Int. J. Contemp. Math. Sciences, Vol. 5, 2010, no. 49, 2409 - 2417

The Minimality and Maximality of Left (Right) Ideals in Ternary Semigroups

Aiyared Iampan

Department of Mathematics, School of Science and Technology
Phayao University, Phayao 56000, Thailand
aiyaredi@nu.ac.th

Abstract

The motivation mainly comes from the conditions of left ideals to be (0-)minimal or maximal that are of importance and interest in (ordered) semigroups. In 1932, Lehmer [4] gave the definition of a ternary semigroup. We can see that any semigroup can be reduced to a ternary semigroup. In this paper, we give some auxiliary results are also necessary for our considerations and characterize the relationship between minimality and maximality of left ideals in ternary semigroups, and left simple and left 0-simple ternary semigroups analogous to the characterizations of the minimality and maximality of left ideals in ordered semigroups considered by Cao and Xu [2].

Mathematics Subject Classification: 20M10

Keywords: (0-)minimal and maximal left ideal; left simple and left 0-simple ternary semigroup

1 Introduction and Prerequisites

In 1995, Dixit and Dewan [3] introduced and studied the properties of quasi-ideals and bi-ideals in ternary semigroups. In 2000, Cao and Xu [2] characterized the minimal and maximal left ideals in ordered semigroups and gave some characterizations of minimal and maximal left ideals in ordered semigroups. In 2002, Arslanov and Kehayopulu [1] characterized the minimal and maximal ideals in ordered semigroups. They proved two theorems as follow: In an ordered semigroup S, for which there exists an element $a \in S$ such that the ideal of S generated by a is S, there is at most one maximal ideal which is the union of all proper ideals of S. In an ordered semigroup S containing unit, there is at most one maximal ideal which is the union of all proper ideals of S.

The concept of the minimality and maximality of left ideals is the really interested and important thing about (ordered) semigroups. Now we also characterize the minimality and maximality of left ideals in ternary semigroups and give some characterizations of the minimality and maximality of left ideals in ternary semigroups. Similar results hold if we replace the word "left" by "right".

Our aim in this paper is fourfold.

1. To introduce the concept of left simple and left 0-simple ternary semigroups.

2. To characterize the properties of left ideals in ternary semigroups.

3. To characterize the relationship between (0-)minimal left ideals and left (0-)simple ternary semigroups.

4. To characterize the relationship between maximal left ideals and left simple and left 0-simple ternary semigroups.

To present the main theorems we first recall the definition of a ternary semigroup which is important here.

A nonempty set T is called a *ternary semigroup* [4] if there exists a ternary operation $T \times T \times T \to T$, written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$, satisfying the following identity for any $x_1, x_2, x_3, x_4, x_5 \in T$,

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]].$$

Example 1. [3] Let $T = \{-i, 0, i\}$. Then T is a ternary semigroup under the multiplication over complex number while T is not a semigroup under complex number multiplication.

Example 2. [3] Let $O = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), A_3 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $A_4 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $T = \{O, I, A_1, A_2, A_3, A_4\}$ is a ternary semigroup under matrix multiplication.

For nonempty subsets A, B and C of T, let

$$[ABC] := \{[abc] \mid a \in A, b \in B \text{ and } c \in C\}.$$

If $A = \{a\}$, then we also write $[\{a\}BC]$ as $[aBC]$, and similarly if $B = \{b\}$ or $C = \{c\}$ or $A = \{a\}$ and $B = \{b\}$ or $A = \{a\}$ and $C = \{c\}$ or $B = \{b\}$ and $C = \{c\}$. A nonempty subset S of a ternary semigroup T is called a *ternary subsemigroup* [3] of T if $[SSS] \subseteq S$. A nonempty subset L (R) of a ternary semigroup T is called a *left ideal* (*right ideal*) [3] of T if $[TTL] \subseteq L$ ($[RTT] \subseteq R$). A left ideal L of a ternary semigroup T is called a *proper left ideal* of T if $L \neq T$. The intersection of all left ideals of a ternary subsemigroup S of a ternary semigroup T containing a nonempty subset A of S is the *left ideal of S generated by A*. For $A = \{a\}$, let $L_S(a)$ denote the left ideal of S generated by $\{a\}$. If $S = T$, then we also write $L_T(a)$ as $L(a)$. An element a of a ternary semigroup T with at least two elements is called a *zero element* of T if $[at_1t_2] = [t_1at_2] = [t_1t_2a] = a$ for all $t_1, t_2 \in T$ and denote it by 0. If T is a ternary semigroup with zero, then every left ideal of T contains a zero element. A ternary semigroup T without zero is called *left simple* if it has no proper left ideals. A ternary semigroup T with zero is called *left 0-simple* if it has no nonzero proper left ideals and $[TTT] \neq \{0\}$. A left ideal L of a ternary semigroup T without zero is called a *minimal left ideal* of T if there is no left ideal A of T such that $A \subset L$. Equivalently, if for any left ideal A of T such that $A \subseteq L$, we have $A = L$. A nonzero left ideal L of a ternary semigroup T with zero is called a *0-minimal left ideal* of T if there is no nonzero left ideal A of T such that $A \subset L$. Equivalently, if for any nonzero left ideal A of T such that $A \subseteq L$, we have $A = L$. Equivalently, if for any left ideal A of T such that $A \subset L$, we have $A = \{0\}$. A proper left ideal L of a ternary semigroup T is called a *maximal left ideal* of T if for any left ideal A of T such that $L \subset A$, we have $A = T$. Equivalently, if for any proper left ideal A of T such that $L \subseteq A$, we have $A = L$.

Throughout this paper, T stands for a ternary semigroup. The following two lemmas are also necessary for our considerations and easy to verify.

Lemma 1.1 *[3] For any nonempty subset A of T, $[TTA] \cup A$ is the smallest left ideal of T containing A. Furthermore, for any $a \in T$,*

$$L(a) = [TTa] \cup \{a\}.$$

Lemma 1.2 *For any nonempty subset A of T, $[TTA]$ is a left ideal of T.*

Lemma 1.3 *If T has no zero element, then the following statements are equivalent:*

(i) *T is left simple.*

(ii) *$[TTa] = T$ for all $a \in T$.*

(iii) *$L(a) = T$ for all $a \in T$.*

Proof. By Lemma 1.2 and T is left simple, we have $[TTa] = T$ for all $a \in T$. Therefore (i) implies (ii). By Lemma 1.1, $L(a) = [TTa] \cup \{a\} = T \cup \{a\} = T$. Thus (ii) implies (iii). Now let L be a left ideal of T and let $a \in L$. Then $T = L(a) \subseteq L \subseteq T$, so $L = T$. Hence T is left simple, we have that (iii) implies (i).

Hence the proof is completed. $\square$

Lemma 1.4 *If T has a zero element, then the following statements hold:*

(i) *If T is left 0-simple, then $L(a) = T$ for all $a \in T \setminus \{0\}$.*

(ii) *If $L(a) = T$ for all $a \in T \setminus \{0\}$, then either $[TTT] = \{0\}$ or T is left 0-simple.*

Proof. (i) Assume that T is left 0-simple. Then $L(a)$ is a nonzero left ideal of T for all $a \in T \setminus \{0\}$. Hence $L(a) = T$ for all $a \in T \setminus \{0\}$.

(ii) Assume that $L(a) = T$ for all $a \in T \setminus \{0\}$ and let $[TTT] \neq \{0\}$. Now let L be a nonzero left ideal of T and put $a \in L \setminus \{0\}$. Then $T = L(a) \subseteq L \subseteq T$, so $L = T$. Therefore T is left 0-simple. $\square$

The next lemma is easy to verify.

Lemma 1.5 *Let $\{L_\gamma \mid \gamma \in \Gamma\}$ be a family of left ideals of T. Then $\bigcup_{\gamma\in\Gamma} L_\gamma$ is a left ideal of T and $\bigcap_{\gamma\in\Gamma} L_\gamma$ is also a left ideal of T if $\bigcap_{\gamma\in\Gamma} L_\gamma \neq \emptyset$.*

Lemma 1.6 *If L is a left ideal of T and S is a ternary subsemigroup of T, then the following statements hold:*

(i) *If S is left simple such that $S \cap L \neq \emptyset$, then $S \subseteq L$.*

(ii) *If S is left 0-simple such that $S \setminus \{0\} \cap L \neq \emptyset$, then $S \subseteq L$.*

Proof. (i) Assume that S is left simple such that $S \cap L \neq \emptyset$. Then, let $a \in S \cap L$. By Lemma 1.2, we have $[SSa] \cap S$ is a left ideal of S. This implies that $[SSa] \cap S = S$. Hence $S \subseteq [SSa] \subseteq [TTL] \subseteq L$, so $S \subseteq L$.

(ii) Assume that S is left 0-simple such that $S \setminus \{0\} \cap L \neq \emptyset$. Then, let $a \in S \setminus \{0\} \cap L$. By Lemmas 1.1 and 1.4 (i), we have $S = L_S(a) = ([SSa] \cup \{a\}) \cap S \subseteq [SSa] \cup \{a\} \subseteq [TTa] \cup \{a\} = L(a) \subseteq L$. Therefore $S \subseteq L$.

Hence the proof of the lemma is completed. $\square$

Lemma 1.7 *For a nonempty subsets A of a left ideal L of T, $[LLA]$ is a left ideal of T.*

Proof. Let A be a nonempty subset of a left ideal L of T. Then $[TTL] \subseteq L$, this implies that $[TT[LLA]] = [[TTL]LA] \subseteq [LLA]$. Hence $[LLA]$ is a left ideal of T. $\square$

2 Minimality of Left Ideals

In this section, we characterize the relationship between minimality of left ideals and left simple and left 0-simple ternary semigroups.

Theorem 2.1 *If T has no zero element and L is a left ideal of T, then the following statements hold:*

(i) *L is a minimal left ideal without zero of T if and only if L is left simple.*

(ii) *If L is a minimal left ideal with zero of T, then L is left 0-simple.*

Proof. (i) Assume that L is a minimal left ideal without zero of T. Now let A be a left ideal of L. Then $[LLA] \subseteq A \subseteq L$. By Lemma 1.7, we have $[LLA]$ is a left ideal of T. Since L is a minimal left ideal of T, $[LLA] = L$. Therefore $A = L$, so we conclude that L is left simple.

Conversely, assume that L is left simple. Let A be a left ideal of T such that $A \subseteq L$. Then $A \cap L \neq \emptyset$, it follows from Lemma 1.6 (i) that $L \subseteq A$. Hence $A = L$, so L is a minimal left ideal of T.

(ii) It is similar to the proof of necessary condition of statement (i).

Therefore we complete the proof of the theorem. □

Using the similar proof of Theorem 2.1 (i) and the Lemma 1.6 (ii), we have Theorem 2.2.

Theorem 2.2 *If T has a zero element and L is a nonzero left ideal of T, then the following statements hold.*

(i) *If L is a 0-minimal left ideal of T, then either $[LLA] = \{0\}$ for some nonzero left ideal A of L or L is left 0-simple.*

(ii) *If L is left 0-simple, then L is a 0-minimal left ideal of T.*

Theorem 2.3 *If T has no zero element but it has proper left ideals, then every proper left ideal of T is minimal if and only if T contains exactly one proper left ideal or T contains exactly two proper left ideals L_1 and L_2, $L_1 \cup L_2 = T$ and $L_1 \cap L_2 = \emptyset$.*

Proof. Assume that every proper left ideal of T is minimal. Now let L be a proper left ideal of T. Then L is a minimal left ideal of T. We consider the following two cases:

Case 1: $T = L(a)$ for all $a \in T \setminus L$.

If K is also a proper left ideal of T and $K \neq L$, then $K \setminus L \neq \emptyset$ because L is a minimal left ideal of T. Thus there exists $a \in K \setminus L \subseteq T \setminus L$. Hence

$T = L(a) \subseteq K \subseteq T$, so $K = T$. It is impossible, so $K = L$. In this case, L is the unique proper left ideal of T.

Case 2: There exists $a \in T \setminus L$ such that $T \neq L(a)$.

Then $L(a) \neq L$ and $L(a)$ is a minimal left ideal of T. By Lemma 1.5, $L(a) \cup L$ is a left ideal of T. By hypothesis and $L \subset L(a) \cup L$, we get $L(a) \cup L = T$. Since $L(a) \cap L \subset L(a)$ and $L(a)$ is a minimal left ideal of T, $L(a) \cap L = \emptyset$. Now let K be an arbitrary proper left ideal of T. Then K is a minimal left ideal of T. We observe that $K = K \cap T = K \cap (L(a) \cup L) = (K \cap L(a)) \cup (K \cap L)$. If $K \cap L \neq \emptyset$, then $K = L$ because K and L are minimal left ideals of T. If $K \cap L(a) \neq \emptyset$, then $K = L(a)$ because K and $L(a)$ are minimal left ideals of T. In this case, T contains exactly two proper left ideals L and $L(a)$, $L(a) \cup L = T$ and $L(a) \cap L = \emptyset$.

The converse is obvious. □

Using the same proof of Theorem 2.3, we have Theorem 2.4.

Theorem 2.4 *If T has a zero element and nonzero proper left ideals, then every nonzero proper left ideal of T is 0-minimal if and only if T contains exactly one nonzero proper left ideal or T contains exactly two nonzero proper left ideals L_1 and L_2, $L_1 \cup L_2 = T$ and $L_1 \cap L_2 = \{0\}$.*

3 Maximality of Left Ideals

In this section, we characterize the relationship between maximality of left ideals and the union $\mathcal{U}$ of all (nonzero) proper left ideals in ternary semigroups.

Theorem 3.1 *If T has no zero element but it has proper left ideals, then every proper left ideal of T is maximal if and only if T contains exactly one proper left ideal or T contains exactly two proper left ideals L_1 and L_2, $L_1 \cup L_2 = T$ and $L_1 \cap L_2 = \emptyset$.*

Proof. Assume that every proper left ideal of T is maximal. Now let L be a proper left ideal of T. Then L is a maximal left ideal of T. We consider the following two cases:

Case 1: $T = L(a)$ for all $a \in T \setminus L$.

If K is also a proper left ideal of T and $K \neq L$, then K is a maximal left ideal of T. This implies that $K \setminus L \neq \emptyset$, so there exists $a \in K \setminus L \subseteq T \setminus L$. Thus $T = L(a) \subseteq K \subseteq T$, so $K = T$. It is impossible, so $K = L$. In this case, L is the unique proper left ideal of T.

Case 2: There exists $a \in T \setminus L$ such that $T \neq L(a)$.

Then $L(a) \neq L$ and $L(a)$ is a maximal left ideal of T. By Lemma 1.5, $L(a) \cup L$ is a left ideal of T. Since $L \subset L(a) \cup L$ and L is a maximal left ideal of T, $L(a) \cup L = T$. By hypothesis and $L(a) \cap L \subset L(a)$, we get $L(a) \cap L = \emptyset$. Now let K be an arbitrary proper left ideal of T. Then K is a maximal left ideal of T. We observe that $K = K \cap T = K \cap (L(a) \cup L) = (K \cap L(a)) \cup (K \cap L)$. If $K \cap L \neq \emptyset$, then $K = L$ because $K \cap L$ and L are maximal left ideals of T. If $K \cap L(a) \neq \emptyset$, then $K = L(a)$ because $K \cap L(a)$ and $L(a)$ are maximal left ideals of T. In this case, T contains exactly two proper left ideals L and $L(a)$, $L(a) \cup L = T$ and $L(a) \cap L = \emptyset$.

The converse is obvious. □

Using the same proof of Theorem 3.1, we have Theorem 3.2.

Theorem 3.2 *If T has a zero element and nonzero proper left ideals, then every nonzero proper left ideal of T is maximal if and only if T contains exactly one nonzero proper left ideal or T contains exactly two nonzero proper left ideals L_1 and L_2, $L_1 \cup L_2 = T$ and $L_1 \cap L_2 = \{0\}$.*

Theorem 3.3 *A proper left ideal L of T is maximal if and only if*

(i) *$T \setminus L = \{a\}$ and $[aTa] \subseteq L$ for some $a \in T$ or*

(ii) *$T \setminus L \subseteq [TTa]$ for all $a \in T \setminus L$.*

Proof. Assume that L is a maximal left ideal of T. Then we consider the following two cases:

Case 1: There exists $a \in T \setminus L$ such that $[TTa] \subseteq L$.

Then $[aTa] \subseteq [TTa] \subseteq L$. By Lemma 1.1, we have $L \cup \{a\} = (L \cup [TTa]) \cup \{a\} = L \cup ([TTa] \cup \{a\}) = L \cup L(a)$. Thus $L \cup \{a\}$ is a left ideal of T because $L \cup L(a)$ is a left ideal of T. Since L is a maximal left ideal of T and $L \subset L \cup \{a\}$, we have $L \cup \{a\} = T$. Hence $T \setminus L = \{a\}$. In this case, the condition (i) is satisfied.

Case 2: $[TTa] \not\subseteq L$ for all $a \in T \setminus L$.

If $a \in T \setminus L$, then $[TTa] \not\subseteq L$ and $[TTa]$ is a left ideal of T by Lemma 1.2. By Lemma 1.5, we have $L \cup [TTa]$ is a left ideal of T and $L \subset L \cup [TTa]$. Since L is a maximal left ideal of T, $L \cup [TTa] = T$. Hence we conclude that $T \setminus L \subseteq [TTa]$ for all $a \in T \setminus L$. In this case, the condition (ii) is satisfied.

Conversely, let J be a left ideal of T such that $L \subset J$. Then $J \setminus L \neq \emptyset$. If $T \setminus L = \{a\}$ and $[aTa] \subseteq L$ for some $a \in T$, then $J \setminus L \subseteq T \setminus L = \{a\}$. Thus $J \setminus L = \{a\}$, so $J = L \cup \{a\} = T$. Hence L is a maximal left ideal of T. If $T \setminus L \subseteq [TTa]$ for all $a \in T \setminus L$, then $T \setminus L \subseteq [TTx] \subseteq [TTJ] \subseteq J$ for all $x \in J \setminus L$. Hence $T = (T \setminus L) \cup L \subseteq J \cup J = J \subseteq T$, so $J = T$. Therefore L is a maximal left ideal of T.

Hence the theorem is now completed. □

For a ternary semigroup T, let $\mathcal{U}$ denote the union of all nonzero proper left ideals of T if T has a zero element and let $\mathcal{U}$ denote the union of all proper left ideals of T if T has no zero element. Then it is easy to verify Lemma 3.4.

Lemma 3.4 *$\mathcal{U} = T$ if and only if $L(a) \neq T$ for all $a \in T$.*

As a consequence of Theorem 3.3 and Lemma 3.4, we obtain

Theorem 3.5 *If T has no zero element, then one and only one of the following four conditions is satisfied:*

(i) *T is left simple.*

(ii) *$L(a) \neq T$ for all $a \in T$.*

(iii) *There exists $a \in T$ such that $L(a) = T, a \notin [TTa], [aTa] \subseteq \mathcal{U} = T \setminus \{a\}$ and $\mathcal{U}$ is the unique maximal left ideal of T.*

(iv) *$T \setminus \mathcal{U} = \{x \in T \mid [TTx] = T\}$ and $\mathcal{U}$ is the unique maximal left ideal of T.*

Proof. Assume that T is not left simple. Then there exists a proper left ideal L of T, so $\mathcal{U}$ is a left ideal of T. We consider the following two cases:

Case 1: $\mathcal{U} = T$.

By Lemma 3.4, we have $L(a) \neq T$ for all $a \in T$. In this case, the condition (ii) is satisfied.

Case 2: $\mathcal{U} \neq T$.

Then $\mathcal{U}$ is a maximal left ideal of T. Now assume that L is a maximal left ideal of T. Then $L \subseteq \mathcal{U} \subset T$ because L is a proper left ideal of T. Since L is a maximal left ideal of T, we have $L = \mathcal{U}$. Hence $\mathcal{U}$ is the unique maximal left ideal of T. By Theorem 3.3, we get

(i) $T \setminus \mathcal{U} = \{a\}$ and $[aTa] \subseteq \mathcal{U}$ for some $a \in T$ or

(ii) $T \setminus \mathcal{U} \subseteq [TTa]$ for all $a \in T \setminus \mathcal{U}$.

Suppose that $T \setminus \mathcal{U} = \{a\}$ and $[aTa] \subseteq \mathcal{U}$ for some $a \in T$. Then $[aTa] \subseteq \mathcal{U} = T \setminus \{a\}$. Since $a \notin \mathcal{U}$, we have $L(a) = T$. If $a \in [TTa]$, then $\{a\} \subseteq [TTa]$. By Lemma 1.1, we have $T = L(a) = [TTa] \cup \{a\} = [TTa]$. Thus $a = [t_1t_2a]$ and $t_1 = [t_3t_4a]$ for some $t_1, t_2, t_3, t_4 \in T$. Hence $a = [t_1t_2a] = [[t_3t_4a]t_2a] = [t_3t_4[at_2a]]$. Since $[at_2a] \in \mathcal{U}$ and $\mathcal{U}$ is a left ideal of T, we get $a = [t_3t_4[at_2a]] \in \mathcal{U}$. It is impossible, so $a \notin [TTa]$. In this case, the condition (iii) is satisfied.

Now suppose that $T \setminus \mathcal{U} \subseteq [TTa]$ for all $a \in T \setminus \mathcal{U}$. To show that $T \setminus \mathcal{U} = \{x \in T \mid [TTx] = T\}$, let $x \in T \setminus \mathcal{U}$. Then $x \in [TTx]$, so $\{x\} \subseteq [TTx]$. By Lemma 1.1, we have $L(x) = [TTx] \cup \{x\} = [TTx]$. Since $x \notin \mathcal{U}$, we have $L(x) = T$. Hence $T = L(x) = [TTx]$. Conversely, let $x \in T$ be such that $T = [TTx]$. If $x \in \mathcal{U}$, then $L(x) \subseteq \mathcal{U} \subset T$. By Lemma 1.1, we have $L(x) = [TTx] \cup \{x\} = T \cup \{x\} = T$. It is impossible, so $x \in T \setminus \mathcal{U}$. Hence $T \setminus \mathcal{U} = \{x \in T \mid [TTx] = T\}$. In this case, the condition (iv) is satisfied.

Hence the proof of the theorem is completed. □

Using the same proof of Theorem 3.5, we have Theorem 3.6.

Theorem 3.6 *If T has a zero element and $[TTT] \neq \{0\}$, then one and only one of the following four conditions is satisfied:*

(i) *T is left 0-simple.*

(ii) *$L(a) \neq T$ for all $a \in T$.*

(iii) *There exists $a \in T$ such that $L(a) = T, a \notin [TTa], [aTa] \subseteq \mathcal{U} = T \setminus \{a\}$ and $\mathcal{U}$ is the unique maximal left ideal of T.*

(iv) *$T \setminus \mathcal{U} = \{x \in T \mid [TTx] = T\}$ and $\mathcal{U}$ is the unique maximal left ideal of T.*

Acknowledgment

The author would like to thank the referee for the useful comments and suggestions given in an earlier version of this paper.

References

[1] M. Arslanov and N. Kehayopulu, *A note on minimal and maximal ideals of ordered semigroups*, Lobachevskii J. Math. **11** (1995), 3–6.

[2] Y. Cao and X. Xu, *On minimal and maximal left ideals in ordered semigroups*, Semigroup Forum **60** (2000), 202–207.

[3] V. N. Dixit and S. Dewan, *A note on quasi and bi-ideals in ternary semigroups*, Int. J. Math. Math. Sci. **18** (1995), 501–508.

[4] D. H. Lehmer, *A ternary analogue of abelian groups*, Am. J. Math. **54** (1932), 329–338.

Received: June, 2010