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Some Theorems on the General Summability Methods

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Abstract

In this paper a general theorem concerning the $|N, p_n, q_n; \delta|_k$ summability has been proved.

Keywords: Cesáro Summability, Infinite Series, Summability Factors

1 DEFINITIONS AND NOTATIONS

Let $\sum a_n$ be an infinite series with partial sums s_n. Let σ_n^δ and η_n^δ denote the n^{th} Cesáro means of order $\delta(\delta > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$, respectively. The series $\sum a_n$ is said to be summable (C, δ) with index k, or simply summable $|C, \delta|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\sigma_n^\delta - \sigma_{n-1}^\delta\right|^k < \infty,$$

or equivalently

$$\sum_{n=1}^{\infty} n^{-1} \left|\eta_n^\delta\right|^k < \infty.$$

Let $\{p_n\}$ be a sequence of real or complex constants with

$$P_n = p_0 + p_1 + p_2 + ... + p_n, \qquad p_{-r} = P_{-r} = 0, \qquad r = 1, 2, ...$$

The series $\sum a_n$ is said to be summable $|N, p_n|$, if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty, \tag{1.1}$$

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where

$$t_n = P_n^{-1} \sum_{v=0}^{n} p_{n-v} s_v. \qquad (t_{-1} = 0)$$

We write $p = \{p_n\}$ and

$$M = \left\{ \; p : p_n > 0 \quad and \quad \tfrac{p_{n+1}}{p_n} \leq \tfrac{p_{n+2}}{p_n} \leq \tfrac{p_{n+2}}{p_{n+1}} \leq 1, \quad n = 0, 1, ... \; \right\}$$

It is known that for $p \in M$, (1.1) holds if and only if (DAS [4])

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{v=1}^{n} p_{n-v} v a_v \right| < \infty.$$

Definition 1.1: For $p \in M$, we say that $\sum a_n$ is summable $|N, p_n; \delta|_k$, $k \geq 1$, $\delta > 0$, if

$$\sum_{n=1}^{\infty} \frac{1}{nP_n^{k+\delta k}} \left| \sum_{v=1}^{n} p_{n-v} v a_v \right|^k < \infty.$$

In the special case in which $p_n = A_n^{r-1}$, $r > -1$, where A_n^r is the coefficient of x^n in the power series expansion of $(1-x)^{-r-1}$ for $|x| < 1$, $|N, p_n; \delta|_k$ summability reduces to $|C, r; \delta|_k$ summability.

The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n; \delta\right|_k$, $k \geq 1$, $\delta > 0$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_n - T_{n-1}|^k < \infty$$

Where $T_n = P_n^{-1} \sum_{v=0}^{n} p_v s_v$.

If we take $p_n = 1$, $\delta = 0$ then $\left|\bar{N}, p_n; \delta\right|_k$ summability is equivalent to $|C, 1|_k$ summability. In general, these two summabilities are not comparable.

We set

$$\begin{aligned}
\Delta f_n &= f_n - f_{n+1} \\
Q_n &= q_0 + q_1 + ... + q_n, q_{-1} = Q_{-1} = 0 \\
U_n &= u_0 + u_1 + ... + u_n, u_{-1} = U_{-1} = 0 \\
V_n &= v_0 + v_1 + ... + v_n, v_{-1} = V_{-1} = 0 \\
R_n &= p_0 q_n + p_1 q_{n-1} + ... + p_n q_0 \\
W_n &= u_0 v_n + u_1 v_{n-1} + ... + u_n v_0
\end{aligned}$$

and assume that P_n, U_n, R_n and W_n all tend to ∞.

Definition 1.2: Let $\{p_n\}$, $\{q_n\}$ be sequences of positive real constants such that $q \in M$. We say that $\sum a_n$ is summable $|N, p_n, q_n; \delta|_k$, $k \geq 1$, $\delta > 0$, if

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n R_{n-1}^{\delta k+k}} \left| \sum_{v=1}^{n} P_{v-1} q_{n-v} a_v \right|^k < \infty.$$

Clearly $|N, p_n, 1; \delta|_k$ and $|N, 1, q_n; \delta|_k$ are equivalent to $|N, p_n; \delta|_k$ and $\left|\bar{N}, q_n; \delta\right|_k$ respectively.

2 INTRODUCTION

SULAIMAN [6] proved the following theorem for $|N, p_n, q_n|_k$ summability.

Theorem 2.1: Let $\{p_n\}$, $\{q_n\}$, $\{u_n\}$ and $\{v_n\}$ be sequences of positive real constants such that $q, v \in M$, $q_n = O(v_n)$, $\{p_n / P_n R_{n-1}^{\delta k+k} v_n^k\}$ non-increasing and that $a_n \geq 0$ if $v_n \neq c$. Suppose $\{\varepsilon_n\}$ is a sequence of constants and write $W_{n-1} G_n = \sum_{r=1}^{n} U_{r-1} v_{n-r}$. If

$$\sum_{n=r+1}^{\infty} \frac{p_r}{P_r R_{r-1}} \frac{q_{n-r-1}}{v_{n-r-1}^k} = O\left(\frac{1}{P_r v_r^k}\right)$$

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} \left(\frac{W_{n-1}}{v_n U_{n-1}}\right)^k |\varepsilon_n|^k |G_n|^k < \infty$$

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\frac{u_n}{U_n}\right)^k \left(\frac{W_{n-1}}{v_n U_{n-1}}\right)^k |\varepsilon_n|^k |G_n|^k < \infty$$

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\frac{W_{n-1}}{v_n U_{n-1}}\right)^k |\Delta \varepsilon_n|^k |G_n|^k < \infty$$

$$\sum_{n=1}^{\infty} \frac{p_r}{P_r} \left(\frac{P_{r-1}}{R_{r-1}}\right) \left(\frac{W_{r-1}}{v_r U_{r-1}}\right)^k |\varepsilon_r|^k |G_r|^k < \infty$$

then the series $\sum a_n \varepsilon_n$ is summable $|N, p_n, q_n|_k$, $k \geq 1$.

3

Generalizing the theorems of SULAIMAN [6] for $|N, p_n, q_n; \delta|_k$ summability, we shall prove the following theorems in this chapter.

Theorem 3.1: Let $\{p_n\}$, $\{q_n\}$, $\{u_n\}$ and $\{v_n\}$ be sequences of positive real constants such that $q, v \in M$, $q_n = O(v_n)$, $\{p_n / P_n R_{n-1}^{\delta k+k} v_n^k\}$ non-increasing and that $a_n \geq 0$ if $v_n \neq c$. Suppose $\{\varepsilon_n\}$ is a sequence of constants and write $W_{n-1}G_n = \sum_{r=1}^{n} U_{r-1}v_{n-r}a_r$. If

$$\sum_{n=r+1}^{\infty} \frac{p_n}{P_n R_{r-1}^{\delta k+1}} \frac{q_{n-r-1}}{v_{n-r-1}^k} = O\left(\frac{1}{P_r v_r^k}\right), \tag{3.1}$$

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} \left(\frac{W_{n-1}}{v_n U_{n-1}}\right)^k |\varepsilon_n|^k |G_n|^k < \infty, \tag{3.2}$$

$$\sum_{n=1}^{\infty} \left(\frac{p_n}{P_n}\right)^{k-1} \left(\frac{u_n}{U_n}\right)^k \left(\frac{W_{n-1}}{v_n U_{n-1}}\right)^k |\varepsilon_n|^k |G_n|^k < \infty, \tag{3.3}$$

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\frac{W_{n-1}}{v_n U_{n-1}}\right)^k |\Delta\varepsilon_n|^k |G_n|^k < \infty, \tag{3.4}$$

$$\sum_{n=1}^{\infty} \frac{p_r}{P_r} \left(\frac{P_{r-1}^k}{R_{r-1}^{\delta k+k}}\right) \left(\frac{W_{r-1}}{v_r U_{r-1}}\right)^k |\varepsilon_r|^k |G_r|^k < \infty, \tag{3.5}$$

then the series $\sum a_n \varepsilon_n$ is summable $|N, p_n, q_n; \delta|_k$, $k \geq 1$and $\delta \geq 0$.

4

For the proof of our theorem, we require the following lemmas.

Lemma 4.1: (SULAIMAN [7]). Let $q \in M$, then for $0 < v \leq 1$,

$$\sum_{n=r}^{\infty} \frac{q_{n-r}}{n^v Q_r} = O\left(r^{-v}\right).$$

Lemma 4.2: $\left\{\frac{p_n}{P_n R_{n-1}^{\delta k+k} v_n^k}\right\}$ non-increasing implies

$$\sum_{n=r+1}^{m} \frac{p_n}{P_n R_{n-1}^{\delta k+k}} \frac{|\Delta_r q_{n-r}|}{v_{n-r}^k} = O\left\{\frac{p_r}{P_r R_{r-1}^{\delta k+k} v_r^k} \sum_{n=1}^{m} |\Delta q_n|\right\}.$$

Proof. Since

$$\begin{aligned} \frac{p_n}{P_n R_{n-1}^{\delta k+k}} &= \frac{p_n v_n^k}{P_n R_{n-1}^{\delta k+k} v_n^k} \\ &\leq \frac{p_{n-1} v_n^k}{P_{n-1} R_{n-2}^{\delta k+k} v_{n-1}^k} \end{aligned}$$

$$\leq \frac{p_n}{P_{n-1}R_{n-2}^{\delta k+k}}$$

therefore $\left\{\frac{p_n}{P_n R_{n-1}^{\delta k+k}}\right\}$ is non-increasing.

We have

$$\sum_{n=r+1}^{m} \frac{p_n}{P_n R_{n-1}^{\delta k+k}} \frac{|\Delta_r q_{n-r}|}{v_{n-r}^k} = \left\{\sum_{n=r+1}^{2r} + \sum_{n=2r+1}^{m}\right\}$$
$$= J_1 + J_2, \text{ say}$$

$$J_1 = O\left\{\frac{p_r}{P_r R_{r-1}^{\delta k+k}}\right\} O\left(\frac{1}{v_r^k}\right) \sum_{n=r+1}^{2r} |\Delta_r q_{n-r}|$$
$$= O\left\{\frac{p_r}{P_r R_{r-1}^{\delta k+k} v_r^k} \sum_{n=1}^{m} |\Delta q_n|\right\}$$
$$J_2 = \sum_{\mu=r+1}^{m-r} \frac{p_{r+\mu}}{P_{r+\mu} R_{r+\mu-1}^{\delta k+k}} \frac{|\Delta q_\mu|}{v_\mu^k}$$
$$= O\left\{\sum_{\mu=r+1}^{m} \frac{p_\mu}{P_\mu R_{\mu-1}^{\delta k+k}} \frac{|\Delta q_\mu|}{v_\mu^k}\right\}$$
$$= O\left\{\frac{p_r}{P_r R_{r-1}^{\delta k+k} v_r^k} \sum_{\mu=1}^{m} |\Delta q_\mu|\right\}.$$

□

5 PROOF OF THE THEOREM

Write $F_n = \sum_{r=1}^{n} P_{r-1} q_{n-r} a_r \varepsilon_r$,

then, by Abel's transformation

$$F_n = \sum_{r=1}^{n} U_{r-1} v_{n-r} a_r \left(\frac{P_{r-1}}{U_{r-1}} \frac{q_{n-r}}{v_{n-r}} \varepsilon_r\right)$$
$$= \sum_{r=1}^{n-1} \left(\sum_{s=1}^{r} U_{s-1} v_{n-s} a_s\right) \Delta_r \left(\frac{P_{r-1}}{U_{r-1}} \frac{q_{n-r}}{v_{n-r}} \varepsilon_r\right) + W_{n-1} G_n \frac{P_{n-1}}{U_{n-1}} \frac{q_0}{v_0} \varepsilon_n$$
$$\leq \sum_{r=1}^{n-1} W_{r-1} |G_r| \left\{\frac{|\Delta_r q_{n-r}|}{v_{n-r}} \frac{P_{r-1}}{U_{r-1}} |\varepsilon_r| + q_{n-r-1} \left|\Delta_r \left(\frac{1}{v_{n-r}}\right)\right| \frac{P_{r-1}}{U_{r-1}} |\varepsilon_r| + \right.$$

$$+\frac{q_{n-r-1}}{v_{n-r-1}}\frac{p_r}{U_{r-1}}|\varepsilon_r| + \frac{q_{n-r-1}}{v_{n-r-1}}\frac{u_r P_r}{U_r U_{r-1}}|\varepsilon_r| + \frac{q_{n-r-1}}{v_{n-r-1}}\frac{p_r}{U_r}|\Delta\varepsilon_r|\Bigg\} +$$
$$+ W_{n-1}|G_n|\frac{P_{n-1}}{U_{n-1}}\frac{q_0}{v_0}|\varepsilon_n|$$
$$= F_{n,1} + F_{n,2} + F_{n,3} + F_{n,4} + F_{n,5} + F_{n,6} \text{ say.}$$

In order to prove the theorem, by Minkowski's inequality, it is therefore sufficient to show that

$$\sum_{n=1}^{\infty}\frac{p_n}{P_n R_{n-1}^{\delta k+k}}F_{n,r}^k < \infty, \qquad r = 1,2,3,4,5,6,$$

where $k > 1$. Applying Hölder's inequality,

$$\sum_{n=2}^{m+1}\frac{p_n}{P_n R_{n-1}^{\delta k+k}}F_{n,1}^k =$$
$$= \sum_{n=2}^{m+1}\frac{p_n}{P_n R_{n-1}^{\delta k+k}}\left\{\sum_{r=1}^{n-1}\frac{|\Delta_r q_{n-r}|}{v_{n-r}}\frac{P_{r-1}}{U_{r-1}}W_{r-1}|\varepsilon_r||G_r|\right\}^k$$
$$\le \sum_{n=2}^{m+1}\frac{p_n}{P_n R_{n-1}^{\delta k+k}}\sum_{r=1}^{n-1}\frac{|\Delta_r q_{n-r}|}{v_{n-r}^k}\frac{P_{r-1}^k}{U_{r-1}^k}W_{r-1}^k|\varepsilon_r|^k|G_r|^k\left\{\sum_{r=1}^{n-1}|\Delta_r q_{n-r}|\right\}^{k-1}$$
$$= O(1)\sum_{r=1}^{m}\frac{P_{r-1}^k}{U_{r-1}^k}W_{r-1}^k|\varepsilon_r|^k|G_r|^k\sum_{n=r+1}^{m+1}\frac{p_n}{P_n^k R_{n-1}^{\delta k+k}}\frac{|\Delta_r q_{n-r}|}{v_{n-r}^k}$$
$$= O(1)\sum_{r=1}^{m}\frac{p_r}{P_r}\frac{P_{r-1}^k}{R_{r-1}^{\delta k+k}}\left(\frac{W_{r-1}}{v_r U_{r-1}}\right)^k|\varepsilon_r|^k|G_r|^k$$
$$= O(1).$$

$$\sum_{n=2}^{m+1}\frac{p_n}{P_n R_{n-1}^{\delta k+k}}F_{n,2}^k =$$
$$\le \sum_{n=2}^{m+1}\frac{p_n}{P_n R_{n-1}^{\delta k+k}}\sum_{r=1}^{n-1}\frac{q_{n-r-1}^k}{v_{n-r-1}^k}\frac{|\Delta_r v_{n-r}|}{v_{n-r}^k}\frac{P_{r-1}^k}{U_{r-1}^k}W_{r-1}^k|\varepsilon_r|^k|G_r|^k \times$$
$$\times\left\{\sum_{r=1}^{n-1}|\Delta_r v_{n-r}|\right\}^{k-1}$$
$$= O(1)\sum_{r=1}^{m}\left(\frac{P_{r-1}}{U_{r-1}}\right)^k W_{r-1}^k|\varepsilon_r|^k|G_r|^k\sum_{n=r+1}^{m+1}\frac{p_n}{P_n R_{n-1}^{\delta k+k}}\frac{|\Delta_r v_{n-r}|}{v_{n-r}^k}$$
$$= O(1)\sum_{r=1}^{m}\frac{p_r}{P_r}\frac{P_{r-1}^k}{R_{r-1}^{\delta k+k}}\left(\frac{W_{r-1}}{v_r U_{r-1}}\right)^k|\varepsilon_r|^k|G_r|^k$$

$$= O(1).$$

$$\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^{\delta k+k}} F_{n,3}^k =$$

$$\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^{\delta k+1}} \sum_{r=1}^{n-1} \frac{q_{n-r-1}}{v_{n-r-1}^k} \frac{p_r}{U_{r-1}^k} W_{r-1}^k |\varepsilon_r|^k |G_r|^k \left\{ \sum_{r=1}^{n-1} \frac{p_r q_{n-r-1}}{R_{n-1}} \right\}^{k-1}$$

$$= O(1) \sum_{r=1}^{m} \frac{p_r}{U_{r-1}^k} W_{r-1}^k |\varepsilon_r|^k |G_r|^k \sum_{n=r+1}^{m+1} \frac{p_n}{P_n R_{n-1}^{\delta k+1}} \frac{q_{n-r-1}}{v_{n-r-1}^k}$$

$$= O(1) \sum_{r=1}^{m} \frac{p_r}{P_r} \left(\frac{W_{r-1}}{v_r U_{r-1}} \right)^k |\varepsilon_r|^k |G_r|^k$$

$$= O(1).$$

$$\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^{\delta k+k}} F_{n,4}^k =$$

$$\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^{\delta k+1}} \sum_{r=1}^{n-1} p_r \frac{q_{n-r-1}}{v_{n-r-1}^k} \frac{P_r^k}{p_r^k} \frac{u_r^k}{U_r^k U_{r-1}^k} W_{r-1}^k |\varepsilon_r|^k |G_r|^k \times$$

$$\times \left\{ \sum_{r=1}^{n-1} \frac{p_r q_{n-r-1}}{R_{n-1}} \right\}^{k-1}$$

$$= O(1) \sum_{r=1}^{m} p_r \frac{P_r^k}{p_r^k} \frac{u_r^k}{U_r^k U_{r-1}^k} W_{r-1}^k |\varepsilon_r|^k |G_r|^k \sum_{n=r+1}^{m+1} \frac{p_n}{P_n R_{n-1}^{\delta k+1}} \frac{q_{n-r-1}}{v_{n-r-1}^k}$$

$$= O(1) \sum_{r=1}^{m} \left(\frac{P_r}{p_r} \right)^{k-1} \left(\frac{u_r}{U_r} \right)^k \left(\frac{W_{r-1}}{v_r U_{r-1}} \right)^k |\varepsilon_r|^k |G_r|^k$$

$$= O(1).$$

$$\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^{\delta k+k}} F_{n,5}^k =$$

$$\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^{\delta k+1}} \sum_{r=1}^{n-1} p_r \frac{q_{n-r-1}}{v_{n-r-1}^k} \frac{P_r^k}{p_r^k} \frac{W_{r-1}^k}{U_r^k} |\Delta\varepsilon_r|^k |G_r|^k \left\{ \sum_{r=1}^{n-1} \frac{p_r q_{n-r-1}}{R_{n-1}} \right\}^{k-1}$$

$$= O(1) \sum_{r=1}^{m} p_r \left(\frac{P_r}{p_r} \right)^k \left(\frac{W_{r-1}}{U_r} \right)^k |\Delta\varepsilon_r|^k |G_r|^k \sum_{n=r+1}^{m+1} \frac{p_n}{P_n R_{n-1}^{\delta k+1}} \frac{q_{n-r-1}}{v_{n-r-1}^k}$$

$$= O(1) \sum_{r=1}^{m} \left(\frac{P_r}{p_r} \right)^{k-1} \left(\frac{W_{r-1}}{v_r U_{r-1}} \right)^k |\Delta\varepsilon_r|^k |G_r|^k$$

$$= O(1).$$

$$\begin{aligned}
&\sum_{n=1}^{m}\frac{p_n}{P_n R_{n-1}^{\delta k+k}}F_{n,6}^{k} = \\
&\quad \leq \sum_{n=1}^{m}\frac{p_n}{P_n R_{n-1}^{\delta k+1}}\left(\frac{q_0}{v_0}\right)^{k}\left(\frac{W_{n-1}}{U_{n-1}}\right)^{k}|\varepsilon_n|^{k}|G_n|^{k}R_{n-1}^{\delta k+k}P_{n-1}^{k} \\
&\quad = O(1)\sum_{n=1}^{m}\frac{p_n}{P_n}\left(\frac{P_{n-1}^{k}}{R_{n-1}^{\delta k+k}}\right)\left(\frac{W_{n-1}}{v_n U_{n-1}}\right)^{k}|\varepsilon_n|^{k}|G_n|^{k} \\
&\quad = O(1).
\end{aligned}$$

This completes the proof of the Theorem 3.1.

6 APPLICATIONS

Theorem 6.1: If $nu_n = O(U_n)$, $U_n = O(nu_n)$,then the series $\sum a_n$ is summable $|C,1;\delta|_k$ if and only if it is summable $|N,u_n;\delta|_k$, $k \geq 1$, $\delta > 0$.

Proof. ($\Rightarrow$)follows from our Theorem 3.1 by putting $p_n = 0$, $q_n = 1$, $v_n = 1$ and $\varepsilon_n = 1$.

($\Leftarrow$)follows from our Theorem 3.1 by putting $u_n = 1$, $q_n = 1$, $v_n = 1$ and $\varepsilon_n = 1$. □

Theorem 6.2: Let $\{p_n\}$, $\{u_n\}$ be sequences of positive real constants. If $p_n U_n = O(P_n u_n)$ and $P_n u_n = O(p_n U_n)$then the series $\sum a_n$ is summable $\left|\bar{N},p_n;\delta\right|_k$ whenever it is summable $\left|\bar{N},u_n;\delta\right|_k$, $k \geq 1$, $\delta > 0$.

Proof. Follows from our Theorem 3.1 by putting $q_n = 1$, $v_n = 1$ and $\varepsilon_n = 1$. □

Theorem 6.3: If the sequences $\{p_n\}$, $\{q_n\}$, $\{u_n\}$, $\{v_n\}$, satisfy the conditions of our Theorem 3.1 except (3.2)-(3.4) and if $p_n U_n = O(P_n u_n)$, $P_n u_n = O(p_n U_n)$ and $W_{n-1} = O(v_n U_{n-1})$, then the series $\sum a_n$ is summable $|N,p_n,q_n;\delta|_k$ whenever it is summable $|N,u_n,v_n;\delta|_k$, $k \geq 1$, $\delta > 0$.

Proof. Follows from our Theorem 3.1 by putting $\varepsilon_n = 1$. □

Corollary 1: If we take $\delta = 0$ in Theorem 6.1 then we find theorem of BOR [1] and [2].

Corollary 2: If we take $\delta = 0$ in Theorem 6.2 then our theorem 1 reduces to the theorem of BOR and THORPE [3].

Corollary 3: If we take $\delta = 0$ in Theorem 3.1 then our theorem reduces to the theorem of SULAIMAN [6]

Corollary 4: Let $\{q_n\}$, $\{u_n\}$, be sequences of positive real constants such that $q \in M$, $U_n = O(nu_n)$ and $nu_n = O(U_n)$. Then the series $\sum a_n$ is summable $|N, q_n; \delta|_k$ whenever it is summable $\left|\bar{N}, u_n; \delta\right|_k$, $k \geq 1$, $\delta > 0$.

Proof. Follows from Theorem 6.3, by putting $p_n = 1$, $v_n = 1$ and making use of Lemma 4.1. □

Corollary 5: If the sequences $\{p_n\}$, $\{q_n\}$, $\{u_n\}$, $\{v_n\}$, satisfy the conditions of our Theorem 3.1 except (3.2)-(3.4) and if $p_n U_n = O(P_n u_n)$ and $P_n u_n = O(p_n U_n)$, then sufficient conditions that $\sum a_n \varepsilon_n$ is summable $|N, p_n, q_n; \delta|_k$ whenever it is summable $|N, u_n, v_n; \delta|_k$, $k \geq 1$, $\delta > 0$ are

(i) $|\Delta \varepsilon_n| = O\left\{\dfrac{p_n}{P_n}\dfrac{v_n U_{n-1}}{W_{n-1}}\right\}$,

(ii) $|\varepsilon_n| = O\left\{\dfrac{v_n U_{n-1}}{W_{n-1}}\right\}$.

Proof. Follows from our Theorem 3.1. □

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