Three-Step Fixed Point Iteration for Multivalued Mapping in Banach Spaces

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Abstract

In this paper, we consider the convergence of three-step fixed point iterative processes for multivalued nonexpansive mapping, under some different conditions, the sequences of three-step fixed point iterates strongly or weakly converge to a fixed point of the multivalued nonexpansive mapping. Our results extend and improve some recent results.

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1 Introduction

Let $X$ be a Banach space and $K$ a nonempty subset of $X$. We shall denote by $2^X$ the family of all subsets of $X$, $CB(X)$ the family of all nonempty closed bounded subsets of $X$ and denote $C(X)$ by the family of nonempty compact subsets of $X$. A multivalued mapping $T : K \to 2^X$ is said to be nonexpansive (resp, contractive) if

$$H(Tx,Ty) \leq \|x - y\|, \quad x, y \in K,$$

(resp, $H(Tx,Ty) \leq k\|x - y\|$, for some $k \in (0, 1)$).

where $H(\cdot, \cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A, B) := \max\{\sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\|\}, \quad A, B \in CB(X).$$

A point $x$ is called a fixed point of $T$ if $x \in Tx$. Since Banach’s Contraction Mapping Principle was extended nicely to multivalued mappings by Nadler
in 1969 (see [4]), many authors have studied the fixed point theory for multivalued mappings (e.g. see [1]). For single-valued nonexpansive mappings, Mann [3] and Ishikawa [2] respectively introduced a new iteration procedure for approximating its fixed point in a Banach space as follows:

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \]  

and

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha ny_n, y_n = (1 - b_n)x_n + b_nTx_n, \]  

where \( \{\alpha_n\} \) and \( \{b_n\} \) are sequences in \([0, 1]\). Obviously, Mann iteration is a special case of Ishikawa iteration. Recently Song in [7] and [8] introduce the following algorithms for multivalued nonexpansive mapping,

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha sn, \]  

where \( s_n \in Tx_n \), such that \( \|s_{n+1} - s_n\| \leq H(Tx_{n+1}, Tx_n) \).

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha r_n, y_n = (1 - b_n)x_n + b_ns_n, \]  

where \( \|s_n - r_n\| \leq H(Tx_n, Ty_n) \) and \( \|s_{n+1} - r_n\| \leq H(Tx_{n+1}, Ty_n) \) for \( s_n \in Tx_n \) and \( r_n \in Ty_n \). He show some strong convergence results of the above iterates for multivalued nonexpansive mapping \( T \) under some appropriate conditions.

In this paper, we introduced the following algorithm, which can be generalized as the above algorithms (3), (4):

**Algorithm.** For a given \( x_0 \in K \) and \( s_0 \in Tx_0 \). Let

\[ z_0 = (1 - a_0)x_0 + a_0s_0. \]

There exists \( t_0 \in Tz_0 \) such that \( \|t_0 - s_0\| \leq H(Tz_0, Tx_0) \). Let

\[ y_0 = (1 - b_0 - c_0)x_0 + b_0t_0 + c_0s_0. \]

There exists \( r_0 \in Ty_0 \) such that \( \|r_0 - t_0\| \leq H(Ty_0, Tz_0) \) and \( \|r_0 - s_0\| \leq H(Ty_0, Tx_0) \). Let

\[ x_1 = (1 - \alpha_0 - \beta_0 - \gamma_0)x_0 + \alpha_0r_0 + \beta_0t_0 + \gamma_0s_0. \]

There exists \( s_1 \in Tx_1 \) such that \( \|s_1 - r_0\| \leq H(Tx_1, Ty_0) \), \( \|s_1 - t_0\| \leq H(Tx_1, Tz_0) \) and \( \|s_1 - s_0\| \leq H(Tx_1, Tx_0) \). Inductively, we can get the sequence \( \{x_n\} \) as follows:

\[ z_n = (1 - a_n)x_n + a_ns_n \]
\[ y_n = (1 - b_n - c_n)x_n + b_nt_n + c_ns_n \]
\[ x_{n+1} = (1 - \alpha_n - \beta_n - \gamma_n)x_n + \alpha_nr_n + \beta_nt_n + \gamma_ns_n, \]  

(5)
where \( \{a_n\}, \{b_n\}, \{c_n\}, \{b_n + c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) \( \text{and} \{\alpha_n + \beta_n + \gamma_n\} \) are appropriate sequence in \([0, 1]\), furthermore \( s_n \in Tx_n, t_n \in Tz, r_n \in Ty_n \) such that \( \|t_n - s_n\| \leq H(Tz, Tx_n), \|r_n - t_n\| \leq H(Ty_n, Tz), \|r_n - s_n\| \leq H(Ty_n, Tx_n), \|s_{n+1} - r_n\| \leq H(Tx_{n+1}, Ty_n), \|s_{n+1} - t_n\| \leq H(Tx_{n+1}, Tz) \) \( \text{and} \|s_{n+1} - s_n\| \leq H(Tx_{n+1}, Tx_n) \). The iterative scheme (5) is called the three-step mean multivalued iterative scheme. If \( a_n = c_n = \beta_n = \gamma_n = 0 \), then iterative scheme (5) reduces to (4). If \( a_n = b_n = c_n = \beta_n = \gamma_n = 0 \), then iterative scheme (5) reduces to (3). In fact let \( \gamma_n \equiv 0 \) or \( c_n = \beta_n = \gamma_n \equiv 0 \) or \( b_n = c_n = \alpha_n = \gamma_n \equiv 0 \), we also have the other three Algorithms.

We consider the convergence of iterative scheme (5) for multivalued nonexpansive mapping, under some different conditions, we show that the sequences of iterative scheme (5) converge to a fixed point of the multivalued nonexpansive mapping \( T \). In particular, the following Theorem 3.3, Theorem 3.4 and Theorem 3.5 extend some results in [8], and also give some new results are different from the [7]. The following definition was introduced in [6].

**Definition 1.1** A multivalued mapping \( T : K \rightarrow CB(K) \) is said to satisfy Condition (A) if there is a nondecreasing function \( f : [0, \infty) \rightarrow [0, \infty) \) with \( f(0) = 0, f(x) > 0 \) for \( x \in (0, \infty) \) such that

\[
d(x, Tx) \geq f(d(x, F(T)) \quad \text{for all} \quad x \in K.
\]

Where \( F(T) \neq \emptyset \) is the fixed point set of the multivalued mapping \( T \). From now on, \( F(T) \) stands for the fixed point set of the multivalued mapping \( T \).

### 2 Preliminaries

A Banach space \( X \) is said to be satisfy Opial’s condition [5] if, for any sequence \( \{x_n\} \) in \( X \), \( x_n \rightharpoonup x (n \to \infty) \) implies the following inequality

\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|
\]

for all \( y \in X \) with \( y \neq x \). We know that Hilbert spaces and \( l_p(1 < p < \infty) \) have the Opial’s condition. The following Lemmas will be useful in this paper.

**Lemma 2.1** (see [9]) Let \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) be sequence in uniformly convex Banach space \( X \). Suppose that \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are sequence in \([0, 1]\) with \( \alpha_n + \beta_n + \gamma_n = 1 \), \( \limsup_n \|x_n\| \leq d, \limsup_n \|y_n\| \leq d, \limsup_n \|z_n\| \leq d \), and \( \lim_n \|\alpha_n x_n + \beta_n y_n + \gamma_n z_n\| = d \). If \( \liminf_n \alpha_n > 0 \) and \( \liminf_n \beta_n > 0 \), then \( \lim_n \|x_n - y_n\| = 0 \).

**Lemma 2.2** (see [9]) Let \( X \) be a uniformly convex Banach space and \( B_r := \{x \in X : \|x\| \leq r\}, r > 0 \). Then there exists a continuous strictly increasing
convex function \( g : [0, \infty) \to [0, \infty) \) with \( g(0) = 0 \) such that
\[
\|\lambda x + \mu y + \xi z + \vartheta \omega\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \vartheta\|\omega\|^2 \\
- \frac{1}{3} \vartheta(\lambda g(\|x - \omega\|) + \mu g(\|y - \omega\|) + \xi g(\|z - \omega\|)),
\]
for all \( x, y, z, \omega \in B_r \) and \( \lambda, \mu, \xi, \vartheta \in [0, 1] \) with \( \lambda + \mu + \xi + \vartheta = 1 \).

3 Main results

Lemma 3.1 Let \( X \) be a real Banach space and \( K \) be a nonempty convex subset of \( X \). Let \( T : K \to CB(K) \) be a multivalued nonexpansive mapping for which \( F(T) \neq \emptyset \) and for which \( T(p) = \{p\} \) for any fixed point \( p \in F(T) \). Let \( \{x_n\} \) be a sequence in \( K \) defined by (5), then we have the following conclusions:
\[
\lim_n \|x_n - p\| \text{ exists for any } p \in F(T).
\]

Proof. Let \( p \in F(T) \), from iterative scheme (5), note that \( T(p) = \{p\} \) for any fixed point \( p \in F(T) \), we have
\[
\|z_n - p\| \leq (1 - a_n)\|x_n - p\| + a_n\|s_n - p\| \\
\leq (1 - a_n)\|x_n - p\| + a_nH(Tx_n, Tp) \\
\leq \|x_n - p\|,
\]
similarly \( \|y_n - p\| \leq \|x_n - p\| \), and so we have
\[
\|x_{n+1} - p\| \leq (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\| + \alpha_n\|r_n - p\| \\
+ \beta_n\|t_n - p\| + \gamma_n\|s_n - p\| \\
\leq (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\| + \alpha_n d(Ty_n, Tp) \\
+ \beta_n H(Tz_n, Tp) + \gamma_n H(Tx_n, Tp) \\
\leq \|x_n - p\|.
\]
Then \( \{|\|x_n - p\|\| \} \) is a decreasing sequence and hence \( \lim_n \|x_n - p\| \) exists for any \( p \in F(T) \).

Lemma 3.2 Let \( X \) be a uniformly convex Banach space and \( K \) be a nonempty convex subset of \( X \). Let \( T : K \to CB(K) \) be a multivalued nonexpansive mapping for which \( F(T) \neq \emptyset \) and for which \( T(p) = \{p\} \) for any fixed point \( p \in F(T) \). Let \( \{x_n\} \) be a sequence in \( K \) defined by (5), if the coefficient satisfy one of the following control conditions:
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(i) \( \lim \inf_n \alpha_n > 0 \) and one of the following holds:
   (a) \( \lim \sup_n (\alpha_n + \beta_n + \gamma_n) < 1 \) and \( \lim \sup_n (b_n + c_n) < 1 \);
   (b) \( 0 < \lim \inf_n \beta_n \leq \lim \sup_n (\alpha_n + \beta_n + \gamma_n) < 1 \) and \( \lim \sup_n c_n < 1 \);
   (c) \( 0 < \lim \inf_n b_n \leq \lim \sup_n (b_n + c_n) < 1 \) and \( \lim \sup_n a_n < 1 \);
   (d) \( 0 < \lim \inf_n c_n \leq \lim \sup_n (b_n + c_n) < 1 \);

(ii) \( 0 < \lim \inf_n \beta_n \leq \lim \sup_n (\alpha_n + \beta_n + \gamma_n) < 1 \) and \( \lim \sup_n a_n < 1 \);

(iii) \( 0 < \lim \inf_n \gamma_n \leq \lim \sup_n (\alpha_n + \beta_n + \gamma_n) < 1 \);

(iv) \( 0 < \lim \inf_n (\alpha_n b_n + \beta_n) \) and \( 0 < \lim \inf_n a_n \leq \lim \sup_n a_n < 1 \).

then we have \( \lim_n d(x_n, T x_n) = 0 \).

**Proof.** By lemma 3.1, we know that \( \lim_n \|x_n - p\| \) exists for any \( p \in F(T) \), then it follows that \( \{s_n - p\}, \{t_n - p\}, \) and \( \{r_n - p\} \) are all bounded. We may assume that these sequences belong to \( B_r \) where \( r > 0 \). Note that \( T(p) = \{p\} \) for any fixed point \( p \in F(T) \). By Lemma 2.2, we get

\[
\|z_n - p\|^2 \leq (1 - a_n)\|x_n - p\|^2 + a_n\|s_n - p\|^2
\]

\[
\|y_n - p\|^2 \leq (1 - b_n - c_n)\|x_n - p\|^2 + b_n\|t_n - p\|^2 + c_n\|s_n - p\|^2
\]

\[
\leq (1 - b_n - c_n)\|x_n - p\|^2 + b_nH(Tz_n, Tp)^2 + c_nH(Tx_n, Tp)^2
\]

\[
-\frac{1}{3}(1 - b_n - c_n)H(Tz_n, Tp)^2 + \frac{1}{3}(1 - b_n - c_n)b_nH(Tx_n, Tp)^2 + c_nH(Tx_n, Tp)^2
\]

\[
\leq \|x_n - p\|^2 - \frac{1}{3}(1 - b_n - c_n)b_nH(Tx_n, Tp)^2 + c_nH(Tx_n, Tp)^2
\]

and therefore we have

\[
\|x_{n+1} - p\|^2 \leq (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\|^2 + \alpha_n\|r_n - p\|^2 + \beta_n\|t_n - p\|^2
\]

\[
+ \gamma_n\|s_n - p\|^2 - \frac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)[\alpha_n g(\|x_n - r_n\|) + \beta_n g(\|x_n - t_n\|) + \gamma_n g(\|x_n - s_n\|)]
\]

\[
\leq (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\|^2 + \alpha_nH(Ty_n, Tp)^2 + \beta_nH(Tz_n, Tp)^2
\]

\[
+ \gamma_nH(Tx_n, Tp)^2 - \frac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)[\alpha_n g(\|x_n - r_n\|) + \beta_n g(\|x_n - t_n\|) + \gamma_n g(\|x_n - s_n\|)]
\]

\[
\leq \|x_n - p\|^2 - \frac{\alpha_n}{3}(1 - b_n - c_n)b_nH(Tx_n, Tp)^2 + \frac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)[\alpha_n g(\|x_n - r_n\|) + \beta_n g(\|x_n - t_n\|) + \gamma_n g(\|x_n - s_n\|)].
\]
follows from (9) that

\[ (1 - \alpha_n - \beta_n - \gamma_n)\alpha_n g(\|x_n - r_n\|) \leq 3(\|x_n - p\|^2 - \|x_{n+1} - p\|^2), \]  
(6)

\[ (1 - \alpha_n - \beta_n - \gamma_n)\beta_n g(\|x_n - t_n\|) \leq 3(\|x_n - p\|^2 - \|x_{n+1} - p\|^2), \]  
(7)

\[ (1 - \alpha_n - \beta_n - \gamma_n)\gamma_n g(\|x_n - s_n\|) \leq 3(\|x_n - p\|^2 - \|x_{n+1} - p\|^2), \]  
(8)

and

\[ \alpha_n(1 - b_n - c_n)b_n g(\|t_n - x_n\|) \leq 3(\|x_n - p\|^2 - \|x_{n+1} - p\|^2). \]  
(9)

Since \( \lim_n \|x_n - p\| \) exists for any \( p \in F(T) \), it follows from (6) that \( \lim_n (1 - \alpha_n - \beta_n - \gamma_n)\alpha_n g(\|x_n - r_n\|) = 0 \). From \( g \) is continuous strictly increasing with \( g(0) = 0 \) and \( 0 < \lim \inf \alpha_n \leq \lim \sup (\alpha_n + \beta_n + \gamma_n) < 1 \), then

\[ \lim_n \|x_n - r_n\| = 0. \]  
(10)

Using a similarly method together with inequalities (7) and \( 0 < \lim \inf \beta_n \leq \lim \sup (\alpha_n + \beta_n + \gamma_n) < 1 \), then

\[ \lim_n \|x_n - t_n\| = 0. \]  
(11)

Similarly, from (8) and \( 0 < \lim \inf \gamma_n \leq \lim \sup (\alpha_n + \beta_n + \gamma_n) < 1 \), we have \( \lim_n \|x_n - s_n\| = 0 \), since \( s_n \in Tx_n \), then \( 0 \leq \lim_n d(x_n, Tx_n) \leq \lim_n \|x_n - s_n\| = 0 \), thus we get the (iii). In the sequence we prove the (i) (a). From iterative scheme (5), we have

\[ \|s_n - x_n\| \leq \|s_n - r_n\| + \|r_n - x_n\| \leq H(Tx_n, Ty_n) + \|r_n - x_n\| \]
\[ \leq \|x_n - y_n\| + \|r_n - x_n\| \]
\[ \leq b_n\|x_n - t_n\| + c_n\|x_n - s_n\| + \|r_n - x_n\|. \]  
(12)

To show that \( \lim_n \|x_n - s_n\| = 0 \), it suffices to show that there exist a subsequence \( \{n_j\} \) of \( \{n\} \) such that \( \lim_{n_j} \|x_{n_j} - s_{n_j}\| = 0 \). If \( \lim \inf b_{n_j} > 0 \), it follows from (9) that

\[ \alpha_{n_j}(1 - b_{n_j} - c_{n_j})b_{n_j} g(\|t_{n_j} - x_{n_j}\|) \leq 3(\|x_{n_j} - p\|^2 - \|x_{n_j+1} - p\|^2). \]

Since \( \lim_n \|x_n - p\| \) exists for any \( p \in F(T) \), we have

\[ \lim_{n_j} \alpha_{n_j}(1 - b_{n_j} - c_{n_j})b_{n_j} g(\|t_{n_j} - x_{n_j}\|) = 0. \]

From \( g \) is continuous strictly increasing with \( g(0) = 0 \), \( \lim \inf \alpha_{n_j} > 0 \) and \( 0 < \lim \inf b_{n_j} \leq \lim \sup (b_{n_j} + c_{n_j}) < 1 \), we have

\[ \lim_{n_j} \|t_{n_j} - x_{n_j}\| = 0. \]  
(13)
This together with (10), (12), (13) gives

\[ \lim_j (1 - c_{n_j}) \| s_{n_j} - x_{n_j} \| = 0. \]

Since \( \lim \inf_{n_j} (1 - c_{n_j}) = 1 - \lim \sup_{n_j} c_{n_j} > 0 \), we have \( \lim_j \| s_{n_j} - x_{n_j} \| = 0 \).

On the other hand, if \( \lim \inf_{j} b_{n_j} = 0 \), then we may extract a subsequence \( \{b_{n_k}\} \) of \( \{b_{n_j}\} \) so that \( \lim_k b_{n_k} = 0 \). This together with (i) (a) and (10), (12) gives

\[ \lim_k (1 - c_{n_k}) \| s_{n_k} - x_{n_k} \| = 0, \quad \text{and so} \quad \lim_k \| s_{n_k} - x_{n_k} \| = 0. \]

By Double Extract Subsequence Principle, we obtain the result.

If \( 0 < \lim \inf_n \beta_n \leq \lim \sup_n (\alpha_n + \beta_n + \gamma_n) < 1 \) and \( \lim \sup_n a_n < 1 \), we will prove (ii).

\[
\|s_n - x_n\| \leq \|s_n - t_n\| + \|t_n - x_n\| \leq H(Tx_n, Tz_n) + \|t_n - x_n\|
\leq \|x_n - z_n\| + \|t_n - x_n\|
\leq a_n \|x_n - s_n\| + \|t_n - x_n\|. \tag{14}
\]

Since \( \lim \sup_n a_n < 1 \), then

\[ \lim_n \inf (1 - a_n) = 1 - \lim_n \sup a_n > 0. \]

This together with (11), (14), we obtain the result.

We will prove (i) (b), let \( p \in F(T) \). By Lemma 3.1, we let \( \lim_n \| x_n - p \| = d \) for some \( d \geq 0 \). From iterative scheme (5), we know

\[
d = \lim_n \| x_{n+1} - p \| = \lim_n \|(1 - \alpha_n - \beta_n - \gamma_n)(x_n - p) + \alpha_n (r_n - p) + \beta_n (t_n - p) + \gamma_n (s_n - p)\|. \tag{15}
\]

From Lemma 3.1, we have known that \( \|z_n - p\| \leq \|x_n - p\| \) and \( \|y_n - p\| \leq \|x_n - p\| \), then

\[
\lim_n \|r_n - p\| \leq \lim_n \sup H(Ty_n, Tp) \leq \lim_n \sup \|y_n - p\| \leq d
\]
\[
\lim_n \|t_n - p\| \leq \lim_n \sup H(Tz_n, Tp) \leq \lim_n \sup \|z_n - p\| \leq d
\]
\[
\lim_n \|s_n - p\| \leq \lim_n \sup H(Tx_n, Tp) \leq \lim_n \sup \|x_n - p\| \leq d
\]

From (15) and Lemma 2.1, we have

\[ \lim_n \| x_n - t_n \| = \lim_n \| r_n - x_n \| = 0. \]

Notice that

\[
\|x_n - s_n\| \leq \|x_n - r_n\| + \|r_n - s_n\| \leq \|x_n - r_n\| + H(Ty_n, Tx_n)
\leq \|x_n - y_n\| + \|x_n - r_n\|
\leq b_n \|x_n - t_n\| + c_n \|x_n - s_n\| + \|x_n - r_n\|. \]
Since \( \limsup_n c_n < 1 \), we have \( \lim_n \| s_n - x_n \| = 0 \), therefore \( 0 \leq \lim_n d(x_n, Tx_n) \leq \lim_n \| x_n - s_n \| = 0 \).

We will prove (i) (c). From iterative scheme (5) and Lemma 3.1, we have

\[
\| x_{n+1} - p \| \leq (1 - \alpha_n - \beta_n - \gamma_n) \| x_n - p \| + \alpha_n \| y_n - p \|
+ \beta_n \| z_n - p \| + \gamma_n \| x_n - p \|.
\]

\[
\leq (1 - \alpha_n) \| x_n - p \| + \alpha_n \| y_n - p \|.
\]

Which implies

\[
\| x_{n+1} - p \| - \| x_n - p \| + \alpha_n \| x_n - p \| \leq \alpha_n \| y_n - p \|. \tag{16}
\]

Notice that \( \liminf_n \alpha_n > 0 \) and \( \lim_n \| x_n - p \| \) exists. Hence from (16) we have

\[
d = \lim_n \| x_n - p \| \leq \liminf_n \| y_n - p \| \leq \limsup_n \| y_n - p \| \leq d.
\]

Therefore from iterative scheme (5) we have

\[
d = \lim_n \| y_n - p \| = \lim_n \| (1 - b_n - c_n)(x_n - p) + b_n(t_n - p) + c_n(s_n - p) \| \tag{17}
\]

From Lemma 2.1, we have

\[
\lim_n \| x_n - t_n \| = 0.
\]

Notice that

\[
\| s_n - x_n \| \leq \| s_n - t_n \| + \| t_n - x_n \| \leq H(Tx_n, Tz_n) + \| t_n - x_n \|
\leq \| x_n - z_n \| + \| t_n - x_n \|
\leq a_n \| x_n - s_n \| + \| t_n - x_n \|.
\]

Since \( \limsup_n a_n < 1 \), then \( 0 \leq \lim_n d(x_n, Tx_n) \leq \lim_n \| x_n - s_n \| = 0 \).

By (17) and Lemma 2.1, we can similarly prove (i) (d).

Finally, we will prove (iv). From iterative scheme (5) and Lemma 3.1, we have

\[
\| x_{n+1} - p \| \leq (1 - \alpha_n - \beta_n - \gamma_n) \| x_n - p \| + \alpha_n \| r_n - p \|
+ \beta_n \| t_n - p \| + \gamma_n \| s_n - p \|
\leq (1 - \alpha_n - \beta_n - \gamma_n) \| x_n - p \| + \alpha_n \| y_n - p \|
+ \beta_n \| z_n - p \| + \gamma_n \| x_n - p \|.
\]

\[
\leq (1 - \alpha_n - \beta_n) \| x_n - p \| + \alpha_n \| (1 - b_n) \| x_n - p \|
+ b_n \| z_n - p \| + \beta_n \| z_n - p \|.\]
Which implies
\[ \|x_{n+1} - p\| - \|x_{n} - p\| + (\alpha_n b_n + \beta_n)\|x_n - p\| \leq (\alpha_n b_n + \beta_n)\|z_n - p\|. \]

Notice that
\[ 0 < \liminf_n (\alpha_n b_n + \beta_n) \text{ and } \lim_n \|x_n - p\| \text{ exists.} \]

Hence we have
\[ d = \lim_n \|x_n - p\| \leq \liminf_n \|z_n - p\| \leq \limsup_n \|z_n - p\| \leq d. \]

Thus we have
\[ d = \lim_n \|z_n - p\| = \lim_n \|(1 - a_n)\|x_n - p\| + a_n\|s_n - p\|. \]

By Lemma 2.1 and \(0 < \liminf_n a_n \leq \limsup_n a_n < 1\), we have \(0 \leq \lim_n d(x_n, Tx_n) \leq \lim_n \|x_n - s_n\| = 0\).

**Theorem 3.3** Let \(X, T\) and \(\{x_n\}\) be the same as in Lemma 3.2, if \(K\) be a nonempty compact convex subset of a Banach space \(X\), then \(\{x_n\}\) converges strongly to a fixed point of \(T\).

**Proof.** By Lemma 3.2, we have \(\lim_n d(x_n, Tx_n) = 0\). Since \(K\) be a nonempty compact convex subset, then there exist a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) such that \(\lim_{k \to \infty} \|x_{n_k} - q\| = 0\) for some \(q \in K\). Thus,
\[ d(q, Tq) \leq \|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tq) \leq 2\|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) \to 0. \]

Hence \(q\) is a fixed point of \(T\). Now on take on \(q\) in place of \(p\), we get that \(\lim_{n \to \infty} \|x_n - q\| \) exists. So the desired conclusion follows.

**Theorem 3.4** Let \(X, K, T\) and \(\{x_n\}\) be the same as in Lemma 3.2, if \(T\) satisfies Condition (A) with respect to the sequence \(\{x_n\}\), then \(\{x_n\}\) converges strongly to a fixed point of \(T\).

**Proof.** By Lemma 3.2, we have \(\lim_n d(x_n, Tx_n) = 0\). Since \(T\) satisfies Condition (A) with respect to \(\{x_n\}\). Then
\[ f(d(x_n, F(T))) \leq d(x_n, Tx_n) \to 0. \]

Thus we get \(\lim_n d(x_n, F(T)) = 0\). The remainder of the proof is the same as Theorem 2.4 in [8], we omit it.
Theorem 3.5 Let $X, T$ and $\{x_n\}$ be the same as in Lemma 3.2 and $T : K \to C(K)$. If $K$ be a nonempty weakly compact convex subset of a Banach space $X$ and $X$ satisfies Opial’s condition, then $\{x_n\}$ converges weakly to a fixed point of $T$.

Proof. The proof of the Theorem is the same as Theorem 2.5 in [8], we omit it.

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