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Graceful Mirror-Staircase Graphs

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Abstract. A Graceful labeling on a graph with p vertices and q edges is a one-to-one map taking the vertices into the integers {0,1,2, …, q } with the property that each edge uv is assigned the label | f(u)-f(v) |. In this paper we prove that **mirror** - Staircase graphs and double-mirror staircase graphs are graceful.

Mathematics Subject Classification: 05C78, 05C30

Keywords: staircase graph, double- staircase graph, mirror-staircase graph, double-mirror-staircase graph

Introduction

Graceful labelings of graphs trace their origin to one introduced by Rosa [4] in 1967 or one given by Graham and Sloane [3] in 1980. Rosa [4] called a function f, a valuation of a graph G with q edges if f is an injection from the vertices of G to the set {0, 1,2,…, q } such that when each edge xy is assigned the label | f(x) – f(y) | , the resulting edge labels are distinct. Golomb [2] subsequently called such labeling graceful. In the cycle related graphs Rosa [4] showed that the n-cycle Cn is graceful if and only if n is 0 or 3 (mod 4) . The n-cone $C_m + K_n$ has been shown to be graceful when m is 0 or 3 (mod 12) by Bhat- Nayak and Selvam [1], when n is even and m is 2,6 or 10(mod 12). A.Solairaju and M.Antony Arockiasamy proved that (i) any Staircase graph $G(S_1, k)$ of order k is graceful (ii) any Double-Staircase graph $G(S_2, k)$ of order k is graceful [5].

Definition: 2.1 A staircase graph with k-steps is the graph obtained by merging the Cartesian products of P_2 and P_2 , P_2 and P_3 , P_2 and P_4 ... P_2 and P_k as given in thefollowing fig.1.

Definition: 2.2 The corner -vertex of a stair case graph is defined as a vertex ‘v ‘of degree 2, such that no adjacent vertex to ‘v ‘is of degree 4. For any stair case graph of finite order it is obvious that the number of corner vertices exactly 3. It is denoted by C_1, C_2, and C_3.

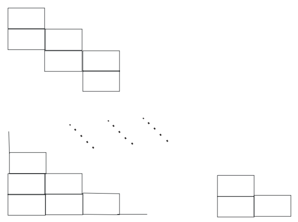

Fig.1 k-step staircase graphs

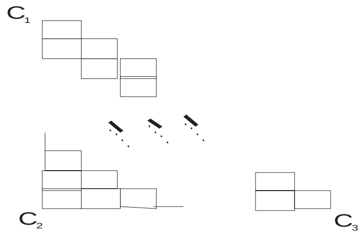

Fig.2.Corner vertices of Staircase graph

Definition: 2.3 Let G be a staircase graph defined as in 2.2. The order of G is defined as the number of two degree vertices in G, excluding the corner- vertices. Any staircase graph G of order k is denoted by G (S_2, k). Further by using the finite difference formula, we can easily prove that any staircase graph of order k will have $(K^2+5k+2)/ 2$ number of vertices and k^2+3k number of edges.

Definition: 2.4 Let G_1 (S_1 , k) and G_2(S_1 , k) be any two staircase graphs of order k. A graph G(S_m , 2k) is obtained by identifying the k+1 vertices of degree 3 and degree 2, in the path C_2 C_1 (or C_2C_3) of G_1(S_1 , k) with the k+1 vertices of degree 3 and degree 2 in the path C_2C_1 (or C_2 C_3) of G_2(S_1 , k) respectively in order.(see fig.3). The new staircase G(S_m , 2k) is called mirror-staircase graph of order 2k. Note that mirror- staircase graphs can be obtained, only from two staircase graphs of same order and resulting mirror-staircase graph will contain only two corner vertices.and naturally the order will be 2k, as shown in the following fig-3.

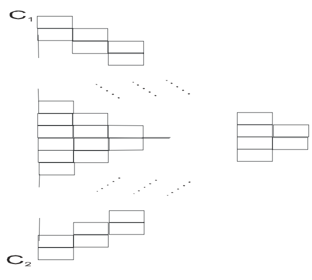

Fig.3 Mirror- staircase graphs

Section 3 - Gracefulness of mirror-staircase graphs

Theorem 3.1 Any mirror staircase graph of order 2k is graceful.

Proof: Let G(S_m , 2k) be a mirror- staircase of order 2k defined as in 2.4 . Note that k is either odd or even. The theorem is proved by giving the vertex labeling values of k of mirror staircase graphs. The vertex labeling for the mirror-staircase graph G(S_m, 2k) is given in the fig-4(k is odd) and fig-5(k is even). For the sake of easy labeling in figures, both in fig-4 and fig-5, we take $V_a = V_{r(2k+1)}$, $V_b = V_{r(2k-1)}$, $V_c = V_{r(2k-2)}$, $V_d = V_{r(2k-3)}$, $V_y = V_{r(k+4)}$, $V_x = V_{r(k+3)}$, $V_z = V_{r(k+2)}$, $V_w = V_{r(k+1)}$, $V_u = V_{r(k-1)}$ and $V_v = V_{r(k-2)}$.

Here $p = k^2+4k+1$ is the number of vertices and $q=2k^2+5k$ is the number of edges in any mirror-staircase graph G(S_m , 2k). Now, a vertex labeling ‘ f ‘ is defined on the vertex set of the mirror- staircase graph G(S_m ,2 k), (for both cases of k=odd and k=even) by $f(v_{r0}) = 0$, $f(v_{r1})= 2k^2 +5k$, $f(v_{ri}) = f(v_{rj})-i$ for i=3,5,7...2k+1, j=1,3,5...2k-1 ; $f(v_{ri}) = f(v_{rj})+i$ for i=2,4,6...2k+1, j=0,2,4...2k-1; And $f(v_{ri+l})=f(v_{ri}) + l$ if i is even, $1\leq l \leq k$; $f(v_{ri-m})= f(v_{ri}) - m$ if i is odd, $1\leq m \leq k$; where i = 1,2,3, ... , 2k+1 as shown in fig.4 and fig.5.

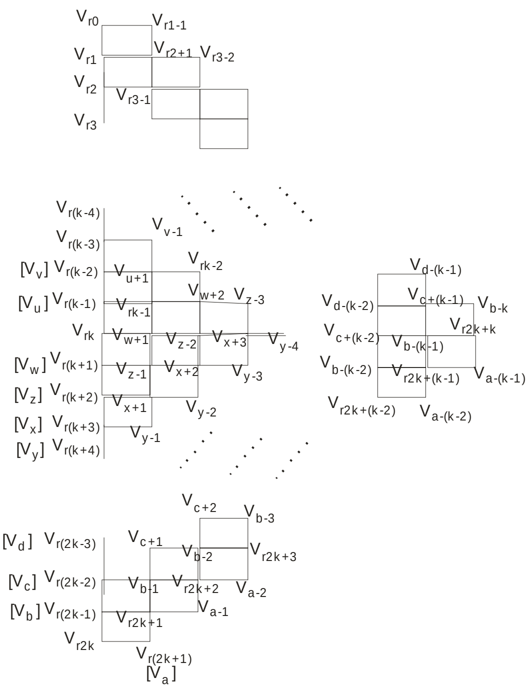

Fig.4 Vertex labeling for G(S_m , 2k) for odd k

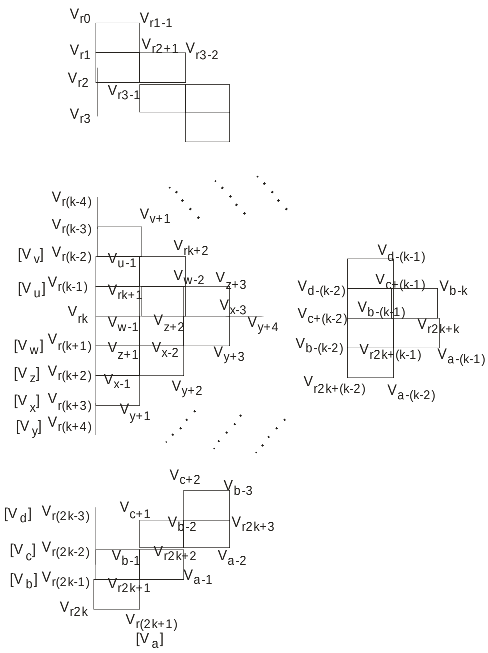

Fig.5 Vertex labeling for G(S_m , 2k) for even k

The edge set labeling f_+ is defined by $f_+(uv) = |f(u) – f(v)|$ for any edge uv in the mirror-stair case graph G(S_m , 2k).Thus G(S_m,2k), with the vertex labeling f and edge labeling f_+ satisfies the conditions for the graceful labeling .

Example: 3.2 The gracefulness of mirror staircases G(S_m, 10) and G(S_m ,12) are given in fig.6 and fig.7 respectively.

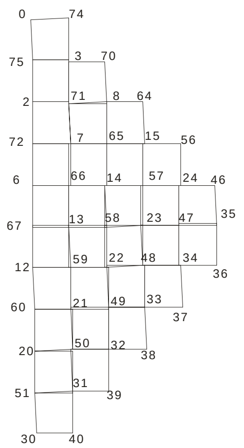

Fig.6 Graceful labeling of G(S_m , 10)

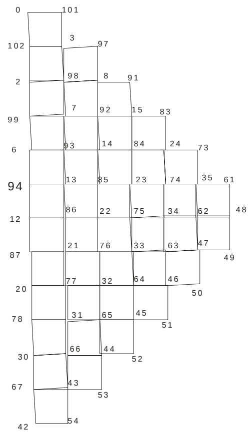

Fig.7 Graceful labeling of G(S_m , 12

Definition: 3.3 A double- staircase graph of same order is obtained ,by joining an edge between, any one of the corner vertices of a stair case graph of order k, with any one of the corner vertices of any other staircase graph, of same order. The resulting double- staircase graph will be of order k. It contains 4 corner vertices. As there are 3 corner vertices in each of the staircase graph joined, there are 9 possible ways of getting a double- stair case graph of same order. They are $C_{1\text{-}G1}$-$C_{1\text{-}G2}$ (i.e., C_1 of G_1(S, k) is joined with $C_1$of G_2 (S, k) by an

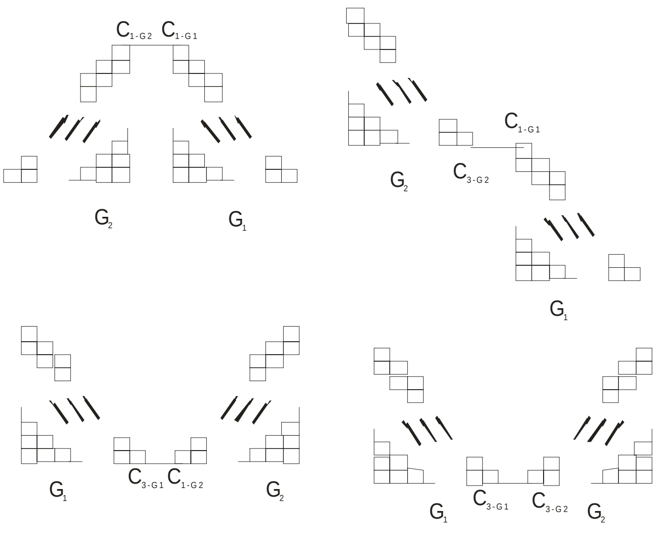

Fig.8 Category-1 double staircase graphs

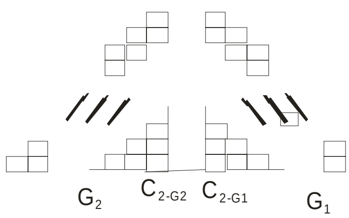

Fig.9 Category-2 double staircase graphs

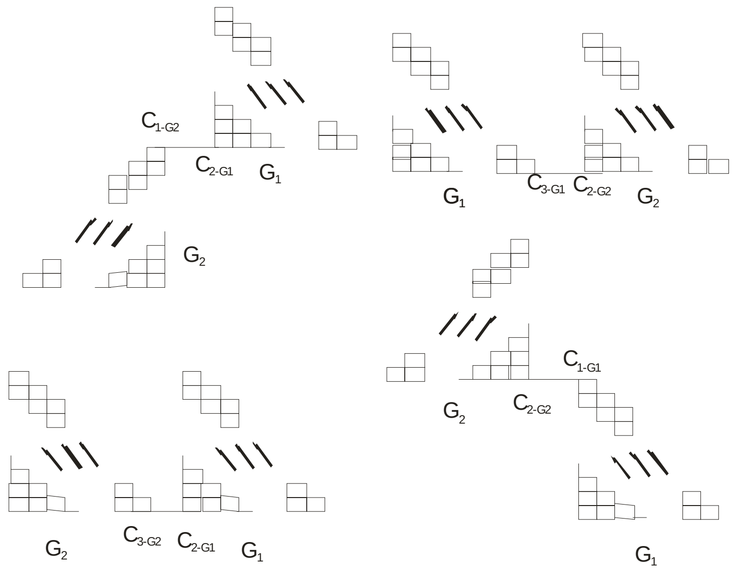

Fig.10 Category-3 double staircase graphs

edge), $C_{1\text{-}G1}$-$C_{2\text{-}G2}$, $C_{1\text{-}G1}$-$C_{3\text{-}G2}$, $C_{2\text{-}G1}$-$C_{1\text{-}G2}$, $C_{2\text{-}G1}$-$C_{2\text{-}G2}$, $C_{2\text{-}G1}$-$C_{3\text{-}G2}$, $C_{3\text{-}G1}$-$C_{1\text{-}G2}$, $C_{3\text{-}G1}$-$C_{2\text{-}G2}$ and $C_{3\text{-}G1}$-$C_{3\text{-}G2}$. These 9 combinations reduce to three combinations namely, $C_{3\text{-}G1}$- $C_{3\text{-}G2}$, $C_{2\text{-}G1}$-$C_{2\text{-}G2}$, $C_{2\text{-}G1}$-$C_{1\text{-}G2}$ as all the other combinations are isomorphic to any one of these three categories as shown in the following figures 8,9,10. A double staircase graph G of order k is denoted by G (S_2, k).

Definition: 3.4 Let $G_1(S_2$, k) and $G_2(S_2$, k) be any two double-staircase graphs of order k. By identifying the 2k+2 vertices , in the path C_2 C_3 of $G_1(S_2$, k) with (2k+2) vertices, in the path C_2C_3 of $G_2(S_1$, k) respectively in order, a new double-staircase graph is obtained which is called double-mirror-staircase graph. Note that double-mirror staircase graphs can be obtained, only from two double-staircase graphs of same order and by identifying the C_2 C_3 paths only. Since double-staircase graphs can be classified into three categories, their mirror images also have three categories of graphs, as given in figures 11, 12,13. A double- mirror- staircase graph obtained from $G_1(S_2$, k) and $G_2(S_2$, k) is denoted by $G(S_{2m}$, 2k). Out of these three categories of mirror images of double staircase graphs, only first category has graceful labelings whereas the second and third categories cannot be obtained as graceful graphs.

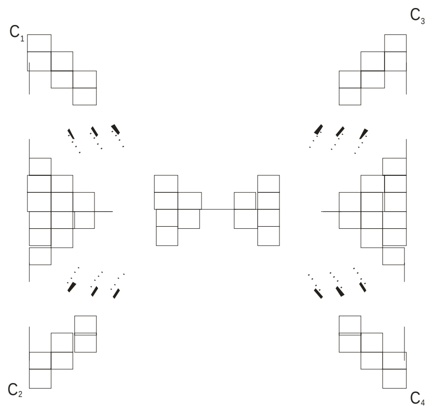

Fig.11 Category -1 Mirror double-staircase graph

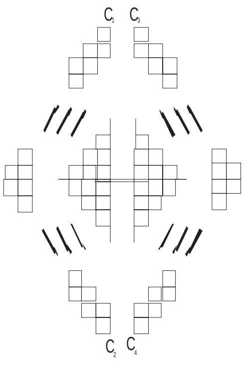

Fig.12 Category -2 Mirror double-staircase graph

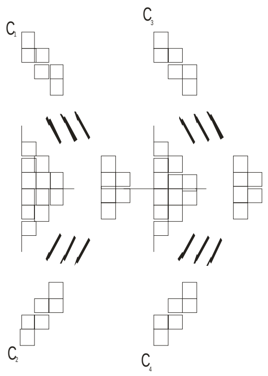

Fig.13 Category -3 Mirror double-staircase graph

Theorem: 3.5 Any Double-mirror-staircase graph of category-1 is graceful.

Proof: Let $G(S_{2m}, 2k)$ be a double-mirror- staircase of order 2k defined as in 3.4. The theorem is proved by giving the vertex labeling as shown in the following figures 14 and 15. For the sake of easy labeling in figures, both in fig-14 and fig-15, we take $V_a = V_{r(2k+1)}$, $V_b = V_{r(2k-1)}$, $V_c = V_{r(2k-2)}$, $V_d = V_{r(2k-3)}$, $V_y = V_{r(k+4)}$, $V_x = V_{r(k+3)}$, $V_z = V_{r(k+2)}$, $V_w = V_{r(k+1)}$, $V_u = V_{r(k-1)}$, $V_v = V_{r(k-2)}$, $V_l = V_{t(k-2)}$, $V_m = V_{t(k-1)}$, $V_n = V_{t(k+1)}$, $V_o = V_{t(k+2)}$, $V_p = V_{t(k+3)}$, $V_q = V_{t(k+4)}$, $V_e = V_{t(2k-3)}$, $V_f = V_{t(2k-2)}$ and $V_g = V_{t(2k-1)}$. Here $p = 2k^2+8k+2$ is the number of vertices and $q=4k^2+10k+1$ is the number of edges in any double-mirror-staircase-graph $G(S_{2m}, 2k)$. Now, a labeling f is defined on the vertex set of the double-mirror- staircase graph, $G(S_{2m}, 2k)$ by $f(v_{r0}) = 0$, $f(v_{r1}) = 4k^2 + 10k+1$, $f(v_{ri}) = f(v_{rj})-i$ for i=3,5,7...2k+1, j=1,3,5...2k-1; $f(v_{ri}) = f(v_{rj})+i$ for i=2,4,6...2k+1, j=0,2,4...2k-1;And $f(v_{ri+l})=f(v_{ri})+l$ if i is even,$1\leq l \leq k$; $f(v_{ri-m})=f(v_{ri})-m$ if i is odd,$1\leq m \leq k$; where i = 1,2,3, ... 2k+1.Similarly, $f(v_{t0}) = 2k^2 +5k+1$, $f(v_{t1})= f(v_{t0})-1$, $f(v_{ti}) = f(v_{tj})-i$ for i=3,5,7...2k+1, j=1,3,5...2k-1 ; $f(v_{ti}) = f(v_{tj})+i$ for i=2,4,6...2k+1, j=0,2,4...2k-1 ; And $f(v_{ti+l}) =f(v_{ti}) +l$ if i is even, $1\leq l \leq k$; $f(v_{ti-m}) = f(v_{ti}) - m$ if i is odd, $1\leq m \leq k$; where i = 1,2,3, ... , 2k+1 .

The edge set labeling f_+ is defined by $f_+(uv) = | f(u) – f(v) |$ for any edge uv in the double- mirror - stair case graph $G(S_{2m,}2k)$.Thus $G(S_{2m}, 2k)$ (where k is odd or even) , with the vertex labeling f and edge labeling f_+ satisfies the conditions for the graceful labeling .

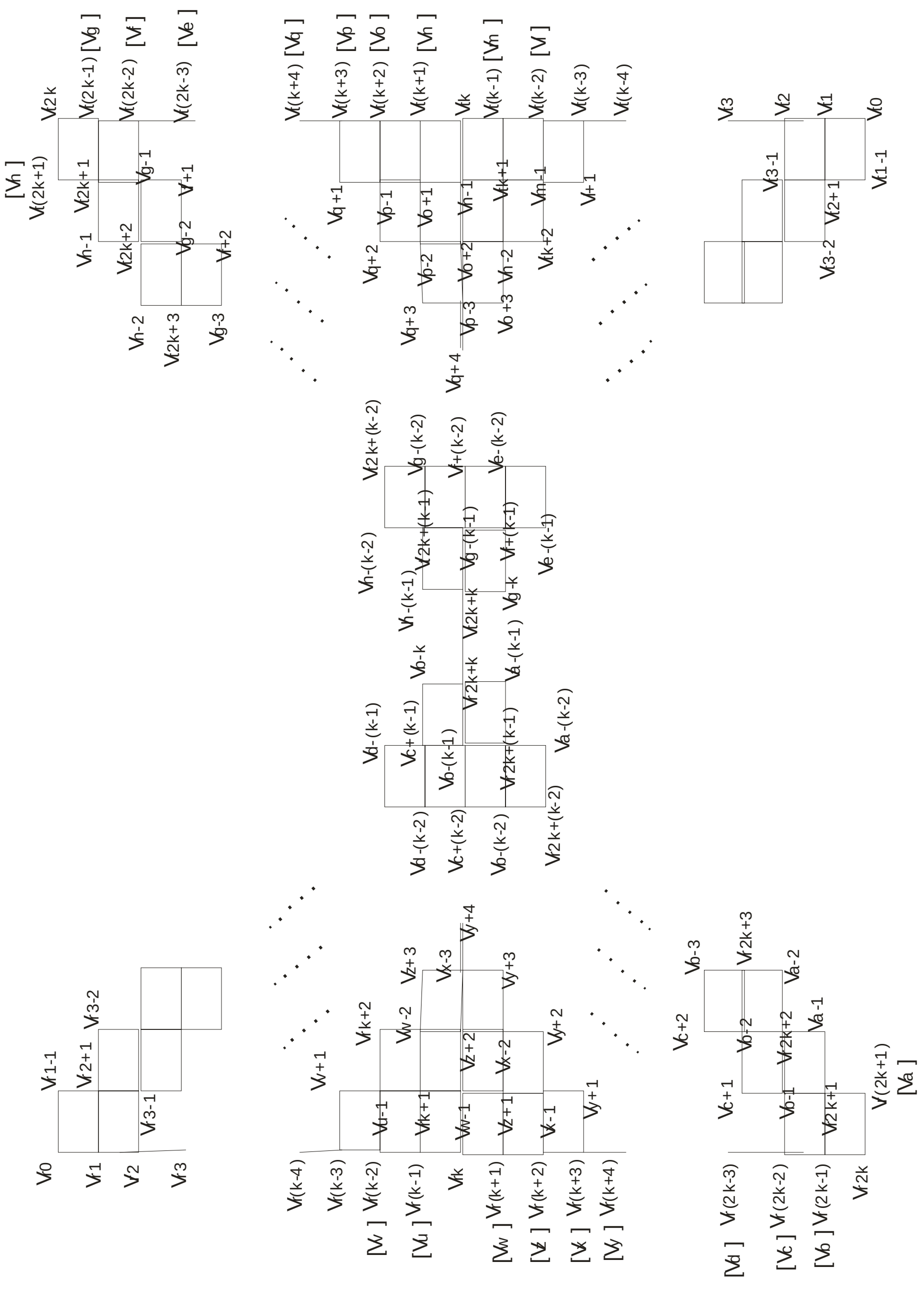

Fig.14- Vertex labellings of $G(S_{2m}, 2k)$ for even k

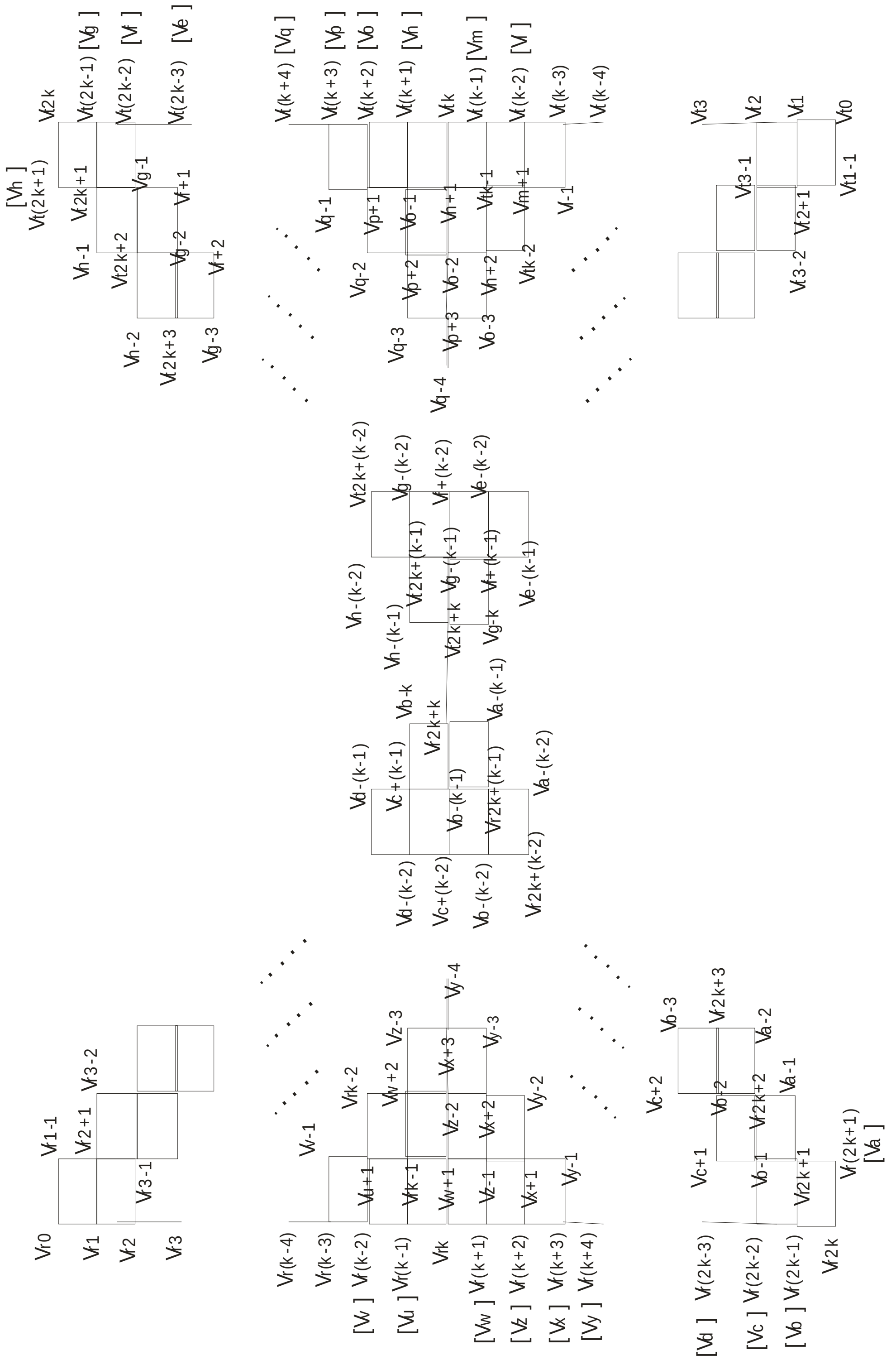

Fig.15- Vertex labellings of $G(S_{2m}, 2k)$ for odd k

Example 3.6 : The gracefulness of double- mirror staircases $G(S_{2m}, 10)$ and $G(S_{2m}, 12)$ are shown in fig.16and fig.17 respectively.

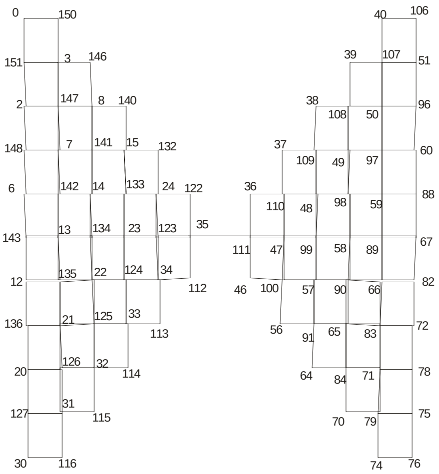

Fig.16 Graceful labeling of G(S_{2m} ,10)

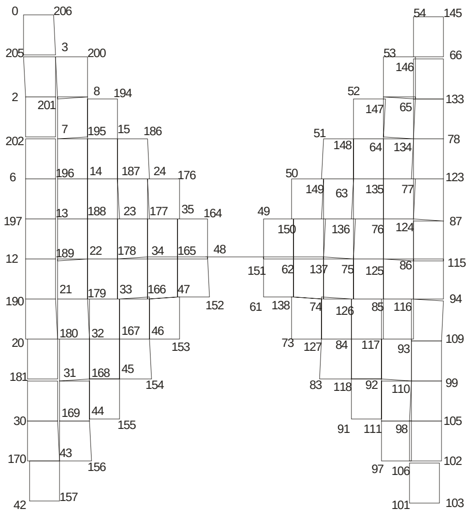

Fig.17 Graceful labeling of G(S_{2m} ,12)

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