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On k-Nearly Uniformly Convex Property in Generalized Cesáro Difference Sequence Space Defined by Weighted Means

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Abstract: The main purpose of this paper is to show that the sequence space $\ell_\Delta[(a_n),(p_n),(q_n)]$ defined by N. Faried and A.A. Bakery [6] is k-nearly uniformly convex (k-NUC) for $k \geq 2$ when $\underset{n\to\infty}{Lim}\inf p_n > 1$. Therefore it is fully k-rotund (kR), NUC and has a drop property.

Keywords: Generalized Cesáro Difference sequence space, H-property, R-property, fully k-rotund (kR), Convex modular, k-nearly uniformly convex, Luxemburg norm

Introduction

Let (X, ||.||) be Banach space over the real numbers $\mathbb{R}$ and let B(X) (respec. S(X)) be the closed unit ball (resp. unit sphere) of X.

A point $x \in S(X)$ is an extreme point of B(X), if for any $y, z \in S(X)$, the equality $x = \frac{y+z}{2}$ implies $y = z$.

A Banach space X is said to be Rotund (R) if for every point of S(X) is an extreme point of B(X).Clarkson [2] who introduced the concept of uniform convexity.

A Banach space X is called uniformly convex (UC) if $\forall \varepsilon > 0\ \exists \delta > 0$ such that for any $x, y \in S(X)$, the inequality $\|x-y\| < \varepsilon$ implies that $\left\|\frac{x+y}{2}\right\| < \delta$. (1.1)

for any $x \notin B(X)$, the drop determined by x is the set

$$D(x, B(X)) = conv(\{x\} \cup B(X)). \quad (1.2)$$

Rolewicz [9], basing on Daneš drop theorem [3], introduced the notation of drop property for Banach spaces. A Banach space X has the drop property (D) if for every closed set C disjoint with B(X) $\exists\, x \in C$ such that $D(x, B(X)) \cap C = \{x\}$. (1.3)

X is said to have the property (H), if for any sequence on the unit sphere of X, weak convergence coincides norm convergence. In [10], Rolewicz proved that if the Banach space X has the drop property (D), then X is reflexive. Montesinos [12] extended this result by showing that X has the drop property if and only if X is reflexive and has the property (H).

A sequence $\{x_n\} \subset X$ is said to be ε -separated sequence for some $\varepsilon > 0$ if

$$sep(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon \quad . \quad (1.4)$$

A Banach space X is called nearly uniformly convex (NUC) if $\forall \varepsilon > 0\ \exists \delta \in (0,1)$ such that for every sequence $(x_n) \subseteq B(X)$ with $sep(x_n) \geq \varepsilon$,

we have $conv(x_n) \cap (1-\delta)B(X) \neq \phi$. (1.5)

Huff [8] proved that every NUC Banach spaces X is reflexive and it has property (H). Kutzarova [1] has defined k-nearly uniformly convex Banach spaces. Let $k \geq 2$ be an integer, a Banach space X is called k-nearly uniformly convex (k-NUC) if

$\forall \varepsilon > 0\ \exists \delta > 0$ such that for any sequence $(x_n) \subset B(X)$ with $sep(x_n) \geq \varepsilon$ there are

$n_1, n_2.n_3, \ldots, n_k \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$. Such that

$$\left\|\frac{x_{n_1} + x_{n_2} + x_{n_3} + \ldots\ldots\ldots\ldots\ldots + x_{n_k}}{k}\right\| < \delta . \quad (1.6)$$

clearly, k-NUC Banach spaces are NUC, however the opposite implication does not hold in general [1].

Fan and Gliksberg [4] have introduced k-Rotund (kR) Banach spaces.

A Banach space X is called fully k-rotund (kR) if for any sequence $(x_n)\subset B(X)$ $\left\|\frac{x_{n_1}+x_{n_2}+x_{n_3}+.................+x_{n_k}}{k}\right\|\to 1$ as $\min\{n_i:1\le i\le k\}\to\infty$ implies that (x_n) is convergent. It is well known that UC implies kR and kR implies (k+1)R, and kR spaces are reflexive and rotund. By ω, we denote the space of all real or complex sequences.

For a real vector space X, a function $\sigma: X\to[0,\infty]$ is called modular, if it satisfies the following conditions:

(i) $\sigma(x)=0 \Leftrightarrow x=0\ \forall x\in X$,

(ii) $\sigma(\lambda x)=\sigma(x)\ \forall\ \lambda\in\mathbb{R}$ with $|\lambda|=1$,

(iii) $\sigma(\lambda x+\beta y)\le\sigma(x)+\sigma(y)\ \forall x,y\in X\ \ \forall\lambda,\beta\ge 0;\ \lambda+\beta=1$.

Further, the modular σ is called convex if

(iv) $\sigma(\lambda x+\beta y)\le\lambda\sigma(x)+\beta\sigma(y)\ \forall x,y\in X\ \ \forall\lambda,\beta\ge 0;\ \lambda+\beta=1$. If σ is a modular on X, we define $X_\sigma=\{x\in X:\lim_{\lambda\to 0^-}\sigma(\lambda x)=0\}$, (1.7)

$X_\sigma^*=\{x\in X:\sigma(\lambda x)<\infty,\exists\lambda>0\}$. It is clear that $X_\sigma\subseteq X_\sigma^*$. If σ is a convex modular $\forall\ x\in X_\sigma$, we define $\|x\|=\inf\left\{\lambda>0:\sigma\left(\frac{x}{\lambda}\right)\le 1\right\}$. (1.8)

Orlicz [10] proved that if σ is a convex modular on X, then $X_\sigma = X_\sigma^*$ and $\|\cdot\|$ is a norm on X_σ for which X_σ is a Banach space. The norm $\|\cdot\|$, defined as in (1.8), is called the Luxemburg norm.

A modular σ is said to satisfy the δ_2-condition $(\sigma\in\delta_2)$ if

$\forall\ \varepsilon>0\ \exists$constants $K\ge 2$ and $a>0$ such that $\sigma(2u)\le K\sigma(u)+\varepsilon$, (1.9)

$\forall\ u\in X_\sigma$ With $\sigma(u)\le a$ .If σ satisfies the δ_2-condition

$\forall\ a>0$ with $K\ge 2$ depending on a,we say that σ satisfies the strong δ_2-condition $(\sigma\in\delta_2^s)$.

The following known results are very important for our consideration.

Theorem1.1. [13]

If $\sigma \in \delta_2^s$, then $\forall\, L > 0$ and $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such

that $|\sigma(u+v) - \sigma(u)| < \varepsilon$, (1.10)

$u, v \in X_\sigma$ With $\sigma(u) \leq L$ and $\sigma(v) \leq \delta$.

Proof. See [13, Lemma 2.1].

Theorem1.2. [13]

(1) If $\sigma \in \delta_2^s$, then $\forall\, x \in X_\sigma$, $\| x \| = 1$ if and only if $\sigma(x) = 1$.

(2) If $\sigma \in \delta_2^s$, then for any sequence (x_n) in X_σ, $\| x_n \| \to 0$ if and only if $\sigma(x_n) \to 0$.

Proof. See [13, Corollary 2.2 and Lemma 2.3].

Theorem 1.3.

If $\sigma \in \delta_2^s$, then $\forall\, \varepsilon \in (0,1)\ \exists\, \delta \in (0,1)$ such that

$\sigma(x) \leq 1 - \varepsilon$ implies $\|x\| \leq 1 - \delta$.

Proof. Suppose that the theorem does not hold, then $\exists\, \varepsilon > 0$ and (x_n) in X_σ such that $\sigma(x_n) \leq 1 - \varepsilon$, and $\frac{1}{2} \leq \| x_n \| \xrightarrow{n\to\infty} 1$. Let $a_n = \frac{1}{\|x_n\|} - 1$.

Then $a_n \xrightarrow{n\to\infty} 0$. Let $L = \sup_n \sigma(2x_n)$ .Since $\sigma \in \delta_2^s$ $\exists\, K \geq 2$ such that,

$$\sigma(2u) \leq K\sigma(u) + 1 \tag{1.11}$$

$\forall\, u \in X_\sigma$ with $\sigma(u) < 1$. By (1.11), we have $\sigma(2x_n) \leq K\sigma(x_n) + 1 < K + 1\ \forall\, n \in \mathbb{N}$.

Hence $0 \leq L < \infty$, by theorem 1.2(1), we have

$$1 = \sigma\left(\frac{x_n}{\|x_n\|}\right) = \sigma(2a_n x_n + (1 - a_n)x_n) \leq a_n\sigma(2x_n) + (1 - a_n)\sigma(x_n) \leq \tag{1.12}$$
$$a_n L + (1 - \varepsilon) \xrightarrow{n\to\infty} 1 - \varepsilon$$

, which is a contradiction.

Definition [6]: Let $(a_n), (q_n)$ and (p_n) are sequences of positive real numbers with $p_n \geq 1$ we define the space $\ell_\Delta((a_n),(p_n),(q_n)) = \{x \in \omega : \sigma(\lambda x) < \infty, \exists\, \lambda > 0\}$,

$\sigma(x)=\sum_{n=1}^{\infty}\left(a_n\sum_{k=1}^{n}q_k\mid\Delta x_k\mid\right)^{p_n}$ and $\Delta x_k = x_k - x_{k+1}$.

The Luxemburg norm on the sequence space $\ell_{\Delta}[(a_n),(p_n),(q_n)]$ is defined as follows: $\|x\|=\inf\left\{\lambda>0:\sigma(\frac{x}{\lambda})\le 1\right\}, \forall x\in\ell_{\Delta}((a_n),(p_n),(q_n))$. In the case when the sequence (p_n) is bounded we can simply write

$\ell_{\Delta}((a_n),(p_n),(q_n))=\left\{x\in\omega:\sum_{n=1}^{\infty}(a_n\sum_{k=1}^{n}q_k|\Delta x_k|)^{p_n}<\infty\right\}$. In [6] we show that

$\ell_{\Delta}[(a_n),(p_n),(q_n)]$ is a Banach space when equipped with the Luxemburg norm, possessing H-property and it is locally uniformly rotund (LUR) when $p_n>1$, for all $n\in\mathbb{N}$.

Throughout this paper, the sequence (p_n) is a bounded sequence of positive real numbers with $\underset{n\to\infty}{Lim}\inf p_n>1$ and $H=\sup_n p_n$. Let (p_k) be a bounded sequence of positive real numbers, we have $|a_k+b_k|^{p_k}\le 2^{H-1}(|a_k|^{p_k}+|b_k|^{p_k})\ \forall k\in\mathbb{N}$.

2. Main results

<u>Proposition2.1.</u>

The functional σ is convex modular on $\ell_{\Delta}[(a_n),(p_n),(q_n)]$ and satisfies the following properties:

(i) If 0<r<1, then

(ii) $r^H\sigma\left(\frac{x}{r}\right)\le\sigma(x)$ and $\sigma(rx)\le r\sigma(x)$.

(ii) If r>1, then $\sigma(x)\le r^H\sigma\left(\frac{x}{r}\right)$.

(iii) If r≥1, then $\sigma(x)\le r\sigma(x)\le\sigma(rx)$.

Proof. All assertions are clearly obtained by the definition and convexity of σ see [11].

Proposition2.2.

For any $x \in \ell_{\Delta}[(a_n),(p_n),(q_n)]$, the following assertions are satisfied:

(i) If $\| x \| < 1$, then $\sigma(x) \le \| x \|$,

(ii) if $\| x \| > 1$, then $\sigma(x) \ge \| x \|$,

(iii) $\| x \| = 1$ if and only if $\sigma(x) = 1$.

Proof: It can be proved with standard techniques in a similar way as in [11].

Proposition2.3. $\forall L > 0$ and $\forall \varepsilon > 0 \; \exists \delta > 0$ such that

$|\sigma(u+v) - \sigma(u)| < \varepsilon$, whenever $u, v \in \ell_{\Delta}[(a_n),(p_n),(q_n)]$ with $\sigma(u) \le L$ and $\sigma(v) \le \delta$.

Proof: Since (p_n) is bounded, it is easy to see that $\sigma \in \delta_2^s$ .Hence the proposition is obtained directly from theorem (1.1).

Proposition2.4. For any sequence

$(x_n) \in \ell_{\Delta}[(a_n),(p_n),(q_n)]$, $\| x_n \| \to 0$ if and only if $\sigma(x_n) \to 0$.

Proof: It follows directly from Theorem (1.2-2) since $\sigma \in \delta_2^s$.

Theorem2.5. $\forall x \in ces[(a_n),(p_n),(q_n)]$ and $\forall \varepsilon \in (0,1) \; \exists \delta \in (0,1)$ such that

$\sigma(x) \le 1 - \varepsilon$ implies $\|x\| \le 1 - \delta$.

Proof: Since $\sigma \in \delta_2^s$, the theorem is obtained directly from theorem (1.3).

Theorem2.6. The space $\ell_{\Delta}[(a_n),(p_n),(q_n)]$ is k-NUC $\forall$ integer $k \ge 2$.

Proof: Let $\varepsilon > 0$ and $(x_n) \in B(\ell_{\Delta}[(a_n),(p_n),(q_n)])$ with $sep(x_n) \ge \varepsilon$ .For each $m \in \mathbb{N}$, let $x_n^m = (0,0,\ldots\ldots, 0, x_n(m), x_n(m+1),\ldots)$. Since for each $i \in \mathbb{N}$, $(x_n(i))_{n=1}^{\infty}$ is bounded, we have that $\forall i \in \mathbb{N}$, $(x_n(i))_{n=1}^{\infty}$ is bounded, by using the diagonal method, we can find a subsequence $(x_{n_j}(i))$ of (x_n) such that $(x_{n_j}(i))$ converges for each $i \in \mathbb{N}$, $1 \le i \le m$ .Therefore, there exists an increasing sequence of positive integer (t_m) such that $sep((x_{n_j}^m)_{j > t_m}) \ge \varepsilon$. Hence, there is a sequence of positive integers $(r_m)_{m=1}^{\infty}$ with $r_1 < r_2 < r_3 < \ldots$ such that $\|x_{r_m}^m\| \ge \frac{\varepsilon}{2}$ $\forall m \in \mathbb{N}$. Then by proposition (2.4), we may assume that there exists $\eta > 0$ such that $\sigma(x_{r_m}^m) \ge \eta$ $\forall m \in \mathbb{N}$. (2.1)

Let $\alpha > 0$ be such that $1 < \alpha < \underset{n\to\infty}{Lim}\inf\ p_n$. For fixed integer $\mathrm{k} \geq 2$,

let $\varepsilon_1 = (\frac{(k^{\alpha-1}-1)}{(k-1)k^{\alpha}})(\frac{\eta}{2})$, then by proposition (2.3) $\exists\, \delta > 0$ such that

$$\left|\sigma(u+v)-\sigma(u)\right| < \varepsilon_1 . \tag{2.2}$$

Whenever $\sigma(u) \leq 1$ and $\sigma(v) \leq \delta$. Since by Proposition (2.2-i) $\sigma(x_n) \leq 1\ \forall\ n \in \mathbb{N}$ $\exists$ positive integers $m_i (i = 1,2,3,.....,k-1)$ with $m_1 < m_2 < m_3 < .... < m_{k-1}$ such that $\sigma(x_i^{m_i}) \leq \delta$ and $\alpha \leq p_j\ \forall\ j \geq m_{k-1}$ .Define $m_k = m_{k-1} + 1$. By (2.1), we have $\sigma(x_{r_{m_k}}^{m_k}) \geq \eta$. Let $s_i = i$ for $1 \leq i \leq k-1$, and $s_k = r_{m_k}$ .Then in virtue of (2.1), (2.2), and Convexity of function $f_i(u) = |u|^{p_i}\ (i \in \mathbb{N})$, we have

$$\sigma(\frac{x_{s_1}+x_{s_2}+x_{s_3}+....+x_{s_k}}{k}) = \sum_{n=1}^{\infty}\left(a_n\sum_{i=1}^{n}q_i\left|\frac{\Delta x_{s_1}(i)+\Delta x_{s_2}(i)+\Delta x_{s_3}(i)+...+\Delta x_{s_k}(i)}{k}\right|\right)^{p_n}$$

$$= \sum_{n=1}^{m_1}\left(a_n\sum_{i=1}^{n}q_i\left|\frac{\Delta x_{s_1}(i)+\Delta x_{s_2}(i)+\Delta x_{s_3}(i)+...+\Delta x_{s_k}(i)}{k}\right|\right)^{p_n} +$$

$$\sum_{n=m_1+1}^{\infty}\left(a_n\sum_{i=1}^{n}q_i\left|\frac{\Delta x_{s_1}(i)+\Delta x_{s_2}(i)+\Delta x_{s_3}(i)+...+\Delta x_{s_k}(i)}{k}\right|\right)^{p_n}$$

$$\leq \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(a_n\sum_{i=1}^{n}q_i\left|\Delta x_{s_j}(i)\right|\right)^{p_n} + \sum_{n=m_1+1}^{m_2}\left(a_n\sum_{i=1}^{n}q_i\left|\frac{\Delta x_{s_2}(i)+\Delta x_{s_3}(i)+...+\Delta x_{s_k}(i)}{k}\right|\right)^{p_n}$$

$$+ \sum_{n=m_2+1}^{\infty}\left(a_n\sum_{i=1}^{n}q_i\left|\frac{\Delta x_{s_3}(i)+...+\Delta x_{s_k}(i)}{k}\right|\right)^{p_n} + 2\varepsilon_1 \leq$$

$$\leq \sum_{n=1}^{m_1} \frac{1}{k} \sum_{j=1}^{k} \left(a_n \sum_{i=1}^{n} q_i \left| \Delta x_{s_j}(i) \right| \right)^{p_n} + \sum_{n=m_1+1}^{m_2} \frac{1}{k} \sum_{j=2}^{k} \left(a_n \sum_{i=1}^{n} q_i \left| \Delta x_{s_j}(i) \right| \right)^{p_n}$$

$$+ \sum_{n=m_2+1}^{m_3} \frac{1}{k} \sum_{j=3}^{k} \left(a_n \sum_{i=1}^{n} q_i \left| \Delta x_{s_j}(i) \right| \right)^{p_n} + \ldots + \sum_{n=m_{k-1}+1}^{m_k} \frac{1}{k} \sum_{j=k-1}^{k} \left(a_n \sum_{i=1}^{n} q_i \left| \Delta x_{s_j}(i) \right| \right)^{p_n} +$$

$$+ \sum_{n=m_k+1}^{\infty} \left(a_n \sum_{i=1}^{n} q_i \left| \frac{\Delta x_{s_j}(i)}{k} \right| \right)^{p_n} + (k-1)\varepsilon_1 \leq \frac{\sigma(x_{s_1}) + \sigma(x_{s_2}) + \ldots.. + \sigma(x_{s_{k-1}})}{k}$$

$$+ \frac{1}{k} \sum_{n=1}^{m_k} \left(a_n \sum_{i=1}^{n} q_i \left| \Delta x_{s_k}(i) \right| \right)^{p_n} + \sum_{n=m_k+1}^{\infty} \left(a_n \sum_{i=1}^{n} q_i \left| \frac{\Delta x_{s_k}(i)}{k} \right| \right)^{p_n} + (k-1)\varepsilon_1 \leq$$

$$\leq \frac{k-1}{k} + \frac{1}{k} \sum_{n=1}^{m_k} \left(a_n \sum_{i=1}^{n} q_i \left| \Delta x_{s_k}(i) \right| \right)^{p_n} + \frac{1}{k^{\alpha}} \sum_{n=m_k+1}^{\infty} \left(a_n \sum_{i=1}^{n} q_i \left| \Delta x_{s_k}(i) \right| \right)^{p_n} + (k-1)\varepsilon_1$$

$$\leq 1 - \frac{1}{k} + \frac{1}{k} \left[1 - \sum_{n=m_k+1}^{\infty} \left(a_n \sum_{i=1}^{n} q_i \left| \Delta x_{s_k}(i) \right| \right)^{p_n} \right] + \frac{1}{k^{\alpha}} \sum_{n=m_k+1}^{\infty} \left(a_n \sum_{i=1}^{n} q_i \left| \Delta x_{s_k}(i) \right| \right)^{p_n} + (k-1)\varepsilon_1$$

$$\leq 1 + (k-1)\varepsilon_1 - \left(\frac{k^{\alpha-1} - 1}{k^{\alpha}} \right) \sum_{n=m_k+1}^{\infty} \left(a_n \sum_{i=1}^{n} q_i \left| \Delta x_{s_k}(i) \right| \right)^{p_n}$$

$$\leq 1 + (k-1)\varepsilon_1 - \left(\frac{k^{\alpha-1} - 1}{k^{\alpha}} \right) \eta = 1 - \left(\frac{k^{\alpha-1} - 1}{k^{\alpha}} \right) \left(\frac{\eta}{2} \right).$$

By theorem (2.5) $\exists\, \gamma > 0$ such that $\left\| \frac{x_{s_1} + x_{s_2} + x_{n_3} + \ldots + x_{s_k}}{k} \right\| < 1 - \gamma$. Therefore, $\ell_{\Delta}\left[(a_n),(p_n),(q_n)\right]$ is k-NUC.

Since k-NUC implies k R and k R implies R and reflexivity holds, and k-NUC implies NUC and NUC implies H-property and reflexivity holds, by theorem (2.6), the following result are obtained.

COROLLARY2.7. For $\underset{n\to\infty}{Lim} \inf p_n > 1$, the space $\ell_{\Delta}\left[(a_n),(p_n),(q_n)\right]$ is kR, NUC, and has a drop property.

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