A. Mihai [14] studied warped submanifolds in complex space forms and established an inequality between the warping function $f$ and the squared mean curvature $\|H\|^2$ and the holomorphic sectional curvature $c$ for warped product submanifolds $M_1 \times_M M_2$ in a complex space form $\tilde{M}(c)$. He also studied [15] warped product submanifolds in generalized complex space forms. In the present paper, we establish a similar inequality for totally real warped product submanifolds in a generalized complex space form.

2. Preliminaries. Let $\tilde{M}$ be an almost Hermitian manifold with an almost complex structure $(J, g)$. If $J$ is integrable, i.e., the Nijenhuis tensor $[J, J]$ of $J$ vanishes then $\tilde{M}$ is called a Hermitian manifold. The fundamental 2-form $\Omega$ of $\tilde{M}$ is defined by
\( \Omega(X, Y) = g(X, JY) \),

for all \( X, Y \in T\tilde{M} \).

An almost Hermitian manifold \( \tilde{M} \) is called an almost Kaehler manifold if the fundamental 2-form \( \Omega \) is closed and it is Kaehler manifold if \( \nabla J = 0 \).

An almost Hermitian manifold with \( J \)-invariant Riemannian curvature tensor \( \tilde{R} \), i.e.,

\[
(2.2) \quad \tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W),
\]

for any vector fields \( X, Y, Z, W \in T\tilde{M} \)
is called an RK-manifold\(^{[20]} \). An almost Hermitian manifold \( \tilde{M} \) is said to have (pointwise) constant type if for each \( p \in \tilde{M} \) and for all vector fields \( X, Y, Z, W \in T\tilde{M} \) such that

\[
(2.3) \quad g(X, Y) = g(JX, Y) = g(X, Z) = g(JX, Z) = 0,
\]

\[
g(Y, Y) = g(Z, Z) = 1,
\]

we have

\[
(2.4) \quad \tilde{R}(X, Y, X, Y) - \tilde{R}(X, Y, JX, JY) = \tilde{R}(X, Z, X, Z) - \tilde{R}(X, Z, JX, JZ).
\]

An RK-manifold \( \tilde{M} \) has (pointwise) constant type if and only if there is a differentiable function \( \alpha \) on \( \tilde{M} \) such that

\[
(2.5) \quad \tilde{R}(X, Y, X, Y) - \tilde{R}(X, Y, JX, JY) = \alpha [g(X, X)g(Y, Y) - g^2(X, Y) - g^2(X, JY)],
\]

for all vector fields \( X, Y \in T\tilde{M} \).

Furthermore, \( \tilde{M} \) has global constant type if \( \alpha \) is constant. The function \( \alpha \) is called the constant type of \( \tilde{M} \).

An RK-manifold of constant holomorphic sectional curvature \( c \) and constant type \( \alpha \) is called a generalized complex space form \( \tilde{M}(c, \alpha) \). The Ricci curvature tensor of generalized complex space \( \tilde{M}(c, \alpha) \) has the following expression \([20]\):
Totally real warped product submanifolds

\[ \tilde{R}(X, Y, Z, W) = \frac{e^{3\alpha}}{4} [g(X, Z)g(Y, W) - g(X, W)g(Y, Z)] + \frac{e^{\alpha}}{4} [g(JX, Z)g(JY, W) - g(JX, W)g(JY, Z)] + 2g(X, JY)g(Z, JW), \]

for any vector fields \( X, Y, Z, W \in T\tilde{M} \).

If \( c = \alpha \), then \( \tilde{M}(c, \alpha) \) is a space of constant curvature. A complex space form \( \tilde{M}(c) \) (a Kaehler manifold of constant holomorphic sectional curvature \( c \)) belongs to the class of almost Hermitian manifold \( \tilde{M}(c, \alpha) \) (with constant type zero).

Let \( M \) be an \( n \)-dimensional submanifold in a generalized complex space form \( \tilde{M}(c, \alpha) \) of dimension \( 2m \). We denote \( K(\pi) \) the sectional curvature of \( M \) associated with a plane section \( \pi \subset T_pM, \ p \in M \), and \( \nabla \) be a Riemannian connection of \( M \), respectively. Also, let \( h \) be the second fundamental form and \( R \) the Riemannian curvature tensor of \( M \).

Then the equation of Gauss is given by

\[ \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \]

for any vector fields \( X, Y, Z, W \) tangent to \( M \).

Let \( p \in M \) and \( \{e_1, \ldots, e_n, \ldots, e_{2m}\} \) be an orthonormal basis of the tangent space \( T_p\tilde{M} \), such that \( \{e_1, \ldots, e_n\} \) are tangent to \( M \).

We denote by \( H(p) \) the mean curvature vector at \( p \in M \), i.e.,

\[ H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i). \]

Also, we set

\[ h_{ij} = g(h(e_i, e_j), e_r), \ i, j \in \{1, \ldots, n\}, \ r \in \{n + 1, \ldots, 2m\} \]

and

\[ \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \]
For any vector $X$ tangent to $M$, we put $JX = PX + FX$, where $PX$ and $FX$ are the tangential and normal parts of $JX$, respectively.

We denote by

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j). \tag{2.11}$$

Let $M$ be a Riemannian $n$-manifold and \{${e_1, \ldots, e_n}$\} be an orthonormal frame field on $M$. For differentiable function $f$ on $M$, the Laplacian $\Delta f$ of $f$ is defined by,

$$\Delta f = \sum_{j=1}^{n} \{(\Delta e_j e_j)f - e_j e_j f\}. \tag{2.12}$$

Now, we recall the following result:-

**Lemma 2.1.** Let $n \geq 2$ and $a_1, \ldots, a_n, b$ be $(n+1)$ real numbers such that

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n - 1)(\sum_{i=1}^{n} a_i^2 + b) \tag{2.13}$$

Then $2a_1 a_2 \geq b$ with equality holding if and only if

$$a_1 + a_2 = a_3 = \ldots = a_n.$$

3. Warped Product Submanifolds

Chen established a sharp relationship between the warping function function $f$ of warped product $M_1 \times_f M_2$ isometrically immersed in a real space form $\tilde{M}(c)$ and the squared mean curvature $\|H\|^2$\cite{6}. A. Mihai obtained a similar inequality for warped product submanifolds of a complex space form.

Let $(M_1, g_1)$ and $(M_2, g_2)$ be two Riemannian manifolds and $f$ a positive differentiable function on $M_1$. The warped product of $M_1$ and $M_2$ is the Riemannian manifold

$$M_1 \times_f M_2 = (M_1 \times M_2, g),$$

where $g = g_1 + f^2 g_2$ \cite{6}.

Let $x : M_1 \times_f M_2 \rightarrow \tilde{M}(c, \alpha)$ be an isometric immersion of a warped product $M_1 \times_f M_2$ into a generalized complex space form $\tilde{M}(c, \alpha)$. We denote by $h$ the second fundamental form of $x$ and $H_i = \frac{1}{n_i} \text{trace } h_i$ where $\text{trace } h_i$ is the trace
of $h$ restricted to $M_i$ and $n_i = \dim M_i$, ($i = 1, 2$).

For a warped product $M_1 \times_f M_2$, we denote by $D_1$ and $D_2$ the distributions
given by the vectors tangent to leaves and fibers, respectively. Thus, $D_1$ is obtained from the tangent vectors of $M_1$ via the horizontal lift and $D_2$ by the
tangent vectors of $M_2$ via the vertical lift.

In this section, we study warped product submanifolds with $JD_1 \perp D_2$ in a
generalized complex space form.

**Theorem 3.1.** Let $x : M_1 \times_f M_2 \to \tilde{M}(c, \alpha)$ be an isometric immersion of an
$n$-dimensional totally real warped product with $JD_1 \perp D_2$ into a $2m$-dimensional
generalized complex space form $\tilde{M}(c, \alpha)$ of constant holomorphic sectional cur-
vature $c$ and of constant type $\alpha$. Then

$$(3.1) \quad \frac{\Delta f}{f} \leq \frac{n_i^2}{4n_2} \|H\|^2 + n_1 c + 3\alpha/4$$

where $n_i = \dim M_i$, $i = 1, 2$ and $\Delta$ is the Laplacian operator of $M_i$.

Moreover, the equally case of (3.1) holds identically if and only if $x$ is a mixed
totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where $H_i$, $i = 1, 2$ are the partial
mean curvature vectors.

**Proof.** Let $M_1 \times_f M_2$ be a warped product submanifold of a generalized
complex space form $\tilde{M}(c, \alpha)$ of constant holomorphic sectional curvature $c$
and of constant type $\alpha$.

We recall the following general formula on a warped product [6]

$$(3.2) \quad \Delta_X Z = \Delta_Z X = \frac{1}{f}(X f)Z,$$

for any vector fields $X, Z$ tangent to $M_1$ and $M_2$ respectively.

If $X$ and $Z$ are unit vector fields, it follows that the sectional curvature $K(X \wedge
Z)$ of the plane section spanned by $X$ and $Z$ is given by

$$(3.3) \quad K(X \wedge Z) = g(\Delta_Z \Delta_X X - \Delta_X \Delta_Z X, Z) = \frac{1}{f}(\{\Delta_X X\} f - X^2 f).$$

We choose a local orthonormal frame $\{e_1, \ldots, e_{n_1}, \ldots, e_{2m}\}$, such that $e_1, \ldots, e_{n_1}$
are tangent to $M_1$ and $e_{n_1+1}, \ldots, e_n$ are tangent to $M_2$, $e_{n_1+1}$ is parallel to the
mean curvature vector $H$. 
Then using (3.3), we obtain

\[(3.4) \quad \Delta f = \sum_{j=1}^{n_1} K(e_j \wedge e_s), \text{ for each } s \in \{n_1 + 1, \ldots, n\}.
\]

On the other hand, the Gauss equation implies

\[(3.5) \quad n^2 \| H \|^2 = 2\tau + \| h \|^2 - n(n - 1)\frac{c^3}{4}.
\]

If we put

\[(3.6) \quad \epsilon = 2\tau - \frac{n^2}{2} \| H \|^2 - n(n - 1)\frac{c^3}{4}.
\]

Then, (3.6) can be written as,

\[(3.7) \quad n^2 \| H \|^2 = 2(\epsilon + \| h \|^2).
\]

With respect to orthogonal frame, (3.7) takes the following form:

\[(3.8) \quad \left( \sum_{i=1}^{n} h_{ii}^{n+1} \right)^2 = 2 \left\{ \sum_{r=n+1}^{2m} \sum_{i,j=1}^{n} (h_{ij}^{r})^2 + \epsilon \right\}.
\]

Or equivalently,

\[(3.9) \quad \left( \sum_{i=1}^{n} h_{ii}^{n+1} \right)^2 = 2 \left\{ \sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{1 \leq i \neq j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^{r})^2 + \epsilon \right\}.
\]

If we put \( a_1 = h_{11}^{n+1}, \ a_2 = \sum_{i=2}^{n_1} \) and \( a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}, \) the above equation becomes

\[(3.10) \quad \left( \sum_{i=1}^{3} a_i \right)^2 = 2 \left\{ \sum_{i=1}^{3} a_i^2 + \sum_{1 \leq i \neq j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^{r})^2
\]

\[- \sum_{2 \leq j \neq k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1 \leq s \neq t \leq n} h_{ss}^{n+1} h_{tt}^{n+1} + \epsilon \right\}.
\]

Thus, \( a_1, a_2, a_3 \) satisfy the Lemma 2.1, (for \( n = 3 \)), i.e.,

\[(\sum_{i=1}^{3} a_i)^2 = 2(b + \sum_{i=1}^{3} a_i^2),
\]

with \( b = \epsilon + \sum_{1 \leq i \neq j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^{r})^2
\]

\[- \sum_{2 \leq j \neq k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1 \leq s \neq t \leq n} h_{ss}^{n+1} h_{tt}^{n+1}.
\]
Then \(2a_1a_2 \geq b\), with equality holding if and only if \(a_1 + a_2 = a_3\).

In the case under consideration, this means

\[
(3.11) \quad \sum_{1 \leq j < k \leq n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n_1+1 \leq s < t \leq n} h_{ss}^{n+1}h_{tt}^{n+1} \geq \frac{\epsilon}{2} + \sum_{1 \leq a < b \leq n} (h_{\alpha\beta}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{n} \sum_{a,\beta=1}^{n} (h_{\alpha\beta}^{r})^2.
\]

Equality holds if and only if,

\[
(3.12) \quad \sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}.
\]

Using again the Gauss equation, we have

\[
(3.13) \quad n_2 \frac{\Delta f}{f} = \tau - \sum_{1 \leq j < k \leq n_1} K(e_j \wedge e_k) - \sum_{n_1+1 \leq s < t \leq n} K(e_s \wedge e_t)
\]

\[
= \tau - \frac{n_1(n_1-1)(c+3\alpha)}{8} - \frac{2m}{2} \sum_{r=n+1}^{n} \sum_{1 \leq j < k \leq n_1} (h_{jj}^r h_{kk}^r - (h_{jk}^r)^2)
\]

\[
- \frac{n_2(n_2-1)(c+3\alpha)}{8} - \frac{2m}{2} \sum_{r=n+1}^{n} \sum_{n_1+1 \leq s < t \leq n} (h_{ss}^r h_{tt}^r - (h_{st}^r)^2)
\]

Combining (3.11) and (3.13), and using (3.4) we obtain

\[
(3.14) \quad n_2 \frac{\Delta f}{f} \leq \tau - \frac{n(n-1)(c+3\alpha)}{8} + n_1 n_2 \frac{c+3\alpha}{4} - \frac{\epsilon}{2}
\]

\[
- \sum_{1 \leq j \leq n_1; 1 \leq l \leq n} (h_{jl}^{n+1})^2 - \frac{1}{2} \sum_{r=n+2}^{n} \sum_{a,\beta=1}^{n} (h_{\alpha\beta}^{r})^2
\]

\[
+ \frac{2m}{2} \sum_{r=n+1}^{n} \sum_{j \leq k \leq n_1} ((h_{jk}^r)^2 - h_{jj}^r h_{kk}^r)
\]

\[
+ \frac{2m}{2} \sum_{r=n+1}^{n} \sum_{n_1+1 \leq s < t \leq n} ((h_{st}^r)^2 - h_{ss}^r h_{tt}^r)
\]

\[
= \tau - \frac{n(n-1)(c+3\alpha)}{8} + n_1 n_2 \frac{c+3\alpha}{4} - \frac{\epsilon}{2}
\]

\[
- \frac{2m}{2} \sum_{r=n+1}^{n} \sum_{j \leq 1} (h_{jj}^r)^2 - \frac{1}{2} \sum_{r=n+2}^{n} \sum_{t=n+1}^{n} (h_{tt}^r)^2
\]

\[
\leq \tau - \frac{n(n-1)(c+3\alpha)}{8} + n_1 n_2 \frac{c+3\alpha}{4} - \frac{\epsilon}{2}.
\]

Since we assume that \(JD_1 \perp D_2\), (3.14) implies

\[
(3.15) \quad n_2 \frac{\Delta f}{f} \leq \tau - \frac{n(n-1)(c+3\alpha)}{8} + n_1 n_2 \frac{c+3\alpha}{4} - \frac{\epsilon}{2}
\]
\[ \frac{n_1^2}{4} \| H \|^2 + n_1 n_2 \frac{c + 3\alpha}{4} \]

which implies (3.1).

The equality case of (3.14) holds if and only if

\[ h_{jt}^r = 0, \ 1 \leq j \leq n_1, \ n_1 + 1 \leq t \leq n, \ n + 1 \leq r \leq 2m, \]  

and

\[ \sum_{i=1}^{n_1} h_{ii}^r = \sum_{t=n_1+1}^{n} h_{tt}^r = 0, \ n + 2 \leq r \leq 2m. \]  

Obviously (3.16) is equivalent to the mixed totally geodesicness of the warped product \( M_1 \times f M_2 \) and (3.12) and (3.17) implies \( n_1 H_1 = n_2 H_2 \).

As applications, we derive certain obstructions to the existence of minimal warped product submanifolds in generalized complex space forms.

**Corollary 3.2.** Let \( M_1 \times f M_2 \) be a warped product whose warping function \( f \) is Harmonic. Then,

(i) \( M_1 \times f M_2 \) admits no minimal immersion with \( JD_1 \perp D_2 \) into a generalized complex space form \( \tilde{M}(c, \alpha) \) with \( c < -3\alpha \).

(ii) Every minimal immersion with \( JD_1 \perp D_2 \) of \( M_1 \times f M_2 \) in the standard Euclidean space \( \mathbb{R}^{2m} \) is a warped product immersion.

**Proof.** We assume that \( f \) is a Harmonic function on \( M_1 \) and \( M_1 \times f M_2 \) admits a minimal immersion with \( JD_1 \perp D_2 \) in a generalized complex space form \( \tilde{M}(c, \alpha) \). Then, the inequality (3.1) becomes \( c \geq -3\alpha \). If \( c = -3\alpha \), the equality case of (3.1) holds. By theorem (3.1), it follows that \( M_1 \times f M_2 \) is mixed totally geodesic and \( H_1 = H_2 = 0 \).

A well known result of Nolker [17] implies that the immersion is a warped product immersion.

**Corollary 3.3.** If the warping function \( f \) of a warped product \( M_1 \times f M_2 \) is an eigen function of the Laplacian on \( M_1 \) with corresponding eigenvalue \( \lambda > 0 \), then \( M_1 \times f M_2 \) does not admit a minimal immersion with \( JD_1 \perp D_2 \) into a generalized complex space form \( \tilde{M}(c, \alpha) \) with \( c \leq -3\alpha \).
Proof. If $f$ is an eigen function of the Laplacian on $M_1$ with eigenvalue $\lambda > 0$, then inequality (3.1) implies that $\frac{n_1(c + 3\alpha)}{4} \geq \lambda > 0$.

References


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