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# An Iteratively Regularized Projection Method for Nonlinear Ill-posed Problems

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**Abstract**

An iterative regularization method in the setting of a finite dimensional subspace $X_h$ of the real Hilbert space $X$ has been considered for obtaining stable approximate solution to nonlinear ill-posed operator equations $F(x) = y$ where $F : D(F) \subseteq X \longrightarrow X$ is a nonlinear monotone operator on $X$. We assume that only a noisy data $y^\delta$ with $\|y - y^\delta\| \leq \delta$ are available. Under the assumption that the Fréchet derivative $F'$ of $F$ is Lipschitz continuous, a choice of the regularization parameter using an adaptive selection of the parameter and a stopping rule for the iteration index using a majorizing sequence are presented. We prove that under a general source condition on $x_0 - \hat{x}$, the error $\|x_{n,\alpha}^{h,\delta} - \hat{x}\|$ between the regularized approximation $x_{n,\alpha}^{h,\delta}, (x_{0,\alpha}^{h,\delta} := P_h x_0$ where $P_h$ is an orthogonal projection on to $X_h$) and the solution $\hat{x}$ is of optimal order. The results of computational experiments are provided which shows the reliability of our method.

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**Keywords:** Nonlinear ill-posed operator, Monotone operator, Majorizing sequence, Regularized Projection method

# 1 Introduction

Let $F : D(F) \subseteq X \mapsto X$ is a nonlinear monotone operator defined on a real Hilbert space $X$ with inner product $\langle .,. \rangle$ and norm $\|.\|$. Recall that $F$ is a monotone operator if

$$\langle F(x_2) - F(x_1), x_2 - x_1 \rangle \geq 0, \qquad \forall x_1, x_2 \in D(F) \subset X.$$

We consider the problem of solving the nonlinear ill-posed operator equation

$$F(x) = y \tag{1}$$

approximately when the data $y$ is not known exactly. Further we assume that $y^\delta \in X$ are the available noisy data with

$$\|y - y^\delta\| \leq \delta \tag{2}$$

and that (1) has a solution $\hat{x}$. The equation (1) is ill-posed in the sense that the Fréchet derivative $F'(.)$ is not boundedly invertible (see [9], page 26).

Nonlinear ill-posed problems arise in a number of applications (see, [4, 5, 9]). Since (1) is ill-posed, one has to replace the equation (1) by a nearby equation whose solution is less sensitive to perturbation in the right side $y$. This replacement is known as regularization. A well known method for regularizing (1), when $F$ is monotone is the method of Lavrentiev regularization (see [12]). In this method approximation $x_\alpha^\delta$ is obtained by solving the singularly perturbed operator equation

$$F(x) + \alpha(x - x_0) = y^\delta. \tag{3}$$

In [2], George and Elmahdy considered an iterative regularization method;

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - (F'(x_0) + \alpha I)^{-1}(F(x_{n,\alpha}^\delta) - y^\delta + \alpha(x_{n,\alpha}^\delta - x_0)), \tag{4}$$

where $x_{0,\alpha}^\delta := x_0$ and proved that $(x_{n,\alpha}^\delta)$ converges to the unique solution $x_\alpha^\delta$ of (3) under the following Assumptions.

**Assumption 1.1** *There exists $r_0 > 0$ such that $B_{r_0}(\hat{x}) \subseteq D(F)$ and $F$ is Fréchet differentiable at all $x \in B_{r_0}(\hat{x})$.*

**Assumption 1.2** *There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|F'(\hat{x})\|$ satisfying $\lim_{\lambda \to 0} \varphi(\lambda) = 0$ and a vector $v \in X$ with $\|v\| \leq 1$ such that*

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))v$$

*and*

$$\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq c_\varphi \varphi(\alpha), \forall \alpha \in (0, a].$$

**Assumption 1.3** *There exists a constant $k_0 > 0$ such that for every $x, u \in B_{r_0}(\hat{x})$ and $v \in X$, there exists an element $\Phi(x, u, v) \in X$ satisfying*

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\| \leq k_0\|v\|\|x - u\|$$

*for all $x, u \in B_{r_0}(\hat{x})$ and $v \in X$.*

**REMARK 1.4** *It can be seen that functions*

$$\varphi(\lambda) = \lambda^\nu, \lambda > 0$$

*for $0 < \nu \leq 1$ and*

$$\varphi(\lambda) = \begin{cases} (ln\frac{1}{\lambda})^{-p} & , 0 < \lambda \leq e^{-(p+1)} \\ 0 & , otherwise \end{cases}$$

*for $p \geq 0$ satisfy the above assumption (see [10]).*

The convergence analysis in [2] as well as in this paper is based on majorizing sequences. Recall (see [1], Definition 1.3.11) that a nonnegative sequence $(t_n)$ is said to be a majorizing sequence of a sequence $(x_n)$ in $X$ if

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \forall n \geq 0.$$

In applications, one looks for a sequence $(x_{n,\alpha}^{h,\delta})$ in a finite dimensional subspace $X_h$ of $X$ such that $x_{n,\alpha}^{h,\delta} \to x_\alpha^\delta$ as $h \to 0$ and $n \to \infty$. After providing some preparatory results in Section 2, in section 3 we considered an iteratively regularized projection method for obtaining a sequence $(x_{n,\alpha}^{h,\delta})$ in a finite dimensional subspace $X_h$ of $X$ and proved that $x_{n,\alpha}^{h,\delta}$ converges to $x_\alpha^\delta$. Also in section 3 we obtained an estimate for $\|x_{n,\alpha}^{h,\delta} - x_\alpha^\delta\|$.

Using an error estimate for $\|x_\alpha^\delta - \hat{x}\|$ (see [2, 12]), we obtained an error estimate for $\|x_{n,\alpha}^{h,\delta} - \hat{x}\|$ in section 4. The error analysis for the order optimal result using an adaptive selection of the parameter $\alpha$ and a stopping rule using a majorizing sequence are also given in section 4. Implementation of the adaptive choice of the parameter and the choice of the stoping rule are given in section 5. Examples and the results of computational experiments are given in section 6. Finally the paper ends with some concluding remarks in section 7.

## 2 Preparatory Results

For proving the results in [2] as well as the results in this paper we use the following Lemma on majorization, which is a reformulation of Lemma 1.3.12 in [1].

**LEMMA 2.1** *Let* $(t_n)$ *be a majorizing sequence for* $(x_n)$ *in* $X$. *If* $\lim_{n\longrightarrow\infty} t_n = t^*$, *then* $x^* = \lim_{n\longrightarrow\infty} x_n$ *exists and*

$$\|x^* - x_n\| \le t^* - t_n, \forall n \ge 0. \tag{5}$$

Let $(\tilde{t}_n), n \ge 0$, be defined iteratively by $\tilde{t}_0 = 0, \tilde{t}_1 = \eta$,

$$\tilde{t}_{n+1} = \tilde{t}_n + \frac{k_0 \eta}{(1-\tilde{r})}(\tilde{t}_n - \tilde{t}_{n-1}) \tag{6}$$

where $\tilde{r} \in [0, 1)$.

**LEMMA 2.2** *([2], Lemma 2.2) Assume there exist nonnegative numbers* $k_0, \eta$ *and* $\tilde{r} \in [0, 1)$ *such that*

$$\frac{k_0}{(1-\tilde{r})}\eta \le \tilde{r}. \tag{7}$$

*Then the sequence* $(\tilde{t}_n)$ *defined in (6) is increasing, bounded above by* $\tilde{t}^{**} := \frac{\eta}{1-\tilde{r}}$, *and converges to some* $\tilde{t}^*$ *such that* $0 < \tilde{t}^* \le \frac{\eta}{1-\tilde{r}}$. *Moreover, for* $n \ge 0$;

$$0 \le \tilde{t}_{n+1} - \tilde{t}_n \le \tilde{r}(\tilde{t}_n - \tilde{t}_{n-1}) \le \tilde{r}^n \eta, \tag{8}$$

*and*

$$\tilde{t}^* - \tilde{t}_n \le \frac{\tilde{r}^n}{1-\tilde{r}}\eta. \tag{9}$$

The following Lemma based on the Assumption 1.3 will be used in due course.

**LEMMA 2.3** *([2],Lemma 2.3) For* $u, v, x_0 \in B_{r_0}(\hat{x})$

$$F(v) - F(u) - F'(x_0)(v-u) = F'(x_0)\int_0^1 \Phi(u + t(v-u), x_0, v-u)dt.$$

Here after we assume that $\|x_0 - \hat{x}\| \le \rho$ and

$$\frac{k_0}{2}\rho^2 + \rho + \frac{\delta}{\alpha} \le \eta \le min\{\frac{\tilde{r}(1-\tilde{r})}{k_0}, r_0(1-\tilde{r})\}. \tag{10}$$

**THEOREM 2.4** *([2],Theorem 2.4) Suppose (6) holds. Let the assumptions in Lemma 2.2 with* $\eta$ *as in (10) and Assumption 1.3 be satisfied. Then the sequence* $(x^\delta_{n,\alpha})$ *defined in (4) is well defined and* $x^\delta_{n,\alpha} \in B_{\tilde{t}^*}(x_0)$ *for all* $n \ge 0$. *Further* $(x^\delta_{n,\alpha})$ *is a Cauchy sequence in* $B_{\tilde{t}^*}(x_0)$ *and hence converges to* $x^\delta_\alpha \in \overline{B_{\tilde{t}^*}(x_0)} \subset B_{\tilde{t}^{**}}(x_0)$ *and* $F(x^\delta_\alpha) + \alpha(x^\delta_\alpha - x_0) = y^\delta$.

*Moreover, the following estimate hold for all* $n \ge 0$,

$$\|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\| \le \tilde{t}_{n+1} - \tilde{t}_n, \tag{11}$$

*and*

$$\|x^\delta_{n,\alpha} - x^\delta_\alpha\| \le \tilde{t}^* - \tilde{t}_n \le \frac{\tilde{r}^n \eta}{(1-\tilde{r})}. \tag{12}$$

# 3 Iteratively Regularized Projection Method

Let $H$ be a bounded subset of positive reals such that zero is a limit point of $H$, and let $\{P_h\}_{h\in H}$ be a family of orthogonal projections from $X$ into itself. Let

$$\Gamma_h := \|(I-P_h)F'(x_0)\| \tag{13}$$

and

$$\gamma_h := \|F'(P_hx_0)(I-P_h)\|. \tag{14}$$

We assume that

$$b_h := \|(I-P_h)x_0\| \to 0 \tag{15}$$

as $h\to 0$. The above assumption is satisfied if $P_h \to I$ pointwise. Let $(\tilde{t}_{n,h}), n \geq 0$ be defined iteratively by $\tilde{t}_{0,h}=0, \tilde{t}_{1,h}=\eta_h$,

$$\tilde{t}_{n+1,h} = \tilde{t}_{n,h} + (1+\frac{\gamma_h}{\alpha})\frac{k_0\eta_h}{(1-r_h)}(\tilde{t}_{n,h}-\tilde{t}_{n-1,h}) \tag{16}$$

where $k_0, \alpha$ and $r_h \in [0,1)$ are nonnegative numbers, with $(1+\frac{\gamma_h}{\alpha})\frac{k_0}{(1-r_h)}\eta_h \leq r_h$. We need the following Lemma, proof of which is analogous to the proof of Lemma2.2 in [2], so we ignore the proof.

**LEMMA 3.1** *Assume there exist nonnegative numbers $k_0, \alpha$ and $r_h \in [0,1)$ such that*

$$(1+\frac{\gamma_h}{\alpha})\frac{k_0}{(1-r_h)}\eta_h \leq r_h. \tag{17}$$

*Then the sequence $(\tilde{t}_{n,h})$ defined in (16) is increasing, bounded above by $\tilde{t_h}^{**} := \frac{\eta_h}{1-r_h}$, and converges to some $\tilde{t_h}^{*}$ such that $0 < \tilde{t_h}^{*} \leq \frac{\eta_h}{1-r_h}$. Moreover, for $n\geq 0$;*

$$0 \leq \tilde{t}_{n+1,h} - \tilde{t}_{n,h} \leq r_h(\tilde{t}_{n,h}-\tilde{t}_{n-1,h}) \leq r_h^n\eta_h, \tag{18}$$

*and*

$$\tilde{t_h}^{*} - \tilde{t}_{n,h} \leq \frac{r_h^n}{1-r_h}\eta_h. \tag{19}$$

Let

$$x_{n+1,\alpha}^{h,\delta} := x_{n,\alpha}^{h,\delta} - (P_hF'(P_hx_0)+\alpha I)^{-1}P_h(F(x_{n,\alpha}^{h,\delta}) - y^\delta + \alpha(x_{n,\alpha}^{h,\delta} - x_0)), \tag{20}$$

where $x_{0,\alpha}^{h,\delta} := P_hx_0$. Now we shall prove that the sequence $(\tilde{t}_{n,h})$ is a majorizing sequence of the sequence $(x_{n,\alpha}^{h,\delta})$.

Let

$$\begin{aligned}(1+\frac{\gamma_h}{\alpha})(\frac{k_0}{2}(b_h+\rho)^2 + b_h+\rho) + \frac{\delta}{\alpha} &\leq \eta_h \\ &\leq min\{\frac{r_h(1-r_h)}{k_0(1+\gamma_h/\alpha)}, r_0(1-r_h)\}.\end{aligned} \tag{21}$$

**THEOREM 3.2** *Let the assumptions in Lemma 3.1 with $\eta_h$ as in (21) and Assumption 1.3 be satisfied. Then the sequence $(\tilde{t}_{n,h})$ defined in (16) is a majorizing sequence of sequence $(x_{n,\alpha}^{h,\delta})$ defined in (20) and $x_{n,\alpha}^{h,\delta} \in B_{\tilde{t_h}^*}(P_h x_0)$ for all $n \geq 0$.*

**Proof.** Let

$$G(x) = x - R_\alpha(P_h x_0)^{-1}[F(x) - y^\delta + \alpha(x - x_0)]$$

where $R_\alpha(P_h x_0)^{-1} = (P_h F'(P_h x_0) P_h + \alpha P_h)^{-1}$. Then since $R_\alpha(P_h x_0)^{-1} = R_\alpha(P_h x_0)^{-1} P_h = P_h R_\alpha(P_h x_0)^{-1}$; for $u, v \in B_{\tilde{t_h}^*}(P_h x_0)$,

$$\begin{aligned} G(u) - G(v) &= u - v - R_\alpha(P_h x_0)^{-1}[F(u) - y^\delta + \alpha(u - x_0)] \\ &\quad + R_\alpha(P_h x_0)^{-1}[F(v) - y^\delta + \alpha(v - x_0)] \\ &= R_\alpha(P_h x_0)^{-1}[R_\alpha(P_h x_0)(u - v) - (F(u) - F(v))] \\ &\quad + \alpha R_\alpha(P_h x_0)^{-1}(v - u) \\ &= R_\alpha(P_h x_0)^{-1}[F'(P_h x_0) P_h(u - v) - (F(u) - F(v)) + \alpha(u - v)] \\ &\quad + \alpha R_\alpha(P_h x_0)^{-1}(v - u) \\ &= R_\alpha(P_h x_0)^{-1}[F'(P_h x_0) P_h(u - v) - (F(u) - F(v))]. \end{aligned}$$

Now since $G(x_{n,\alpha}^{h,\delta}) = x_{n+1,\alpha}^{h,\delta}$ and $P_h(x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}) = (x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta})$ we have

$$\begin{aligned} (x_{n+1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}) &= G(x_{n,\alpha}^{h,\delta}) - G(x_{n-1,\alpha}^{h,\delta}) \\ &= R_\alpha(P_h x_0)^{-1}[F'(P_h x_0)(x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}) \\ &\quad -(F(x_{n,\alpha}^{h,\delta}) - F(x_{n-1,\alpha}^{h,\delta}))] \\ &= R_\alpha(P_h x_0)^{-1} F'(P_h x_0) \\ &\quad \times \int_0^1 \Phi(x_{n,\alpha}^{h,\delta} + t(x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}), P_h x_0, x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}) dt \\ &= R_\alpha(P_h x_0)^{-1}[F'(P_h x_0) P_h + F'(P_h x_0)(I - P_h)] \\ &\quad \times \int_0^1 \Phi(x_{n,\alpha}^{h,\delta} + t(x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}), P_h x_0, x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}) dt \end{aligned}$$

The last but one step follows from Lemma 2.3. So by Assumption 1.3 and the relation

$$\|R_\alpha(P_h x_0)^{-1}[F'(P_h x_0) P_h + F'(P_h x_0)(I - P_h)]\| \leq 1 + \frac{\gamma_h}{\alpha} \tag{22}$$

we have

$$\begin{aligned} \|x_{n+1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}\| &\leq (1 + \frac{\gamma_h}{\alpha}) k_0 \|x_{n,\alpha}^{h,\delta} + t(x_{n-1,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}) - P_h x_0\| \\ &\quad \times \|x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}\|. \end{aligned} \tag{23}$$

Now we shall prove that the sequence $(\tilde{t}_{n,h})$ defined in (16) is a majorizing sequence of the sequence $(x_{n,\alpha}^{h,\delta})$ and $x_{n,\alpha}^{h,\delta} \in B_{\tilde{t_h}^*}(P_h x_0)$, for all $n \geq 0$. Note that $F(\hat{x}) = y$, so

$$\begin{aligned}
\|x_{1,\alpha}^{h,\delta} - P_h x_0\| &= \|(P_h F'(P_h x_0) + \alpha I)^{-1} P_h (F(P_h x_0) - y^\delta)\| \\
&= \|(P_h F'(P_h x_0) + \alpha I)^{-1} P_h (F(P_h x_0) - y + y - y^\delta)\| \\
&= \|(P_h F'(P_h x_0) + \alpha I)^{-1} P_h (F(P_h x_0) - F(\hat{x}) + y - y^\delta)\| \\
&= \|(P_h F'(P_h x_0) + \alpha I)^{-1} P_h (F(P_h x_0) - F(\hat{x}) - F'(P_h x_0)(P_h x_0 - \hat{x}) \\
&\quad + F'(P_h x_0)(P_h x_0 - \hat{x}) + y - y^\delta)\| \\
&\leq \|(P_h F'(P_h x_0) + \alpha I)^{-1} P_h (F(P_h x_0) - F(\hat{x}) - F'(P_h x_0)(P_h x_0 - \hat{x}))\| \\
&\quad + \|(P_h F'(P_h x_0) + \alpha I)^{-1} P_h F'(P_h x_0)(P_h x_0 - \hat{x})\| \\
&\quad + \|(P_h F'(P_h x_0) + \alpha I)^{-1} P_h (y - y^\delta)\| \\
&\leq \|(P_h F'(P_h x_0) + \alpha I)^{-1} P_h F'(P_h x_0) \\
&\quad \times \int_0^1 \Phi(\hat{x} + t(P_h x_0 - \hat{x}), P_h x_0, (P_h x_0 - \hat{x}))dt\| \\
&\quad + \|(P_h F'(P_h x_0) + \alpha I)^{-1} P_h F'(P_h x_0)(P_h x_0 - \hat{x})\| + \frac{\delta}{\alpha} \\
&\leq \|(P_h F'(P_h x_0) + \alpha I)^{-1} P_h [F'(P_h x_0) P_h + F'(P_h x_0)(I - P_h)] \\
&\quad \times \int_0^1 \Phi(\hat{x} + t(P_h x_0 - \hat{x}), P_h x_0, (P_h x_0 - \hat{x}))dt\| \\
&\quad + \|(P_h F'(P_h x_0) + \alpha I)^{-1} P_h [F'(P_h x_0) P_h \\
&\quad + F'(P_h x_0)(I - P_h)](P_h x_0 - \hat{x})\| + \frac{\delta}{\alpha} \\
&\leq (1 + \frac{\gamma_h}{\alpha})(\frac{k_0}{2}\|P_h x_0 - \hat{x}\|^2 \\
&\quad + \|P_h x_0 - \hat{x}\|) + \frac{\delta}{\alpha} \\
&\leq (1 + \frac{\gamma_h}{\alpha})(\frac{k_0}{2}(b_h + \rho)^2 + b_h + \rho) + \frac{\delta}{\alpha} \\
&\leq \eta_h.
\end{aligned}$$

The last but one step follows from Assumption 1.3, (22) and the inequality $\|P_h x_0 - \hat{x}\| \leq b_h + \rho$. So $\|x_{1,\alpha}^{h,\delta} - P_h x_0\| \leq \tilde{t}_{1,h} - \tilde{t}_{0,h}$. Assume that

$$\|x_{i+1,\alpha}^{h,\delta} - x_{i,\alpha}^{h,\delta}\| \leq \tilde{t}_{i+1,h} - \tilde{t}_{i,h}, \qquad \forall i \leq k \tag{24}$$

for some $k$. Then

$$\begin{aligned}
\|x_{k+1,\alpha}^{h,\delta} - P_h x_0\| &\leq \|x_{k+1,\alpha}^{h,\delta} - x_{k,\alpha}^{h,\delta}\| + \|x_{k,\alpha}^{h,\delta} - x_{k-1,\alpha}^{h,\delta}\| + \cdots + \|x_{1,\alpha}^{h,\delta} - P_h x_0\| \\
&\leq \tilde{t}_{k+1,h} - \tilde{t}_{k,h} + \tilde{t}_{k,h} - \tilde{t}_{k-1,h} + \cdots + \tilde{t}_{1,h} - \tilde{t}_{0,h} \\
&= \tilde{t}_{k+1,h} \leq \tilde{t_h}^*.
\end{aligned}$$

So $x^{h,\delta}_{i+1,\alpha} \in B_{\tilde{t_h}^*}(P_h x_0)$ for all $i \leq k$, and hence, $x^{h,\delta}_{k+1,\alpha} + t(x^{h,\delta}_{k,\alpha} - x^{h,\delta}_{k+1,\alpha}) \in B_{\tilde{t_h}^*}(P_h x_0)$. Therefore by (23) and (24) we have

$$\begin{aligned}\|x^{h,\delta}_{k+2,\alpha} - x^{h,\delta}_{k+1,\alpha}\| &\leq k_0(1 + \frac{\gamma_h}{\alpha})\tilde{t_h}^* \|x^{h,\delta}_{k+1,\alpha} - x^{h,\delta}_{k,\alpha}\| \\ &\leq k_0(1 + \frac{\gamma_h}{\alpha})\frac{\eta_h}{(1-r_h)}(\tilde{t}_{k+1,h} - \tilde{t}_{k,h}) \\ &= \tilde{t}_{k+2,h} - \tilde{t}_{k+1,h}.\end{aligned}$$

Thus by induction $\|x^{h,\delta}_{n+1,\alpha} - x^{h,\delta}_{n,\alpha}\| \leq \tilde{t}_{n+1,h} - \tilde{t}_{n,h}$ for all $n \geq 0$ and hence $(\tilde{t}_{n,h}), n \geq 0$ is a majorizing sequence of the sequence $(x^{h,\delta}_{n,\alpha})$. In particular $\|x^{h,\delta}_{n,\alpha} - P_h x_0\| \leq \tilde{t}_{n,h} \leq \tilde{t_h}^*$, i.e., $x^{h,\delta}_{n,\alpha} \in B_{\tilde{t_h}^*}(P_h x_0)$, for all $n \geq 0$.

Hence

$$\|x^{h,\delta}_{n,\alpha} - P_h x_0\| \leq \tilde{t_h}^* \leq \frac{\eta_h}{1 - r_h}. \tag{25}$$

This completes the proof.

Let

$$\tilde{\tilde{r}} := max\{\tilde{r}, r_h\}, \tag{26}$$

and

$$q := \frac{1}{2}[2\tilde{\tilde{r}} + k_0 b_h]. \tag{27}$$

Note that for $0 < b_h < \frac{2(1-\tilde{\tilde{r}})}{k_0}$, $q < 1$.

**THEOREM 3.3** *Let $x^{h,\delta}_{n,\alpha}$ be as in (20) and $x^{\delta}_{n,\alpha}$ be as in (4). Let assumptions in Theorem 2.4 and Theorem 3.2 hold. Then we have the following estimate,*

$$\|x^{h,\delta}_{n,\alpha} - x^{\delta}_{n,\alpha}\| \leq q^n b_h + (\frac{\Gamma_h + k_0\|F'(x_0)\|b_h}{\alpha})\frac{q^n}{(q - r_h)}\eta_h.$$

**Proof.**Note that

$$\begin{aligned}x^{h,\delta}_{n,\alpha} - x^{\delta}_{n,\alpha} &= x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha} - (P_h F'(P_h x_0) + \alpha I)^{-1} P_h \\ &\quad (F(x^{h,\delta}_{n-1,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{n-1,\alpha} - x_0)) \\ &\quad + (F'(x_0) + \alpha I)^{-1}(F(x^{\delta}_{n-1,\alpha}) - y^\delta + \alpha(x^{\delta}_{n-1,\alpha} - x_0)) \\ &= x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha} - [(P_h F'(P_h x_0) + \alpha I)^{-1} P_h - (F'(x_0) + \alpha I)^{-1}] \\ &\quad (F(x^{h,\delta}_{n-1,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{n-1,\alpha} - x_0)) \\ &\quad - (F'(x_0) + \alpha I)^{-1}[F(x^{h,\delta}_{n-1,\alpha}) - F(x^{\delta}_{n-1,\alpha}) \\ &\quad + \alpha(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha})] \\ &= (F'(x_0) + \alpha I)^{-1}[F'(x_0)(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}) \\ &\quad - (F(x^{h,\delta}_{n-1,\alpha}) - F(x^{\delta}_{n-1,\alpha}))]\end{aligned}$$

$$
\begin{aligned}
& -(F'(x_0)+\alpha I)^{-1}[F'(x_0)P_h - P_hF'(P_hx_0)P_h](P_hF'(P_hx_0)+\alpha I)^{-1} \\
& P_h[(F(x^{h,\delta}_{n-1,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{n-1,\alpha} - x_0))] \\
= \;& (F'(x_0)+\alpha I)^{-1}[F'(x_0)(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}) \\
& -(F(x^{h,\delta}_{n-1,\alpha}) - F(x^{\delta}_{n-1,\alpha}))] \\
& -(F'(x_0)+\alpha I)^{-1}[F'(x_0) - P_hF'(x_0) \\
& +P_hF'(x_0) - P_hF'(P_hx_0)](x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha}) \\
=: \;& \Gamma_1 - \Gamma_2. \qquad (28)
\end{aligned}
$$

where

$$\Gamma_1 = (F'(x_0)+\alpha I)^{-1}[F'(x_0)(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}) - (F(x^{h,\delta}_{n-1,\alpha}) - F(x^{\delta}_{n-1,\alpha}))]$$

and

$$
\begin{aligned}
\Gamma_2 \;=\;& (F'(x_0)+\alpha I)^{-1}[F'(x_0) - P_hF'(x_0) \\
& +P_hF'(x_0) - P_hF'(P_hx_0)](x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha}).
\end{aligned}
$$

Note that by Lemma 2.3

$$
\begin{aligned}
\|\Gamma_1\| \;\leq\;& \|(F'(x_0)+\alpha I)^{-1}F'(x_0)\int_0^1 \Phi(x^{h,\delta}_{n-1,\alpha} + t(x^{\delta}_{n-1,\alpha} \\
& -x^{h,\delta}_{n-1,\alpha}), x_0, x^{\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha})dt\| \\
\leq\;& k_0\int_0^1 \|x_0 - (x^{h,\delta}_{n-1,\alpha} + t(x^{\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}))\|\|x^{\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}\|dt \\
\leq\;& k_0\int_0^1 [t\|x_0 - x^{\delta}_{n-1,\alpha}\| + (1-t)\|P_hx_0 - x^{h,\delta}_{n-1,\alpha}\| \\
& +(1-t)\|P_hx_0 - x_0\|]\|x^{\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}\|dt \\
\leq\;& \frac{k_0}{2}[\frac{\eta}{1-\tilde{r}} + \frac{\eta_h}{1-r_h} + b_h]\|x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}\| \\
\leq\;& \frac{1}{2}[\tilde{r} + r_h + k_0b_h]\|x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}\| \\
\leq\;& \frac{1}{2}[2\tilde{\tilde{r}} + k_0b_h]\|x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}\| \\
\leq\;& q\|x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}\| \qquad (29)
\end{aligned}
$$

and by Assumption 1.3

$$
\begin{aligned}
\|\Gamma_2\| \;=\;& \|(F'(x_0)+\alpha I)^{-1}[(I-P_h)F'(x_0) \\
& -P_h(F'(P_hx_0) - F'(x_0))](x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha})\| \\
\leq\;& \|(F'(x_0)+\alpha I)^{-1}(I-P_h)F'(x_0)\| \\
& +\|(F'(x_0)+\alpha I)^{-1}P_hF'(x_0)\Phi(P_hx_0, x_0, x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha}) \\
\leq\;& (\frac{\Gamma_h + k_0\|F'(x_0)\|b_h}{\alpha})\|x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha}\|. \qquad (30)
\end{aligned}
$$

Therefore by (28), (29)and (30) we have

$$\begin{aligned}
\|x_{n,\alpha}^{h,\delta} - x_{n,\alpha}^{\delta}\| &\leq q\|x_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{\delta}\| + \frac{\Gamma_h + k_0\|F'(x_0)\|b_h}{\alpha}\|x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}\| \\
&\leq q^n b_h + \frac{\Gamma_h + k_0\|F'(x_0)\|b_h}{\alpha}\eta_h(r_h^{n-1} + qr_h^{n-2} + \cdots + q^{n-1}) \\
&\leq q^n b_h + (\frac{\Gamma_h + k_0\|F'(x_0)\|b_h}{\alpha})\frac{q^n}{(q - r_h)}\eta_h.
\end{aligned}$$

This completes the proof.

# 4 Error Bounds Under Source Conditions

It is known (cf.[12], Proposition 3.1) that

$$\|x_\alpha^\delta - x_\alpha\| \leq \frac{\delta}{\alpha} \tag{31}$$

and (cf.[2], Theorem 3.1) that

$$\|x_\alpha - \hat{x}\| \leq (k_0 r_0 + 1)c_\varphi \varphi(\alpha). \tag{32}$$

where $x_\alpha$ is the unique solution of $F(x) + \alpha(x - x_0) = y$. Combining the estimates in Theorem 2.4, Theorem 3.3, (31) and (32) we obtain the following Theorem.

**THEOREM 4.1** *Let $x_{n,\alpha}^{h,\delta}$ be as in (20) and let the assumptions in Theorem 2.4 and Theorem 3.3 be satisfied. Then we have the following;*

$$\|x_{n,\alpha}^{h,\delta} - \hat{x}\| \leq q^n b_h + (\frac{\Gamma_h + k_0\|F'(x_0)\|b_h}{\alpha})\frac{q^n}{(q - r_h)}\eta_h + \frac{\tilde{r}^n \eta}{1 - \tilde{r}} + \frac{\delta}{\alpha} + (k_0 r_0 + 1)c_\varphi \varphi(\alpha). \tag{33}$$

Let

$$n_\delta := min\{n : max\{q^n, \tilde{r}^n\} \leq \delta\} \tag{34}$$

and let

$$C := max\{b_h + \frac{\Gamma_h + k_0\|F'(x_0)\|b_h}{(q - r_h)}\eta_h + \frac{\eta}{1 - \tilde{r}} + 1, (k_0 r_0 + 1)c_\varphi\}. \tag{35}$$

**THEOREM 4.2** *Let $x_{n,\alpha}^{h,\delta}$ be as in (20) and let the assumptions in Theorem 2.4and Theorem 3.3 be satisfied. Let $n_\delta$ be as in (34) and $C$ be as in (35). Then for all $0 < \alpha \leq 1$ we have the following;*

$$\|x_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| \leq C(\varphi(\alpha) + \frac{\delta}{\alpha}). \tag{36}$$

## 4.1 A priori choice of the parameter

Note that the error $\varphi(\alpha) + \frac{\delta}{\alpha}$ in (36) is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies, $\alpha_\delta \varphi(\alpha_\delta) = \delta$. Now using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda), 0 < \lambda \leq a$ we have $\delta = \alpha_\delta\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, so that $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. Hence by (36) we have the following.

**THEOREM 4.3** *Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a$, and assumptions in Theorem 4.2 holds. For $\delta > 0$, let $\alpha =: \alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. Let $n_\delta$ be as in (34). Then*

$$\|x^{h,\delta}_{n_\delta,\alpha} - \hat{x}\| = \bigcirc(\psi^{-1}(\delta)).$$

## 4.2 An adaptive choice of the parameter

In this subsection, we will present a parameter choice rule based on the adaptive method studied in [7, 11].

In practice, the regularization parameter $\alpha$ is often selected from some finite set

$$D_M(\alpha) := \{\alpha_i = \mu^i\alpha_0, i = 0, 1, \cdots, M\} \tag{37}$$

where $\mu > 1$ and $M$ is such that $\alpha_M < 1 \leq \alpha_{M+1}$. We choose $\alpha_0 := \sqrt{\delta}$, because in general $\varphi(\lambda) = \lambda^\nu, 0 < \nu \leq 1$ and in this case the best possible error estimate is order $\bigcirc(\sqrt{\delta})$ and from Theorem 4.3, it follows that such an accuracy cannot be guaranteed for $\alpha < \sqrt{\delta}$.

Let

$$n_M := min\{n : max\{q^n, \tilde{r}^n\} \leq \delta\} \tag{38}$$

and let $x_i := x^{h,\delta}_{n_M,\alpha_i}$. The parameter choice strategy that we are going to consider in this paper, we select $\alpha = \alpha_i$ from $D_M(\alpha)$ and operates only with corresponding $x_i, \quad i = 0, 1, \cdots, M$.

**THEOREM 4.4** *Assume that there exists $i \in \{0, 1, 2, \cdots, M\}$ such that $\varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}$. Let assumptions of Theorem 4.2 and Theorem 4.3 hold and let*

$$l := max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}\} < M,$$

$$k := max\{i : \|x_i - x_j\| \leq 4C\frac{\delta}{\alpha_j}, j = 0, 1, 2, \cdots, i\}. \tag{39}$$

*Then $l \leq k$ and*

$$\|\hat{x} - x_k\| \leq c\psi^{-1}(\delta)$$

*where $c = 6C\mu$.*

**Proof.** To see that $l \leq k$, it is enough to show that, for each $i \in \{1, 2, \cdots, M\}$,

$$\varphi(\alpha_i) \leq \frac{\delta}{\alpha_i} \Longrightarrow \|x_i - x_j\| \leq 4C\frac{\delta}{\alpha_j}, \quad \forall j = 0, 1, \cdots, i.$$

For $j \leq i$, by (36) we have

$$\begin{aligned} \|x_i - x_j\| &\leq \|x_i - \hat{x}\| + \|\hat{x} - x_j\| \\ &\leq C(\varphi(\alpha_i) + \frac{\delta}{\alpha_i}) + C(\varphi(\alpha_j) + \frac{\delta}{\alpha_j}) \\ &\leq 2C\frac{\delta}{\alpha_i} + 2C\frac{\delta}{\alpha_j}. \\ &\leq 4C\frac{\delta}{\alpha_j}. \end{aligned}$$

Thus the relation $l \leq k$ is proved. Next we observe that

$$\begin{aligned} \|\hat{x} - x_k\| &\leq \|\hat{x} - x_l\| + \|x_l - x_k\| \\ &\leq C(\varphi(\alpha_l) + \frac{\delta}{\alpha_l}) + 4C\frac{\delta}{\alpha_l} \\ &\leq 6C\frac{\delta}{\alpha_l}. \end{aligned}$$

Now since $\alpha_\delta \leq \alpha_{l+1} \leq \mu\alpha_l$, it follows that

$$\frac{\delta}{\alpha_l} \leq \mu\frac{\delta}{\alpha_\delta} = \mu\varphi(\alpha_\delta) = \mu\psi^{-1}(\delta).$$

This completes the proof of the theorem.

# 5 Implementation of Adaptive Choice Rule

In this section we provide an algorithm for the determination of a parameter fulfilling the balancing principle (39) and also provide a starting point for the iteration (20) approximating the unique solution $x_\alpha^\delta$ of (3). The choice of the starting point involves the following steps:

- Choose $\alpha_0 = \sqrt{\delta}$, $\mu > 1$ and $q < 1$.
- Choose $x_0 \in D(F)$ such that $\|x_0 - \hat{x}\| \leq \rho$ and $(1 + \frac{\gamma_h}{\alpha_0})(\frac{k_0}{2}(b_h + \rho)^2 + b_h + \rho) + \frac{\delta}{\alpha_0} \leq \eta_h \leq min\{\frac{(1-r_h)r_h}{k_0(1+\frac{\gamma_h}{\alpha_0})}, r_0(1 - r_h)\}$.

Choose $n_M$ such that $n_M = min\{n : max\{q^n, \tilde{r}^n\} \leq \delta\}$.

Finally the adaptive algorithm associated with the choice of the parameter specified in Theorem 4.4 involves the following steps:

### 5.1 Algorithm

- Set $i \leftarrow 0$
- solve $x_i := x^{h,\delta}_{n_M,\alpha_i}$ by using the iteration (20).
- If $\|x_i - x_j\| > 4C\frac{\sqrt{\delta}}{\mu^j}, \quad j \le i$, then take $k = i - 1$.
- Set $i = i + 1$ and return to step 2.

## 6 Examples

In this section we consider some simple examples satisfying the assumptions made in the paper and presents a few computed examples.

We consider the operator $F : L^2[0,1] \to L^2[0,1]$ defined by (cf.[10], Example 6.1)

$$F(x)(s) = K^*K(x)(s) + f(s), \qquad x, f \in L^2[0,1], s \in [0,1] \tag{40}$$

where $K : L^2[0,1] \to L^2[0,1]$ is a compact linear operator such that the range of $K$ denoted by $R(K)$ is not closed in $L^2[0,1]$. Then the equation $F(x) = y$ is ill-posed as $K$ is compact with non-closed range. The Frèchet derivative $F'(.)$ of $F$ is given by

$$F'(x)z = K^*Kz, \qquad \forall x, z \in L^2[0,1]. \tag{41}$$

So $F$ is monotone on $L^2[0,1]$. Further for $x, y, z \in L^2[0,1]$

$$[F'(x) - F'(y)]z = 0. \tag{42}$$

Hence Assumption 1.3 holds trivially. Again note that, since $\Phi(x,y,z) = 0 \le k_0\|z\|\|x-y\|, \ \forall k_0 \ge 0$ we can choose $\eta_h$ large enough in step 2 of the algorithm.

Further , due to (41) the iteration $x^{h,\delta}_{m+1,\alpha}$ needs only one step to compute. This can be seen as follows:

$$x^{h,\delta}_{m+1,\alpha} = x^{h,\delta}_{m,\alpha} - (P_hF'(P_hx_0) + \alpha I)^{-1}P_h[F(x^{h,\delta}_{m,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{m,\alpha} - x_0)]$$

i.e.,

$$\begin{aligned}
(P_hF'(P_hx_0) + \alpha I)P_hx^{h,\delta}_{m+1,\alpha} &= (P_hF'(P_hx_0) + \alpha I)P_hx^{h,\delta}_{m,\alpha} \\
&\quad -P_h[F(x^{h,\delta}_{m,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{m,\alpha} - x_0)] \\
&= (P_hK^*K + \alpha I)P_hx^{h,\delta}_{m,\alpha} - P_h[K^*Kx^{h,\delta}_{m,\alpha} \\
&\quad +f - y^\delta + \alpha(x^{h,\delta}_{m,\alpha} - x_0)] \\
&= -P_h(f - y^\delta - \alpha x_0).
\end{aligned} \tag{43}$$

Now we shall give the details for implementing the algorithm given in the above section. Let $(V_n)$ be a sequence of finite dimensional subspaces of $X$ and let $P_h, h = 1/n$ denote the orthogonal projection on $X$ with range $R(P_h) = V_n$. We assume that $dim V_n = n+1$, and $\|P_h x - x\| \to 0$ as $h \to 0$ for all $x \in X$. Let$\{v_1, v_2, \cdots, v_{n+1}\}$ be a basis of $V_n, n = 1, 2, \cdots$.

Note that $x^{h,\delta}_{m+1,\alpha} \in V_n$. Thus $x^{h,\delta}_{m+1,\alpha}$ is of the form $\sum_{i=1}^{n+1} \lambda_i v_i$ for some scalars $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$. It can be seen that $x^{h,\delta}_{m+1,\alpha}$ is a solution of (43) if and only if $\bar{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_{n+1})^T$ is the unique solution of

$$(M_n + \alpha B_n)\bar{\lambda} = \bar{a} \tag{44}$$

where

$$M_n = (\langle K v_i, K v_j \rangle), i, j = 1, 2, \cdots, n+1$$
$$B_n = (\langle v_i, v_j \rangle), i, j = 1, 2, \cdots, n+1$$

and

$$\bar{a} = (\langle P_h(y^\delta + \alpha x_0 - f), v_i \rangle)^T, i = 1, 2, \cdots, n+1.$$

Note that (44) is uniquely solvable because $M_n$ is a positive definite matrix (i.e., $x M_n x^T > 0$ for all non-zero vector x) and $B_n$ is an invertible matrix.

## 6.1 Numerical Examples

In order to illustrate the method considered in the above section, we consider the space $X = Y = L^2[0,1]$ and consider $K : L^2[0,1] \to L^2[0,1]$ as the Fredholm integral operator

$$K(x)(s) = \int_0^1 k(s,t)x(t)dt \tag{45}$$

with

$$k(t,s) = \begin{cases} 0, & t \le s \\ t - s, & t > s. \end{cases} \tag{46}$$

We apply the Algorithm in section 5 by choosing $V_n$ as the space of linear splines in a uniform grid of $n+1$ points in $[0,1]$. Specifically for fixed $n$ we consider $t_i = \frac{i-1}{n}, i = 1, 2, \cdots, n+1$ as the grid points. We take the basis function $v_i, i = 1, 2, \cdots, n+1$ of $V_n$ as follows:

$$v_1(t) = \begin{cases} \frac{t_2 - t}{t_2}, & 0 = t_1 \le t \le t_2 \\ 0, & t_2 \le t \le t_{n+1} = 1 \end{cases} \tag{47}$$

for $j = 2, 3, \cdots, n$,

$$v_j(t) = \begin{cases} 0, & 0 = t_1 \le t \le t_{j-1}, \\ \frac{t - t_{j-1}}{t_j - t_{j-1}}, & t_{j-1} \le t \le t_j, \\ \frac{t_{j+1} - t}{t_{j+1} - t_j}, & t_j \le t \le t_{j+1}, \\ 0, & t_{j+1} \le t \le t_{n+1} = 1 \end{cases} \tag{48}$$

and

$$v_{n+1}(t) = \begin{cases} 0, & 0 \le t \le t_n \\ \frac{t-t_n}{t_{n+1}-t_n}, & t_n \le t \le t_{n+1}. \end{cases} \tag{49}$$

Let $P_h$ be the orthogonal projection onto $V_n$. We note that for $x \in C[0,1]$

$$\begin{aligned} \|P_h x - x\|_2 &= dist(x, R(P_h)) \\ &\le \|\pi_n x - x\|_2 \\ &\le \|\pi_n x - x\|_\infty \end{aligned}$$

where $\pi_n$ is the (piecewise linear) interpolatory projection onto $V_n$. It is known [6] that $\|\pi_n x - x\|_\infty \to 0$ as $n \to \infty$. Therefore using the fact that $C[0,1]$ is dense in $L^2[0,1]$, it follows that $\|P_h x - x\|_2 \to 0$ for all $x \in L^2[0,1]$.

The elements $Kv_i, i = 1, 2, \cdots, n+1$, the entries of the matrix $B_n, M_n$ and $\bar{a}$ are computed explicitly. For the operator $K$ defined by (45) and (46), $\Gamma_h = \gamma_h = \|(I - P_h)F'(x_0)\| = \|(I - P_h)K^*K\| = O(n^{-2})$ (see [3]).

**EXAMPLE 6.1** *In this example we take* $y = \frac{1}{720}(26 + s^6 - 6s^5 + 15s^4 - 36s) + f(s)$ *where* $f(s) = s^2$ *and* $x_0 = 0$. *Then the exact solution is* $\hat{x} = \frac{1}{2}(s-1)^2$. *Since* $\hat{x} - x_0 = \hat{x} = K^*1 \in R(K^*) = R(F'(\hat{x})^{1/2})$, $\varphi(\lambda) = \lambda^{1/2}$ *and hence* $\psi^{-1}(\delta) = \varphi(\alpha_\delta) = (\delta)^{1/3}$ *and*

$$\|\hat{x} - x_k\| \le c\psi^{-1}(\delta)$$

*where* $c = 6C\mu$. *The result are given in Table 1, Table 2 and figure 1.*
*Here and below* $e_k := \|x_k - \hat{x}\|$ *and* $y^\delta = y + \delta$.

| n | k | $e_k$ | $\frac{e_k}{\psi^{-1}(\delta)}$ |
|---|---|---|---|
| 4 | 88 | 0.0318 | 1.7300 |
| 8 | 87 | 0.0325 | 1.7505 |
| 16 | 87 | 0.0326 | 1.7520 |
| 32 | 87 | 0.0326 | 1.7532 |
| 64 | 87 | 0.0326 | 1.7536 |
| 128 | 87 | 0.0326 | 1.7537 |
| 256 | 87 | 0.0326 | 1.7537 |
| 512 | 87 | 0.0326 | 1.7537 |
| 1024 | 87 | 0.0326 | 1.7538 |

Table 1: $\delta = 0.0011; \mu = 1.01$

**EXAMPLE 6.2** *In this example we take* $y = \frac{1}{720}(s^6 + 15s^5 - 66s + 50) + f(s)$ *where* $f(s) = s^2$ *and* $x_0(s) = s$. *Then the exact solution is* $\hat{x} = \frac{1}{2}(s^2 + 1)$ *and*

| n | k | $e_k$ | $\frac{e_k}{\psi^{-1}(\delta)}$ |
|---|---|---|---|
| 4 | 4 | 0.0278 | 1.4296 |
| 8 | 4 | 0.0276 | 1.4248 |
| 16 | 4 | 0.0280 | 1.4350 |
| 32 | 4 | 0.0281 | 1.4388 |
| 64 | 4 | 0.0282 | 1.4399 |
| 128 | 4 | 0.0282 | 1.4402 |
| 256 | 4 | 0.0282 | 1.4402 |
| 512 | 4 | 0.0282 | 1.4403 |
| 1024 | 4 | 0.0282 | 1.4403 |

Table 2: $\delta = 0.0016, \mu = 1.3$

*$\hat{x} - x_0 = \frac{1}{2}(s-1)^2 = K^*1 \in R(K^*) = R(F'(\hat{x})^{1/2})$, $\varphi(\lambda) = \lambda^{1/2}$ and hence $\psi^{-1}(\delta) = \varphi(\alpha_\delta) = \delta^{1/3}$. According to the theory,*

$$\|\hat{x} - x_k\| \leq c\psi^{-1}(\delta)$$

*where $c = 6C\mu$. The results are given in Table 3, Table 4 and Figure 2.*

| n | k | $e_k$ | $\frac{e_k}{\psi^{-1}(\delta)}$ |
|---|---|---|---|
| 4 | 100 | 0.0287 | 1.6435 |
| 8 | 97 | 0.0261 | 1.5683 |
| 16 | 97 | 0.0257 | 1.5559 |
| 32 | 96 | 0.0255 | 1.5490 |
| 64 | 96 | 0.0254 | 1.5481 |
| 128 | 96 | 0.0254 | 1.5479 |
| 256 | 96 | 0.0254 | 1.5478 |
| 512 | 96 | 0.0254 | 1.5478 |
| 1024 | 96 | 0.0254 | 1.5478 |

Table 3: $\delta = 0.0011; \mu = 1.01$

**REMARK 6.3** *The last column of the tables shows that $e_k = \bigcirc(\psi^{-1}(\delta))$. During computation we observe that due to the round off error k and $e_k$ remains as a constant for large values of n.*

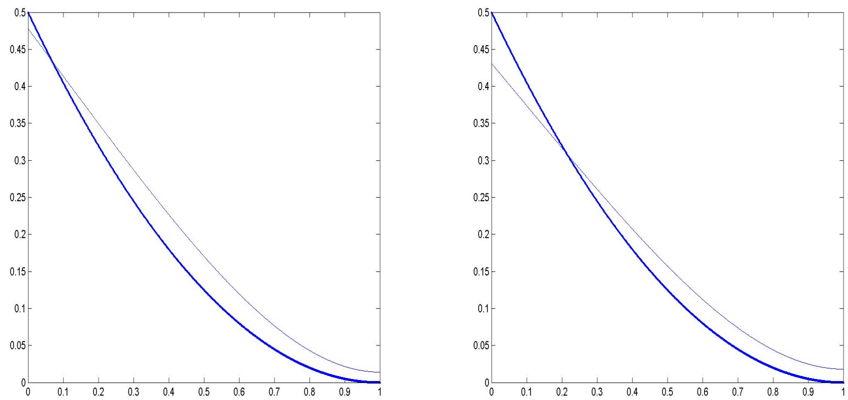


Figure 1: The curve starting from 0.5 represents the actual solution $\hat{x}$ and the other curve represents $x_k$ of Example 6.1. The left figure shows the solution for $n = 1024, \delta = 0.0011; \mu = 1.01$ and the right figure shows the solution for $n = 1024, \delta = 0.0016; \mu = 1.3$

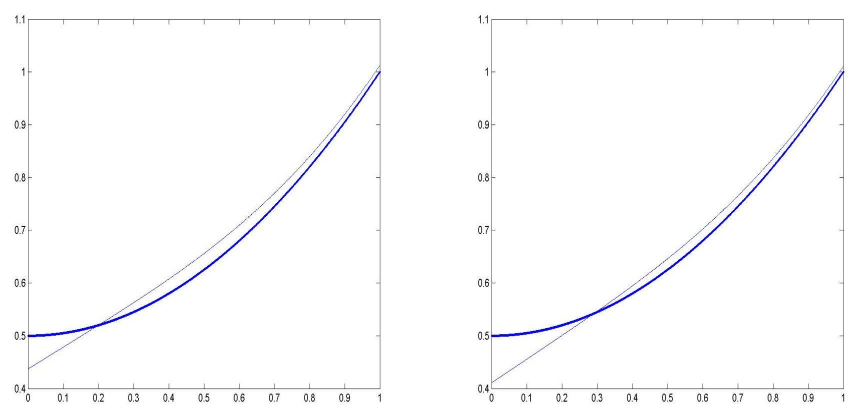


Figure 2: The curve starting from 0.5 represents the actual solution $\hat{x}$ and the other curve represents $x_k$ of Example 6.2. The left figure shows the solution for $n = 1024, \delta = 0.0011; \mu = 1.01$ and the right figure shows the solution for $n = 1024, \delta = 0.001; \mu = 1.3$

| n | k | $e_k$ | $\frac{e_k}{\psi^{-1}(\delta)}$ |
|---|---|---|---|
| 4 | 5 | 0.0301 | 1.7349 |
| 8 | 5 | 0.0290 | 1.7015 |
| 16 | 5 | 0.0287 | 1.6931 |
| 32 | 5 | 0.0286 | 1.6910 |
| 64 | 5 | 0.0286 | 1.6905 |
| 128 | 5 | 0.0286 | 1.6904 |
| 256 | 5 | 0.0286 | 1.6903 |
| 512 | 5 | 0.0286 | 1.6903 |
| 1024 | 5 | 0.0286 | 1.6903 |

Table 4: $\delta = 0.001; \mu = 1.3$

# 7 Concluding Remarks

In this paper we have considered an iteratively regularized projection method for approximately solving the nonlinear ill-posed operator equation $F(x) = y$, when the available data is $y^\delta$ in place of the exact data $y$ with $\|y - y^\delta\| \le \delta$. It is assumed that $F$ is Fréchet differentiable in a neighborhood of some initial guess $x_0$ of the actual solution $\hat{x}$. The procedure involves finding the fixed point of the function

$$G_h(x) := x - (P_h F'(P_h x_0) + \alpha I)^{-1} P_h(F(x) - y^\delta + \alpha(x - x_0)),$$

in an iterative manner in a finite dimensional subspace $X_h$ of X. Here $x_0$ is an initial guess and $P_h$ is the orthogonal projection on to $X_h$. For choosing the regularization parameter $\alpha$ we made use of the adaptive method suggested by Pereversev and Schock in [11] and the stopping rule is based on a majorizing sequence.

The numerical experiments presented in the above section support our claim that if $\alpha$ is chosen according to the balancing principle (39), then $\|x_k - \hat{x}\| \le c\psi^{-1}(\delta)$.

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