Int. J. Contemp. Math. Sciences, Vol. 5, 2010, no. 49, 2443 - 2451

Inclusion Theorems for Absolute Matrix Summability Methods of an Infinite Series (IV)

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Abstract

In this paper a general theorems concerning the $|T, q_n; \delta|_k$ summabiligy has been proved.

Keywords: Cesáro Summability, Infinite Series

1 DEFINITIONS AND NOTATIONS

Let $\sum a_n$ be a given infinite series with the partial sums $\{s_n\}$. By ω_n^δ be denote the n-th Cesáro means of order $\alpha(\alpha > -1)$ of the sequence $\{s_n\}$. The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\omega_n^\delta - \omega_{n-1}^\delta\right|^k < \infty \tag{1.1}$$

An appropriate extension of (1.1) to arbitrary lower triangular matrix (a_{nv}) is

$$\sum_{n=1}^{\infty} n^{k-1} \left|\Delta t_{n-1}\right|^k < \infty, \tag{1.2}$$

where

$$t_n = \sum_{v=0}^{n} a_{nv} s_v$$

and the operator Δ is definined by $\Delta f_n = f_n - f_{n+1}$, this defines the summability $|T_k|$ of $\sum a_n$.

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Let $\{p_n\}$ be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty, \text{ as } n \to \infty, \quad (P_{-1} = p_{-1} = 0)$$

The sequence-to-sequence transformation

$$v_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.3}$$

defines the sequence $\{v_n\}$ of the $\left|\bar{N}, p_n\right|$ means of the sequence $\{s_n\}$, generated by the sequence of coefficient $\{p_n\}$. The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$ if (see[**2**])

$$\sum_{n=1}^{\infty} n^{k-1} \left|v_n - v_{n-1}\right|^k < \infty$$

The series $\sum a_n$ is also summable $\left|\bar{N}, p_n; \delta\right|_k$, $k \geq 1$ and $\delta \geq 0$ if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} \left|v_n - v_{n-1}\right|^k < \infty$$

SULAIMAN [**3**] give the following new definition
A series $\sum a_n$ is summable $|T, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\Delta t_{n-1}\right|^k < \infty$$

where

$$t_n = \sum_{v=0}^{n} a_{nv} s_v$$

A series $\sum a_n$ is also summable $|T, p_n; \delta|_k$, $k \geq 1$, $\delta \geq 0$ if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left|\Delta t_{n-1}\right|^k < \infty$$

For $p_n = 1$ and $\delta = 0$, $|T, p_n; \delta|_k$ summability reduces to $|T|_k$ summability, and for $a_{nv} = \frac{P_v}{p_n}$, $|T, p_n; \delta|_k$ is same as $\left|\bar{N}, p_n; \delta\right|_k$.

2 INTRODUCTION

SULAIMAN [3] proved the following theorems for $|T, q_n|_k$ summability.

Theorem 2.1: Write

$$\sum_{v=1}^{n-1} \frac{p_v}{P_v} |A_{n-1,v-1}| = O(f_n)$$

$$\sum_{v=1}^{n-1} |\Delta_v A_{n-1,v-1}| = O(g_n)$$

if

$$|\in_v|^k \sum_{n=v+1}^{\infty} \left(\frac{Q_n}{q_n} f_n\right)^{k-1} |A_{n-1,v-1}| = O(1)$$

$$\left(\frac{P_v}{p_v}\right) |\in_v|^k \sum_{n=v+1}^{\infty} \left(\frac{Q_n}{q_n} g_n\right)^{k-1} |\Delta_v A_{n-1,v-1}| = O(1)$$

$$\left(\frac{P_v}{p_v}\right)^k |\Delta \in_v|^k \sum_{n=v+1}^{\infty} \left(\frac{Q_n}{q_n} f_n\right)^{k-1} |A_{n-1,v}| = O(1)$$

and

$$\left(\frac{Q_n}{q_n}\right)^{k-1} \frac{P_n}{p_n} |A_{n-1,v-1}|^k |\in_n|^k = O(1) \tag{2.1}$$

then $\sum a_n \lambda_n$ is summable $|T, q_n|_k$ whenever $\sum a_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

Theorem 2.2: Necessary condition that $\sum a_n \lambda_n$ be summable $|T, q_n|_k$ whenever $\sum a_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, are

$$\left(\frac{Q_n}{q_n}\right)^{k-1} \frac{P_n}{p_n} |A_{n-1,n-1}|^k |\in_n|^k = O(1)$$

$$\frac{P_v}{p_v} |\Delta \in_v|^k \sum_{n=v}^{\infty} \left(\frac{Q_n}{q_n}\right)^{k-1} |A_{n-1,v}|^k = O(1).$$

3

We generalizes the following theorems of **SULAIMAN [3]** for $|T, q_n; \delta|_k$ summability.

Theorem 3.1: Write

$$\sum_{v=1}^{n-1} \frac{p_v}{P_v} |A_{n-1,v-1}| = O(s_n)$$

$$\sum_{v=1}^{n-1} |\Delta_v A_{n-1,v-1}| = O(f_n)$$

if

$$|\in_n|^k \sum_{n=v+1}^{\infty} \left(\frac{Q_n}{q_n}\right)^{\delta k+k-1} f_n^{k-1} |A_{n-1,v-1}| = O(1)$$
$$\left(\frac{P_v}{p_v}\right) |\in_v|^k \sum_{n=v+1}^{\infty} \left(\frac{Q_n}{q_n}\right)^{\delta k+k-1} g_n^{k-1} |\Delta A_{n-1,v-1}| = O(1)$$
$$\left(\frac{P_v}{p_v}\right)^k |\Delta \in_v|^k \sum_{n=v+1}^{\infty} \left(\frac{Q_n}{q_n}\right)^{\delta k+k-1} f_n^{k-1} |A_{n-1,v}| = O(1)$$

and

$$\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1} \frac{P_n}{p_n} |A_{n-1,v-1}|^k |\in_n|^k = O(1)$$

then $\sum a_n \in_n$ is summable $|T, q_n; \delta|_k$ whenever $\sum a_n$ is summable $\left|\bar{N}, p_n; \delta\right|_k$, $k \geq 1,\ \delta \geq 0$.

Theorem 3.2: Necessary condition that $\sum a_n \in_n$ be summable $|T, q_n; \delta|_k$ whenever $\sum a_n$ is summable $\left|\bar{N}, p_n; \delta\right|_k$, $k \geq 1,\ \delta \geq 0$, are

$$\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1} |A_{n-1,n-1} \in_n|^k = O\left(\frac{p_n}{P_m}\right)^{1-\delta k}$$

$$\frac{P_v}{p_v} |\Delta \in_v|^k \sum_{n=v}^{\infty} \left(\frac{Q_n}{q_n}\right)^{\delta k+k-1} |A_{n-1,v}|^k = O(1).$$

4 PROOF OF THE THEOREM 3.1 and 3.2

<u>Proof of Theorem 3.1</u> : Define

$$\Delta\mu_n = \left(\frac{P_n}{p_n}\right)^{1-\frac{1}{k}} |v_n - v_{n-1}|^k,$$

then we have

$$\Delta\mu_n = \left(\frac{p_n}{P_n}\right)^{\frac{1}{k}} \frac{1}{P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v$$

We also have

$$\begin{aligned}\Delta t_{n-1} &= \sum_{r=0}^{n-1} a_{n-1,r} s_r - \sum_{r=0}^{n} a_{n,r} s_r \\ &= \sum_{r=0}^{n} (a_{n-1,r} - a_{n,r}) \sum_{v=0}^{r} a_r \in_r \\ &= \sum_{v=0}^{n} a_v \in_v \sum_{r=v}^{n} (a_{n-1,r} - a_{n,r})\end{aligned}$$

Now as,

$$\begin{aligned}\sum_{r=v}^{n} a_{n-1,r} - \sum_{r=v}^{n} a_{n,r} &= \sum_{r=v}^{n-1} a_{n-1,r} - \sum_{r=v}^{n} a_{n,r} \\ &= 1 - \sum_{r=0}^{v-1} a_{n-1,r} - \left(1 - \sum_{r=0}^{v-1} a_{n,r}\right) \\ &= \sum_{r=0}^{v-1} (a_{n,r} - a_{n-1,r}) \\ &= -\sum_{r=0}^{v-1} \Delta_n (a_{n-1,r}) \\ &= A_{n-1,v-1}, \text{ say}\end{aligned}$$

via Abel's transformation

$$\Delta t_{n-1} = \sum_{v=1}^{n} (P_{n-1} a_v) \left(\frac{A_{n-1,v-1} \in_v}{P_{v-1}}\right)$$

$$= \sum_{v=1}^{n-1} \left(\sum_{r=1}^{v} P_{n-1} a_v \right) \Delta_v \left(\frac{A_{n-1,v-1} \in_v}{P_{v-1}} \right) +$$
$$+ \left(\sum_{v=1}^{n} P_{n-1} a_v \right) \left(\frac{A_{n-1,v-1} \in_v}{P_{v-1}} \right)$$
$$= \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^{\frac{1}{k}} P_{v-1} \Delta\mu_v \left(\frac{p_v A_{n-1,v-1} \in_v}{P_v P_{v-1}} + \frac{1}{P_v} \Delta_v (A_{n-1,v-1}) \in_v + \right.$$
$$\left. + \frac{1}{P_v} A_{n-1,v} \Delta \in_v \right) + \left(\frac{P_v}{p_v} \right)^{\frac{1}{k}} \Delta\mu_v A_{n-1,n} \in_n$$
$$= \sum_{v=1}^{n-1} \left[\left(\frac{P_v}{p_v} \right)^{1-\frac{1}{k}} A_{n-1,v-1} \Delta\mu_v \in_v + \left(\frac{P_v}{p_v} \right)^{\frac{1}{k}} \frac{P_{v-1}}{P_v} \Delta_v A_{n-1,v-1} \Delta\mu_v \in_v \right.$$
$$\left. + \left(\frac{P_v}{p_v} \right)^{\frac{1}{k}} \frac{P_{v-1}}{P_v} A_{n-v} \Delta\mu_v \Delta \in_v \right] + \left(\frac{P_n}{p_n} \right)^{\frac{1}{k}} A_{n-1,n-1} \Delta\mu_v \in_v$$
$$= t_1 + t_2 + t_3 + t_4, \text{ say.}$$

In order to prove sufficiency, by Minkowski's inequality, it is therefore sufficient to prove that

$$\sum_{n=1}^{\infty} \left(\frac{Q_n}{q_n} \right)^{\delta k + k - 1} |t_i|^k < 0, \qquad i = 1, 2, 3, 4.$$

Applying Hölder's inequality

$$\sum_{n=2}^{m} \left(\frac{Q_n}{q_n} \right)^{\delta k + k - 1} |t_1|^k =$$
$$= \sum_{n=2}^{m} \left(\frac{Q_n}{q_n} \right)^{\delta k + k - 1} \left| \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^{1-\frac{1}{k}} A_{n-1,v-1} \Delta\mu_v \in_v \right|^k$$
$$\leq \sum_{n=2}^{m} \left(\frac{Q_n}{q_n} \right)^{\delta k + k - 1} \sum_{v=1}^{n-1} |A_{n-1,v-1}| \, |\Delta\mu_v|^k \, |\in_v|^k \left\{ \sum_{v=1}^{n-1} \frac{p_v}{P_v} |A_{n-1,v-1}| \right\}^{k-1}$$
$$= O(1) \sum_{n=2}^{m} \left(\frac{Q_n}{q_n} \right)^{\delta k + k - 1} f_n^{k-1} \sum_{v=1}^{n-1} |A_{n-1,v-1}| \, |\Delta\mu_v|^k \, |\in_v|^k$$
$$= O(1) \sum_{v=1}^{m} |\Delta\mu_v|^k \, |\in_v|^k \sum_{m=v+1}^{m} \left(\frac{Q_n}{q_n} \right)^{\delta k + k - 1} f_n^{k-1} \, |A_{n-1,v-1}|$$
$$= O(1) \sum_{v=1}^{m} |\Delta\mu_v|^k$$

$$= O(1).$$

$$\sum_{n=2}^{m}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}|t_2|^k =$$

$$= \sum_{n=2}^{m}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^{\frac{1}{k}}\frac{P_{v-1}}{P_v}\Delta_v(A_{n-1,v-1})\Delta\mu_v \in_v\right|^k$$

$$\leq \sum_{n=2}^{m}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)|\Delta_v(A_{n-1,v-1})|\,|\Delta\mu_v|^k\,|\in_v|^k \times$$

$$\times\left\{\sum_{v=1}^{n-1}|\Delta_v(A_{n-1,v-1})|\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m}\frac{P_v}{p_v}|\Delta\mu_v|^k\,|\in_v|^k\sum_{n=v+1}^{m}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}g_n^{k-1}\,|\Delta_v(A_{n-1,v-1})|$$

$$= O(1)\sum_{v=1}^{m}|\Delta\mu_v|^k$$

$$= O(1).$$

$$\sum_{n=2}^{m}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}|t_3|^k =$$

$$= \sum_{n=2}^{m}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^{\frac{1}{k}}\frac{P_{v-1}}{P_v}(A_{n-1,v})\Delta\mu_v\Delta \in_v\right|^k$$

$$\leq \sum_{n=2}^{m}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)|A_{n-1,v}|\,|\Delta\mu_v|^k\,|\Delta \in_v|^k$$

$$\times\left\{\sum_{v=1}^{n-1}\frac{p_v}{P_v}|A_{n-1,v}|\right\}^{k-1}$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_v}{p_v}\right)^k|\Delta\mu_v|^k\,|\Delta \in_v|^k\sum_{n=v+1}^{m}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}f_n^{k-1}\,|A_{n-1,v}|$$

$$= O(1)\sum_{v=1}^{m}|\Delta\mu_v|^k$$

$$= O(1).$$

$$\sum_{n=2}^{m}\left(\frac{Q_n}{q_n}\right)^{\delta k+k-1}|t_4|^k =$$

$$= \sum_{n=1}^{m} \left(\frac{Q_n}{q_n}\right)^{\delta k+k-1} \left| \left(\frac{P_v}{p_v}\right)^{\frac{1}{k}} A_{n-1,n-1} \Delta\mu_v \in_v \right|^k$$
$$= O(1) \sum_{n=1}^{m} \left(\frac{Q_n}{q_n}\right)^{\delta k+k-1} \frac{P_n}{p_n} \left|A_{n-1,v-1}\right|^k \left|\Delta\mu_v\right|^k \left|\in_v\right|^k$$
$$= O(1) \sum_{v=1}^{m} \left|\Delta\mu_v\right|^k$$
$$= O(1).$$

Proof of Theorem 3.2 : To prove necessity suppose that $\sum a_n \in_n$ is summable $|T, q_n; \delta|_k$ whenever $\sum a_n$ is summable $\left|\bar{N}, p_n; \delta\right|_k$. Therefore we have

$$\sum_{n=1}^{m} \left(\frac{Q_n}{q_n}\right)^{\delta k+k-1} \left|\Delta t_{n-1}\right|^k = O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \left|\Delta v_{n-1}\right|^k$$
$$\sum_{n=1}^{m} \left(\frac{Q_n}{q_n}\right)^{\delta k+k-1} \left|\sum_{v=1}^{m} A_{n-1,v-1} a_v \in_v\right|^k = O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \left|\frac{1}{P_{n-1}} \sum_{v=1}^{n} a_v P_{v-1}\right|^k$$

Let $a_m = 1$, $a_v = 0$ for $v \neq m$, we have

$$\left(\frac{Q_m}{q_m}\right)^{\delta k+k-1} \left|A_{m-1,m-1} \in_n\right|^k = O\left(\frac{p_n}{P_m}\right)^{1-\delta k}$$

From (2.1), we have

$$\sum_{n=1}^{m} \left[\left(\frac{Q_m}{q_n}\right)^{\delta k+k-1} \left| \sum_{v=1}^{n-1} \left[\left(\frac{P_v}{p_v}\right)^{1-\frac{1}{k}} A_{n-1,v-1} \Delta\mu_v \in_v + \right.\right.\right.$$
$$\left. + \left(\frac{P_v}{p_v}\right)^{\frac{1}{k}} \frac{P_{v-1}}{P_v} \Delta_v A_{n-1,v-1} \Delta\mu_v \in_v + \frac{P_v}{p_v} \frac{P_{v-1}}{P_v} A_{n-1,v} \Delta\mu_v \Delta \in_v \right] +$$
$$\left. + \left(\frac{P_n}{p_n}\right)^{\frac{1}{k}} A_{n-1,n-1} \Delta\mu_v \in_v \right|^k$$
$$= O(1) \sum_{v=1}^{m} \left|\Delta\mu_v\right|^k$$

The independence of $\in_v$ and $\Delta \in_v$ show that the above implies

$$\sum_{n=1}^{m} \left(\frac{Q_n}{q_n}\right)^{\delta k+k-1} \left| \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^{\frac{1}{k}} \frac{P_{v-1}}{P_v} A_{n-1,v} \Delta\mu_v \Delta \in_v \right|^k = O(1) \sum_{n=1}^{m} \left|\Delta\mu_n\right|^k$$

Let $\Delta\mu_v = 1$, $\Delta\mu_r = 0$, $r \neq v$, we have

$$\sum_{n=v}^{m} \left(\frac{Q_n}{q_n}\right)^{\delta k+k-1} \left| \left(\frac{P_v}{p_v}\right)^{\frac{1}{k}} \frac{P_{v-1}}{P_v} A_{n-1,v} \Delta \in_v \right|^k = O(1)$$

as

$$P_v = O(P_{v-1}), \text{ for} \frac{P_v}{P_{v-1}} = \frac{P_v}{P_v - p_v} = \frac{1}{1 - \frac{p_v}{P_v}} = \frac{1}{1 - O(1)} = O(1)$$

then

$$\frac{P_v}{p_v} |\Delta \in_v|^k \sum_{m=v+1}^{m} \left(\frac{Q_n}{q_n}\right)^{\delta k+k-1} |A_{n-1,v}|^k = O(1)$$

This completes the proof of the theorem.

Corollary 1: If we take $\delta = 0$, then our theorem reduces the theorem of **SULAIMAN [3]**.

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Received: May, 2010